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ON A SEMIDISCRETIZATION METHOD FOR THE PSEUDOPARABOLIC VON KÁRMÁN SYSTEM *

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Abstract.
We shall deal with a system of quasistationary von Kármán equations describing great deflections of thin viscoelastic plates made of Zener type material. In the special case of the exponential relaxation function is the original integral stress-strain relation transformed to the first order differential relation. The resulting system for the deflection and the Airy stress function can be considered as the pseudoparabolic generalization of the elastic von Kármán system. The existence of a unique weak solution as the limit of the sequence of segment line functions with respect to time is verified.

Key words. von Kármán system, viscoelastic plate, pseudoparabolic initial value problem, Rothe's method

AMS subject classifications. 74D10, 74K20, 45K05

1. Introduction.

We have dealt in [3] with the integro-differential von Kármán system for describing the geometrically nonlinear behaviour of the long memory isotropic viscoelastic plate. The relaxation function was exponentially decreasing. In the special case of the exponential relaxation function the originally integro-differential stress-strain relation can be transformed to the first order differential initial value problem with a nonzero initial elastic relation. The deflection of the middle surface of the plate and the Airy stress function are to be determined as solution of the generalized von Kármán system which is pseudoparabolic with respect to the deflection. The general n-th order case was investigated in [4], where the stability problem for the linearized case was considered. The dynamic von Kármán system with a viscoelastic damping term in the equation for the deflection has been considered by [11]. The authors in [16] considered the linear memory term with respect to the deflection. Thermo-viscoelastodynamic von Kármán system was investigated in [1], where a lot of references can be found. The fluid-structure interaction problem involving von Kármán system is considered in [9]. The latest paper of Ciarlet and Gratie [6], Ciarlet, Gratie and Sabu CGS present the new results on the justification of the geometrically nonlinear plate theory.

We have considered the anisotropic Voigt-Kelvin material with the zero initial condition in [2]. In the presented case the nonlinear pseudoparabolic term appears also in the right-hand side of the equation for the Airy stress function and the nonzero initial values fulfil the classical elastic von Kármán system. A weak formulation of the problem is equivalent with a nonlinear pseudoparabolic initial value problem with the time derivative appearing also in the nonlinear term. We solve the problem by Rothe's method. A finite sequence of stationary von Kármán-like equations is to be solved. The corresponding sequence of segment-like functions is convergent to a solution of the original pseudoparabolic problem in the case of sufficiently small right-hand side. This condition assures also the uniqueness of a solution.

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2. Formulation of the problem. We assume a thin isotropic plate occupying the domain

$$Q = \{(x,z) \in R^3;\ \mathbf{x} = (x_1,x_2) \in \Omega,\ -h/2 < z < h/2\},$$

where Ω is a bounded simply connected domain in R^2 with a Lipschitz boundary Γ.

We assume the plate subjected both to a perpendicular load f. Its boundary is divided in a form $\Gamma = \overline{\Gamma}_1 \cup \overline{\Gamma}_2 \cup \overline{\Gamma}_3$, where each Γ_i is either empty or $mes(\Gamma_i) > 0$. Further we assume that $\Gamma_1 \neq \emptyset$ or $\Gamma_2 \neq \emptyset$ and Γ_2 is not a segment of a straight line. The part Γ_3 contains only smooth parts. The plate is clamped on Γ_1, simply supported on Γ_2 and free on the part Γ_3.

Considering the great deflections we have the nonlinear strain-displacement relations

$$\varepsilon_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i + \partial_i w \partial_j w) - z\partial_{ij} w, \quad i,j = 1,2;\ \varepsilon_{13} = \varepsilon_{23} = 0$$

with plane displacements $u_1,\ u_2$ and a deflection w.

Let $\{\sigma^{ij}\}$ be the stress tensor fulfilling the condition $\sigma^{33} = 0$. The principle of virtual displacements holds in the form

$$\int_\Omega \left(\int_{-h/2}^{h/2} \sigma^{ij} \delta\varepsilon_{ij} \mathrm{d}z \right) \mathrm{d}x = \int_\Omega f(t,x) v(x) \mathrm{d}x \text{ for all } (\omega_1,\omega_2,v) \in U \times U \times V,$$

where v and ω_i are virtual displacements in the directions z and x_i $(i = 1,2)$ respectively and $U = H_0^1(\Omega),\ V \subset H^2(\Omega)$ are the spaces of admissible displacements. The virtual strains are of the form

$$\delta\varepsilon_{ij} = \frac{1}{2}(\partial_i \omega_j + \partial_j \omega_i + \partial_i w \partial_j v) - z\partial_{ij} v,\ i,j = 1,2.$$

The principle of virtual displacements implies that the stress resultants $N_{ij} = \int_{-h/2}^{h/2} \sigma^{ij} dz$ fulfil the homogeneous equations $\partial_j N_{ij} = 0, \quad i,j = 1,2.$

Then there exists the Airy stress function $\Phi : \Omega \to R$ defined by the equations

$$N_{11} = \partial_{22}\Phi, \quad N_{22} = \partial_{11}\Phi, \quad N_{12} = -\partial_{12}\Phi.$$

The stress-strain relations for the isotropic viscoelastic long memory material of the Boltzmann type are of the form

$$\sigma^{ij} = \frac{E(0)}{1-\mu^2}[(1-\mu)\varepsilon_{ij} + \mu\delta_{ij}\varepsilon_{kk}] + \frac{E'}{1-\mu^2} * [(1-\mu)\varepsilon_{ij} + \mu\delta_{ij}\varepsilon_{kk}](t), \tag{1}$$
$$i,j \in \{1,2\}, \quad \varepsilon_{kk} = \varepsilon_{11} + \varepsilon_{22}, \quad \sigma^{33} = 0$$

with a Poisson ratio $\mu \in (0, \frac{1}{2})$, a positive decreasing relaxation function $E \in C^1(R^+)$ and a convolution product $f * g(t) = \int_0^t f(t-s)g(s)\mathrm{d}s$.

Most of long memory viscoelastic material are modelled by the relaxation function of the form ([5])

$$E(t) = E_0 + \sum_{i=1}^{k} E_i e^{-\beta_i t},\ E_0 > 0,\ E_i > 0,\ \beta_i > 0,\ i = 1,...,k.$$

It is possible in the case $k = 1$ to transform the integro-differential stress-strain relations to the first order differential relations. Let

$$E(t) = E_0 + E_1 e^{-\beta t},\ E_0 > 0,\ E_1 > 0,\ \beta > 0.$$

After differentiating the relation (1) we obtain the following differential stres-strain relation characterizing the Zener viscoelastic model

$$(2)\quad \sigma'_{ij} + \beta\sigma_{ij} = \frac{E_0 + E_1}{1-\mu^2}[(1-\mu)\varepsilon_{ij} + \mu\delta_{ij}\varepsilon_{kk}]' + \frac{\beta E_0}{1-\mu^2}[(1-\mu)\varepsilon_{ij} + \mu\delta_{ij}\varepsilon_{kk}]$$

with the initial conditions

$$(3)\quad \sigma_{ij}(0) = \frac{E_0 + E_1}{1-\mu^2}[(1-\mu)\varepsilon_{ij} + \mu\delta_{ij}\varepsilon_{kk}](0).$$

Let us define the material constants $D_i = \frac{h^3}{12(1-\mu^2)}E_i,\ i = 0,1$ and the expression

$$(4)\quad [v,w] = \partial_{11}v\partial_{22}w + \partial_{22}v\partial_{11}w - 2\partial_{12}v\partial_{12}w, \quad v,\ w \in H^2(\Omega).$$

Let us assume that the perpendicular load f is differentiable with respect to t. Applying the similar approach as in the elastic case (see [8]) the following initial-boundary value problem for a pseudoparabolic von Kármán system for the deflection w and the Airy stress function Φ can be derived:

$$(5)\quad (D_0 + D_1)\Delta^2 w' + \beta D_0\Delta^2 w - [\Phi, w] = f'(t) + \beta f(t),\ \mathbf{x} \in \Omega,$$

$$(6)\quad w = \frac{\partial w}{\partial \nu} = 0 \text{ on } \Gamma_1,\ w = \mathcal{M}(w) = 0 \text{ on } \Gamma_2,\ \mathcal{M}(w) = \mathcal{S}(w) = 0 \text{ on } \Gamma_3,$$

$$(7)\quad \Delta^2\Phi = -\frac{h}{2}\left((E_0 + E_1)[w,w]' + \beta E_0[w,w]\right),\ \mathbf{x} \in \Omega$$

$$(8)\quad \Phi = \phi_0(t),\quad \frac{\partial\Phi}{\partial\nu} = \phi_1(t) \quad \text{on } \Gamma,\ t > 0,$$

$$(9)\quad w(0,\mathbf{x}) = w_0(\mathbf{x}),\ \mathbf{x} \in \Omega,$$

where

$$\mathcal{M}(w) = (D_0 + D_1)M(w') + \beta D_0 M(w),$$
$$M(w) = \mu\Delta w + (1-\mu)(w_{,11}\nu_1^2 + 2w_{,12}\nu_1\nu_2 + w_{,22}\nu_2^2),$$
$$\mathcal{S}(w) = w_{,1}\Phi_{,2\sigma} - w_{,2}\Phi_{,1\sigma} + (D_0 + D_1)S(w') + \beta D_0 S(w),$$
$$S(w) = -\frac{\partial}{\partial\nu}\Delta w + (1-\mu)\frac{\partial}{\partial\sigma}[w_{,11}\nu_1\nu_2 - w_{,12}(\nu_1^2 - \nu_2^2) - w_{,22}\nu_1\nu_2],$$

where $w_{,i} = \frac{\partial w}{\partial x_i},\ w_{,ij} = \frac{\partial^2 w}{\partial x_i \partial x_j},\ \Phi_{,i\sigma} = \frac{\partial}{\partial\sigma}\frac{\partial\Phi}{\partial x_i}$ and $\nu = (\boldsymbol{\nu_1}, \boldsymbol{\nu_2}),\quad \sigma = (-\boldsymbol{\nu_2}, \boldsymbol{\nu_1})$ are the unit outward normal and the unit tangential vector with respect to Ω respectively.

The initial deflection fulfils the stationary von Kármán system with the analogous boundary conditions as above:

$$(10)\ (D_0 + D_1)\Delta^2 w_0 - [\Phi_0, w_0] = f(0),$$

$$(11)\ w_0 = \frac{\partial w_0}{\partial\nu} = 0 \text{ on } \Gamma_1,\ w_0 = M(w_0) = 0 \text{ on } \Gamma_2,\ M(w_0) = S(w_0) = 0 \text{ on } \Gamma_3,$$

$$(12)\ \Delta^2\Phi_0 = -\frac{h}{2}(E_0 + E_1)[w_0, w_0],$$

$$(13)\ \Phi_0 = \phi_0(0),\quad \frac{\partial\Phi_0}{\partial\nu} = \phi_1(0) \quad \text{on } \Gamma, \quad t > 0$$

Let us introduce following Hilbert spaces:

$$H_0^2(\Omega) = \{v \in H^2(\Omega)|\ v = \frac{\partial v}{\partial \nu} = 0 \text{ on } \Gamma\}.$$

$H_0^2(\Omega)$ is the Hilbert space with the inner product $((.,.))_0$ and the norm $\|.\|_0$ defined by

$$((u,v))_0 = \int_\Omega \Delta u \Delta v \mathrm{d}x, \quad \|u\|_0 = ((u,u))_0^{1/2}, \quad u,v \in H_0^2(\Omega).$$

Further we introduce the Hilbert space

$$V = \{v \in H^2(\Omega)|\ v = \frac{\partial v}{\partial \nu} = 0 \text{ on } \Gamma_1, \quad v = 0 \text{ on } \Gamma_2\}$$

with the inner product $((.,.))$ and the norm $\|.\|$ defined by

$$((u,v)) = \int_\Omega [u_{,11}v_{,11} + 2(1-\mu)u_{,12}v_{,12} + u_{,22}v_{,22} + \mu(u_{,11}v_{,22} + u_{,22}v_{,11}]\mathrm{d}x \tag{14}$$

$$\|u\| = ((u,u))^{1/2}, \quad u,v \in V. \tag{15}$$

The norm defined in (15) is in the space V equivalent with the original norm

$$\|u\|_{H^2(\Omega)} = [\int_\Omega (u^2 + u_{,11}^2 + 2u_{,12}^2 + u_{,22}^2)\mathrm{d}x]^{1/2}$$

of the Sobolev space V (see [17], Lemma 11.3.2 for the details).

We denote by V^* the space of all linear bounded functionals over V with the norm $\|f\|_*$ and the duality pairing $\langle f, v\rangle$ for $f \in V^*$ and $v \in V$.

If X is a Banach space, then we denote by $C(0,T;X)$ the Banach space of continuous functions defined on the interval $[0,T]$ with values in X.

We suppose the functions $\phi_i : [0,T] \times \Gamma \to R,\ i = 0,1$ to be sufficiently smooth in order to enable the existence of a function $F \in C([0,T], H^2(\Omega))$ such that

$$F = \phi_0,\ \frac{\partial F}{\partial \nu} = \phi_1 \text{ on } \Gamma, \quad ((F(t),\phi))_0 = 0 \quad \text{for all } \phi \in H_0^2(\Omega). \tag{16}$$

The paper [10] contains the detailed assumptions imposed upon $\phi_0,\ \phi_1$ in order to fulfil (12), (13).

Let us introduce the trilinear form

$$\mathcal{B}(u,v;w) = \int_\Omega [(u_{,12}v_{,2} - u_{,22}v_{,1})w_{,1} + (u_{,12}v_{,1} - u_{,11}v_{,2})w_{,2}]\mathrm{d}x, \quad u,v,w \in H^2(\Omega). \tag{17}$$

The existence of the integral in (17) is assured due to the compact imbedding $H^2(\Omega) \subset W^{1,4}(\Omega)$. The form $\mathcal{B}$ fulfils the inequality

$$|\mathcal{B}(u,v;w)| \le \sqrt{2}|u|_{H^2(\Omega)}|v|_{W^{1,4}(\Omega)}|w|_{W^{1,4}(\Omega)}, \quad u,v,w \in H^2(\Omega) \tag{18}$$

with seminorms

$$|u|_{H^2(\Omega)} = [\int_\Omega (u_{,11}^2 + 2u_{,12}^2 + u_{,22}^2)\mathrm{d}x]^{1/2},$$

$$|v|_{W^{1,4}(\Omega)} = [\int_\Omega (v_{,1}^4 + v_{,2}^4)\mathrm{d}x]^{1/4}.$$

There holds a following crucial relation derived in [8]:

$$\mathcal{B}(u,v;w) = \int_\Omega [u,v]w\mathrm{d}x = \int_\Omega [u,w]v\mathrm{d}x, \tag{19}$$

if $u,\ v,\ w\ \in H^2(\Omega)$ and at least one function $u,\ v,\ w$ belongs to $H_0^2(\Omega)$.

Using the integration by parts after applying the boundary conditions we formulate a weak solution of the problem (5)-(13) in a similar way as in [10] for the elastic case.

DEFINITION 2.1. *The pair $\{w,\Phi\} \in W^{1,\infty}(0,T;V) \times L^\infty(0,T;H^2(\Omega))$ is a weak solution of the problem (5)-(13), if Φ fulfils the boundary conditions (8) for a.e. $t \in [0,T]$ there hold the identities*

$$((\ (D_0+D_1)w' + \beta D_0 w \ , \ v \ )) - \mathcal{B}(\Phi,w;v) = \langle f'(t)+\beta f(t), v\rangle \tag{20}$$

for all $v \in V$,

$$((\Phi(t),\phi))_0 = -\frac{h}{2}\int_\Omega (\ (E_0+E_1)[w,w]'(t) + \beta E_0[w,w](t) \ )v\mathrm{d}x \tag{21}$$

for all $v \in H_0^2(\Omega)$

and w fulfils the initial condition (9) *with $\{w_0,\Phi_0\} \in V \times H_0^2(\Omega)$ defined by the identities*

$$((\ (D_0+D_1)w_0 \ , \ v \ )) - \mathcal{B}(\Phi_0,w_0;v) = \langle f(0),v\rangle \text{ for all } v \in V, \tag{22}$$

$$((\Phi_0,\phi))_0 = -\frac{h}{2}\int_\Omega (\ (E_0+E_1)[w,w] \ )v\mathrm{d}x \text{ for all } v \in H_0^2(\Omega). \tag{23}$$

We introduce the bilinear operators $B : H^2(\Omega)\times H^2(\Omega) \to V$ and $B_0 : V\times V \to H_0^2(\Omega)$ as solutions of equations

$$((B(u,w),v)) = \mathcal{B}(u,w;v) \quad \text{for all } v \in V, \tag{24}$$

$$((B_0(u,w),\phi))_0 = \int_\Omega [u,w]\phi\mathrm{d}x \quad \text{for all } \phi \in H_0^2(\Omega). \tag{25}$$

Both equations are solved uniquelly, because the right-hand sides of both relations belong to the dual spaces V^* and $(H_0^2(\Omega))^*$ respectively.

The operators $B : H^2(\Omega)\times H^2(\Omega) \to V,\ B_0 : V\times V \to H_0^2(\Omega)$ are bounded (as bilinear operators) and fulfil the properties

$$((B(u,v),w)) = ((B(v,u),w)) = ((u,B(v,w))) = \int_\Omega [u,v]w\mathrm{d}x \tag{26}$$

for all $u \in H_0^2(\Omega),\ v,\ w \in V$,

$$B_0(u,v) = B_0(v,u), \tag{27}$$

$$((B(B_0(u,v),w),\phi)) = ((B_0(u,v),B_0(w,\phi) \ ))_0, \tag{28}$$

for all $u,v,w,\phi \in V$.

Using the definition of the operator B_0 we arrive at the nonlinear pseudoparabolic initial-boundary value problem for the deflection w :

$$\begin{aligned} &(D_0+D_1)\Delta^2 w'(t) + \beta D_0\Delta^2 w(t) - [F(t),w(t)] \\ &+\frac{h}{2}[(E_0+E_1)B_0(w,w)' + \beta E_0 B_0(w,w),w] \\ &= f'(t)+\beta f(t),\ x\in\Omega, \end{aligned} \tag{29}$$

$$(D_0 + D_1)\Delta^2 w(0) - [F(0), w(0)] + \frac{h}{2}(E_0 + E_1)[B_0(w(0), w(0)), w(0)] = f(0), \tag{30}$$

$$w = \frac{\partial w}{\partial \nu} = 0 \text{ on } \Gamma_1,\ w = \mathcal{M}(w) = 0 \text{ on } \Gamma_2,\ \mathcal{M}(w) = \mathcal{S}(w) = 0 \text{ on } \Gamma_3,, \tag{31}$$

$$w_0 = \frac{\partial w_0}{\partial \nu} = 0 \text{ on } \Gamma_1,\ w_0 = M(w_0) = 0 \text{ on } \Gamma_2,\ M(w_0) = S(w_0) = 0 \text{ on } \Gamma_3, \tag{32}$$

A weak formulation of the problem can be expressed as a nonlinear pseudoparabolic initial value problem in the Hilbert space V:

$$w'(t) + aw(t) - dB(F(t), w(t)) + bB(B_0(w, w)' + aB_0(w, w), w)(t) = q'(t) + \beta q(t), \tag{33}$$

$$w(0) - dB(F(0), w(0)) + bB(B_0(w(0), w(0)), w(0)) = q(0). \tag{34}$$

where

$$a = \frac{\beta D_0}{D_0 + D_1} = \frac{\beta E_0}{E_0 + E_1}, \quad b = \frac{h(E_0 + E_1)}{2(D_0 + D_1)} = \frac{6(1 - \mu^2)}{h^2}, \quad d = \frac{1}{D_0 + D_1}$$

and the function $q : [0, \infty) \to V$ is uniquely defined as a solution of the identity

$$((q(t), v)) = \frac{1}{D_0 + D_1}\langle f(t), v\rangle \text{ for all } v \in V.$$

3. Approximation by Rothe's Method. We shall verify the existence of a solution of the initial value problem (33),(34) using its discretization with respect to the time variable t by Rothe's method (see [12]).

Before formulating the discrete scheme we set some additional asssumptions on the functions q and F. We assume $q \in C^1([0, \infty), V)$ and $F \in C^1([0, \infty), H^2(\Omega))$.

Further we assume that

$$((B(F(t), v), v)) \leq 0 \text{ for all } v \in V,\ t \geq 0. \tag{35}$$

and

$$e^{-2\alpha t}\|q(0)\|^2 + \frac{1}{a}\int_0^t e^{-2\alpha(t-s)}\|q'(s) + \beta q(s)\|^2 \mathrm{d}s < \frac{1}{b\|B_0\|_0^2},\ t \in [0, T]. \tag{36}$$

for a sufficiently small $\alpha > 0$, which will be precised later.

For a fixed $T > 0$ and the integer N we set

$$\tau = \frac{T}{N},\ t_i = i\tau,\ w_i = w(t_i),\ i = 0, 1, ..., N;$$
$$\delta w_j = \frac{1}{\tau}(w_j - w_{j-1}),\ j = 1, ..., N.$$

Applying the discrete values w_i and the finite differences δw_i in (33), (34) we obtain the nonlinear equations in the space V:

$$w_0 - dB(F_0, w_0) + bB(B(w_0, w_0), w_0) = q_0, \tag{37}$$

$$\delta w_i + aw_i - dB(F_i, w_i) + bB(\delta B(w_i, w_i) + aB(w_i, w_i), w_i) = \delta q_i + \beta q_i,\ i = 1, ...n. \tag{38}$$

Both equations above have solutions w_0 and $w_i,\ i = 1, ..., n$ respectively. They are minimizers of the problems

$$J_i(w_i) = \min_{v \in V} J_i(v),\ i = 0, 1, ..., n, \tag{39}$$

where

$$J_0(v) = \frac{1}{2}[\|v\|^2 - d((B(F_0, w_0), w_0))] + \frac{b}{4}\|B(v, v)\|^2 - ((q_0, v)), \tag{40}$$

$$J_i(v) = \frac{1}{2}(1 + \tau a)[\|v\|^2 + \frac{b}{2}\|B(v, v)\|^2] - d((B(F_i, w_i), w_i))$$

$$+\frac{b}{2}((B(w_{i-1}, w_{i-1}), B(v, v))) - ((w_{i-1} + \tau(\delta q_i + \beta q_i), v)), \quad i = 1, ...n. \tag{41}$$

In order to obtain a priori estimates not depending on the length T of the time interval we express the values w_i in a form

$$w_i = e^{-\alpha\tau i} u_i,\ \alpha > 0,\ i = 0, 1, ..., n. \tag{42}$$

We have the following expression of the difference δw_i :

$$\delta w_i = (\delta e^{-\alpha\tau i}) u_i + e^{-\alpha\tau(i-1)} \delta u_i,\ i = 1, ..., n. \tag{43}$$

After setting $i = j$ in (38) and multiplying with $e^{\alpha\tau j} u_j$ in the Hilbert space V we obtain the identity

$$e^{\alpha\tau}((\delta u_j, u_j)) + \left(a - \frac{e^{\alpha\tau} - 1}{\tau}\right) \|u_j\|^2 + be^{-2\alpha\tau(j-1)}((\delta B_0(u_j, u_j), B_0(u_j, u_j)))_0$$

$$+\ be^{-2\alpha\tau j}\left(a - \frac{e^{2\alpha\tau} - 1}{\tau}\right) \|B(u_j, u_j)\|_0^2 - d((B(F_i, u_i), u_i)) = \ e^{\alpha\tau j}((\delta q_j + \beta q_j, u_j)),$$

Summing up and using the relations

$$2\tau \sum_{j=1}^{i}((\delta u_j, u_j)) = \|u_i\|^2 - \|u_0\|^2 + \sum_{j=1}^{i} \tau^2 \|\delta u_j\|^2,$$

$$\sum_{j=1}^{i} e^{\alpha\tau} e^{-2\alpha\tau j}((\ \delta B_0(u_j, u_j), B_0(u_j, u_j)\ ))_0 =$$

$$\sum_{j=1}^{i}((\delta(e^{-\alpha\tau j} B_0(u_j, u_j)\ ), e^{-\alpha\tau j} B_0(u_j, u_j)\ )) + \frac{e^{\alpha\tau} - 1}{\tau} \sum_{j=1}^{i} e^{-2\alpha\tau j} \|B_0(u_j, u_j)\|_0^2$$

we arrive at the inequality

$$e^{\alpha\tau}(\|u_i\|^2 + be^{-2i\alpha\tau}\|B_0(u_i, u_i)\|_0^2) +$$

$$2(a - \alpha e^{\alpha\tau}) \sum_{j=1}^{i} \tau \|u_j\|^2 + 2b(a - 2\alpha e^{2\alpha\tau}) \sum_{j=1}^{i} \tau e^{-2\alpha\tau j} \|B_0(u_j, u_j)\|_0^2$$

$$\leq\ e^{\alpha\tau}(\|u_0\|^2 + b\|B_0(u_0, u_0)\|_0^2) + 2 \sum_{j=1}^{i} \tau e^{\alpha\tau j}((\delta q_j + \beta q_j, u_j)).$$

We obtain directly from the equation (37) the estimate

$$\|w_0\|^2 + b\|B_0(w_0, w_0)\|_0^2 \le \|q_0\|^2. \tag{44}$$

Setting $\alpha > 0,\ \tau_0 > 0$ such that

$$\alpha e^{\alpha\tau} \le \frac{a}{2} \text{ for all } \tau \in (0, \tau_0) \tag{45}$$

we achieve considering $w_0 = u_0$ the estimate

$$\|u_i\|^2 \le \|q_0\|^2 + \frac{1}{a}\sum_{j=1}^{i} \tau e^{\alpha\tau(2j-1)}\|\delta q_j + \beta q_j\|^2.$$

Implying the expression (42) we obtain the estimate

$$\|w_i\|^2 \le e^{-2\alpha\tau i}\|q_0\|^2 + \frac{1}{a}\sum_{j=1}^{i} \tau e^{-\alpha\tau(1+2i-2j)}\|\delta q_j + \beta q_j\|^2, \tag{46}$$
$$i = 1, ..., N,\ 0 < \tau \le \tau_0.$$

We continue with uniform estimates of the differences. After multiplying the equation (38) with δw_j in V we obtain

$$\|\delta w_i\|^2 - d((B(F_i, w_i), \delta w_i)) + a((w_i, \delta w_i)) + \frac{1}{2}b\|\delta B_0(w_i, w_i)\|_0^2 +$$
$$\frac{1}{2}b\tau((\delta B_0(w_i, w_i), B_0(\delta w_i, \delta w_i))_0 + ab((\ B_0(w_i, w_i), B_0(w_i, \delta w_i)\ ))_0$$
$$= ((\delta q_i + \beta q_i), \delta w_i)), \tag{47}$$

where we have used the relation

$$2B_0(w_i, \delta w_i) = \delta B_0(w_i, w_i) + \tau B_0(\delta w_i, \delta w_i).$$

The a priori estimate (46) and the identity (47) further imply the inequality

$$\|\delta w_i\|^2 \le C_1 + \frac{1}{4}b\|B_0(w_i - w_{i-1}, \delta w_i)\|_0^2, \tag{48}$$

where the constant C_1 depends only on the constants $a,\ b$ and the function q and its derivative. Let us assume that

$$\|w_i\| < \frac{1}{b\|B_0\|_0^2},\ i = 1, ..., n. \tag{49}$$

Comparing with the a priori estimate (46) we can see that the condition (36) is sufficient for fulfilling the estimate (49) for $\tau \in (0, \tau_0)$ with sufficiently small τ_0.

Comparing with (48), (49) we obtain the a priori estimate

$$\|\delta w_j\| \le C_2,\ i = 1, ..., N. \tag{50}$$

Let us further define the following functions determined by values $w_i,\ \delta w_i$

$$w_n : [0, T] \to V,\ w_n(t) = w_{i-1} + (t - t_{i-1})\delta w_i,\ t_{i-1} \le t \le t_i,$$
$$\bar{w}_n : [0, T] \to V,\ \bar{w}_n(0) = w_0,\ \bar{w}_n(t) = w_i,\ t_{i-1} < t \le t_i,\ i = 1, ..., n.$$

The sequence of functions $\{w_n\}$ is due to previous a priori estimates bounded in the Sobolev space $W^{1,\infty}(0,T;V)$:

$$\|w_n\|_{W^{1,\infty}(0,T;V)} \le C_3,\ n = 1, 2, ... \tag{51}$$

Then there exists its subsequence (again denoted by $\{w_n\}$) and a function $w \in W^{\infty,1}(0,T;V)$ such that

$$w_n \rightharpoonup^* w \quad \text{in} \quad W^{1,\infty}(0,T;V), \tag{52}$$

$$w_n(t) \rightharpoonup w(t),\ \bar{w}_n(t) \rightharpoonup w(t) \text{ in } V \text{ for every } t \in [0,T], \tag{53}$$

$$w_n \rightharpoonup^* w,\ \bar{w}_n \rightharpoonup^* w,\ w_n' \rightharpoonup^* w' \text{ in } L^\infty(0,T;V), \tag{54}$$

$$w_n \to w,\ \bar{w}_n \to w \text{ in } L^p(0,T;W^{1,r}(\Omega)),\ p > 1,\ r > 1. \tag{55}$$

Setting $B_0(w_i, w_i) = U_i,\ i = 0, 1, ..., N$ we obtain also the existence of $U \in W^{1,\infty}(0,T;V)$ fulfilling

$$U_n \rightharpoonup^* U \quad \text{in} \quad W^{1,\infty}(0,T;V), \tag{56}$$

$$U_n(t) \rightharpoonup U(t),\ \bar{U}_n(t) \rightharpoonup U(t) \text{ in } V \text{ for every } t \in [0,T], \tag{57}$$

$$U_n \rightharpoonup^* U,\ \bar{U}_n \rightharpoonup^* U,\ U_n' \rightharpoonup^* U' \text{ in } L^\infty(0,T;V). \tag{58}$$

Using the properties of the bilinear operator $B_0 : V \times V \to H_0^2(\Omega)$ we obtain that

$$U(t) = B_0(w(t), w(t)). \tag{59}$$

We express the discrete equations (37), (38) in a differential form

$$\begin{aligned}
&w_n'(t) + a\bar{w}_n(t) - dB(\bar{F}_n(t), \bar{w}_n(t)) + bB(U_n'(t) + a\bar{U}_n(t), \bar{w}_n(t)) \\
&= q_n'(t) + \beta\bar{q}_n(t) \ \text{ for a.e. } \ t \in (0,T] \\
&w_n(0) - dB(F(0), w_n(0)) + bB(B_0(w_n(0), w_n(0)), w_n(0)) = q_n(0).
\end{aligned}$$

We shall verify that the limiting function $w \in W^{1,\infty}(0,T;V)$ is a solution of the initial value problem (33), (34). We have directly from the definition of $w_0 \in V$ in (37) that

$$w_n(0) = w_0 = w(0) \ \text{ for } n = 1, 2, ..., \tag{60}$$

and the initial condition (34) is fulfilled.

Let $v \in L^2(0,T;V)$ be an arbitrary test function. The regularity of the functions $q,\ F$ and the convergence (52), (54) imply

$$\int_0^T ((\ q_n'(t) + \beta\bar{q}_n(t), v(t)\ ))\mathrm{d}t \to \int_0^T ((\ q'(t) + \beta q(t), v(t)\ ))\mathrm{d}t, \tag{61}$$

$$\int_0^T ((\ w_n'(t) + a\bar{w}_n(t), v(t)\ ))\mathrm{d}t \to \int_0^T ((\ w'(t) + aw(t), v(t)\ ))\mathrm{d}t. \tag{62}$$

$$\int_0^T ((\ B(\bar{F}_n(t), \bar{w}_n(t)), v(t)\ ))\mathrm{d}t \to \int_0^T ((\ B(F(t), w(t)), v(t)\ ))\mathrm{d}t. \tag{63}$$

The properties (18), (19) and the convergence (55), (58) imply the convergence

$$\begin{aligned}
&\int_0^T \int_\Omega [U_n'(t) + a\bar{U}_n(t), v(t)]\bar{w}_n(t)\mathrm{d}x\mathrm{d}t \ \to \\
&\int_0^T \int_\Omega [U'(t) + aU(t), v(t)]w(t)dx\mathrm{d}t.
\end{aligned} \tag{64}$$

and it follows the identity

$$\begin{aligned}&\int_0^T ((w'(t) + aw(t) - dB(F(t), w(t)), v(t)))\mathrm{d}t \ + \\ &\int_0^T ((bB(B_0(w(t), w(t))' + aB_0(w(t), w(t)), w(t)), v(t)))\mathrm{d}t \ = \\ &\int_0^T ((q'(t) + \beta q(t), v(t)))\mathrm{d}t \text{ for all } v \in L^2(0, T; V),\end{aligned}$$

which implies together with (60) that w is a solution of the initial value problem (33), (34).

We formulate the existence and uniqueness theorem.

THEOREM 3.1. *Let $q \in C^1([0, T]; V)$ fulfil the condition (36) with $\alpha < \frac{a}{2}$. Then there exists a unique solution $w \in W^{1,\infty}(0, T; V)$ of the initial value problem (33), (34).*

There exists a subsequence of a sequence $\{w_n\}$ of segment line functions defined by discrete values w_i fulfilling the equations (37), (38) such that the convergence (52)-(55) holds.

Proof. We have verified the existence and the convergence above. It remains us to verify the uniqueness.

We derive it even in the case that the bounds (36) is fulfilled only for the initial point $t = 0$:

$$\|q(0)\| < \frac{1}{\sqrt{b}\|B_0\|}. \tag{65}$$

Let $w_i \in W^{1,\infty}(0, T; V), i = 1, 2$ be solutions of the initial value problem (33), (34). The condition (65) enables the uniqueness of a solution of the stationary equation (33) determining the initial condition. Really, setting $w_0 = w_1(0) - w_2(0)$ and using the assumption (35) we have the inequality

$$\|w_0\|^2 \le b((B(B_0(w_2(0), w_2(0)), w_2(0)) - B(B_0(w_1(0), w_1(0)), w_1(0)), w_0)). \tag{66}$$

Let us set $w_{0\xi} = w_2(0) + \xi w_0,\ \xi \in R$. There holds the relation

$$\begin{aligned}&((B(B_0(w_2(0), w_2(0)), w_2(0)) - (B(B_0(w_1(0), w_1(0)), w_1(0)), w_0)) = \\ &- \int_0^1 [((B_0(w_{0\xi}, w_{0\xi}), B_0(w, w)))_0 + 2\|B_0(w_{0\xi}, w)\|_0^2]\mathrm{d}\xi.\end{aligned} \tag{67}$$

The assumption (65) implies the estimates

$$\|w_i(0)\| \le \frac{1}{\sqrt{b}\|B_0\|},\ i = 1, 2.$$

The inequality (66) together with (67) implies $\|w_0\|^2 \le 0$ and the uniqueness of the initial condition follows.

The difference $w = w_1 - w_2$ of two solutions of the equation (33) then fulfils the homogeneous initial value problem

$$\begin{aligned}&w'(t) + aw(t) - dB(F(t), w(t)) + bB(B_0(w_1, w_1)' + aB_0(w_1, w_1), w_1)(t) \\ &-bB(B_0(w_2, w_2)' + aB_0(w_2, w_2), w_2)(t) = 0,\end{aligned} \tag{68}$$

$$w(0) = 0. \tag{69}$$

After multiplying with w in the space V and integrating we obtain the inequality

$$\|w(t)\|^2 + a\int_0^t \|w(s)\|^2 ds \leq \tag{70}$$

$$b\int_0^t ((B(B_0(w_2,w_2)',w_2) - (B(B_0(w_1,w_1)',w_1),w))\mathrm{d}s +$$

$$ba\int_0^t ((B(B_0(w_2,w_2),w_2) - (B(B_0(w_1,w_1),w_1),w))\mathrm{d}s.$$

$w_\xi = w_2 + \xi w,\ \xi \in R$. We can then express the functions in the integrals on the right-hand side of (70) in a following way

$$((B(B_0(w_2,w_2)',w_2) - (B(B_0(w_1,w_1)',w_1),w)) =$$

$$-\int_0^1 [((\ B_0(w_\xi,w_\xi)', B_0(w,w)\ ))_0 + ((\ B_0(w,w_\xi)', B_0(w_\xi,w)\ ))_0]\mathrm{d}\xi,$$

$$((B(B_0(w_2,w_2),w_2) - (B(B_0(w_1,w_1),w_1),w)) =$$

$$-\int_0^1 [((\ B_0(w_\xi,w_\xi), B_0(w,w)\ )) + 2\|B_0(w_\xi,w)\|_0^2]\mathrm{d}\xi.$$

Using the fact that functions $w_i,\ i=1,2$ belong to the space $W^{1,\infty}(0,T;V)$ and the same holds for w_ξ we obtain from (70) the estimate

$$\|w(t)\|^2 \leq C_4\int_0^t \|w(s)\|^2\mathrm{d}s$$

with the constant C_4 depending only on $a,\ b,\ \|B\|$ and the right-hand side $q' + \beta q$. The Gronwall lemma implies $w(t) = 0,\ t \in [0,T]$ and the uniqueness of a solution follows. □

After coming back to the original problem for a couple $\{w,\Phi\}$ of the deflection and the Airy stress function we obtain directly

THEOREM 3.2. *Let $f \in C^1([0,T];V^*)$ fulfil the bound*

$$e^{-2\alpha t}\|f(0)\|_*^2 + \frac{1}{a}\int_0^t e^{-2\alpha(t-s)}\|f'(s)+\beta f(s)\|_*^2 ds < \frac{D_0^2}{b\|B_0\|^2},\ t\in[0,T] \tag{71}$$

with an arbitrary $\alpha \in (0,\frac{a}{2})$.

Then there exists a unique weak solution

$$\{w,\Phi\} \in W^{1,\infty}(0,T;V)\times L^\infty(0,T;H^2(\Omega))$$

of the initial boundary value problem (3)-(13).

REMARK 3.3. *The exponential character of the conditions (36) and (71) implies that the bounds for the right hand sides q, q' or f, f' do not depend on the length T of the time interval.*

REMARK 3.4. *Applying the Rothe method to the weak formulation of the problem (3)-(13) means that we obtain the stationary von Kármán like system at each time level. A solution is a unique minimizer of the functional defined in (41). We can use* *some of the gradient algorithms ([14]) combined with cubic finite elements in order to solve the corresponding minimum problem.*

Another possibility is to use the mixed formulation of the stationary problem due to Miyoshi [15] or [13], [18]. A weak formulation of the problem is converted into the problem involving 8 unknown functions with at most 2-nd order derivatives and using linear finite elements.

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