Proceedings of ALGORITMY 2002
Conference on Scientific Computing, pp. 23–32

EXTENDING THE COX-ROSS-RUBINSTEIN ALGORITHM FOR PRICING OPTIONS WITH EXPONENTIAL BOUNDARIES

MASSIMO COSTABILE *

Abstract. It is a common belief that the standard binomial algorithm of Cox-Ross-Rubinstein (CRR) cannot be used to deal with barrier options with multiple or time-varying boundaries. We propose an extension of the CRR model to evaluate options with exponential boundaries. The essence of the extended binomial model relies upon the construction of a binomial tree for the underlying asset price dynamics, characterized by sets of nodes that mirror the barriers evolution. As a result, a very easy algorithm is derived that produces accurate prices with respect to the corresponding continuous time values. Moreover, numerical results show that the performance of the extended binomial algorithm is superior to that of the trinomial algorithms usually employed to price these options.

Key words. barrier options, binomial algorithm

AMS subject classifications. 05A10, 91B28

1. Introduction. In the traditional binomial method of Cox, Ross and Rubinstein (CRR) [4] for pricing European options, the underlying asset price dynamics is modeled by a random walk. This means that, if $S(t) = S$ is the asset value at time t then, after one period, at time $t+1$, it can rise to uS or decrease to dS, where u and d represent, respectively, the magnitude of one up step and the magnitude of one down step. As a consequence, the underlying asset price evolution can be represented by a binomial tree where each node corresponds to one possible value of the asset price. The usual condition $ud = 1$ implies that the binomial tree is characterized by sets of horizontal nodes.

In 1994 Boyle and Lau [2] pointed out that a naïve application of the traditional CRR algorithm can lead to consistent bias in pricing options with one flat barrier. The way to overcome this drawback is to build up a binomial tree with a set of nodes that is as close as possible to the barrier and mirrors the barrier evolution. In this case the problem of defining a binomial tree with a set of nodes that mimics the barrier dynamics is solved automatically because the barrier is constant and we consider trees with horizontal set of nodes. So we only need to choose a suitable number of steps in price computations such that there exists one layer of the lattice as close as possible to the barrier.

In the case of options with time-varying barriers, this approach fails because of the fact that the sets of horizontal nodes in the binomial tree cannot mirror the barriers evolution. As a consequence, the CRR algorithm computes consistently biased prices that converge very slowly towards the corresponding values obtained in a continuous time setting.

To overcome this problem a trinomial model has been proposed by Ritchken [10] to evaluate barrier options both with one flat boundary and with more complex boundaries of time varying and multiple type.

However the problem of pricing complex barrier options within a binomial setting can be successfully tackled. In Costabile [3] a binomial algorithm has been proposed

*Dipartimento di Organizzazione Aziendale e Amministrazione Pubblica, University of Calabria, Arcavacata di Rende (CS) 87036, Italy (massimo.costabile@unical.it).

to deal with double barrier options. Here we present an extension of the CRR model to price options with curved barriers. It is based on a binomial tree for the evolution of the underlying asset price characterized by sets of nodes that mimic the barrier dynamics. Relaxing the usual condition $ud = 1$ and choosing a particular size for the magnitude of the up moves and down-moves of the asset price process, a binomial tree is derived with layers of nodes that exactly mirror the moving time barriers. Following this way we construct a very simple algorithm that produces accurate prices with respect to continuous time values illustrated in Zhang (1998). As in Lyuu(1998) we compare the performance of the extended binomial algorithm with that of the Ritchken trinomial tree based model. The conclusion is that the extended binomial algorithm is clearly more efficient than the trinomial model in evaluating options with curved boundaries.

The remainder of the paper is organized as follows. In Section 2 we illustrate the problem of pricing options with an exponential boundary both in a continuous time environment and in a discrete time one. In Section 3 we present the extended binomial model to evaluate options with a single exponential boundary. In Section 4 we illustrate how the extended binomial algorithm can be used to evaluate double barrier options. In Section 5 we analyze the performance of the extended binomial algorithm with respect to that of the trinomial tree based model. In Section 6 we draw conclusions.

2. The barrier option pricing problem. We consider a European down-and-out call option with strike price K and time to maturity T, written at time t on an asset with price $S(t) = S$. The option vanishes if the underlying asset price reaches before expiration the barrier $He^{\delta\tau}$ $(H < S, \delta \in \mathbb{R}, \quad 0 \le \tau \le T)$.

In a continuous time setting, under the usual assumptions of the Black-Scholes analysis [1], the price of this option with $K > He^{\delta\tau}$ can be obtained, via the in-out parity, as the difference between the value of a standard European call and the value of a down-and-in European call with one exponential boundary (see Zhang [11])

$$\text{DOC}^c = \text{C}_{\text{bs}}(S, K, T, r, \sigma) - (H/S)^{2v/\sigma^2}\text{C}_{\text{bs}}(H^2/S, K, T, r, \sigma) \tag{1}$$

where $\text{C}_{\text{bs}}(S, K, T, r, \sigma)$ is the Black-Scholes formula for a standard European call option, σ is the volatility of the underlying asset price process, r is the risk-free interest rate (continuously compounded) and $v = r - \delta - \sigma^2/2$.

The most popular and diffused algorithm for pricing options in a discrete time world is the CRR binomial method (Cox, Ross, Rubinstein [4]). According to this model, at the end of each period, the underlying asset price rises by a factor $u = \exp(\sigma\sqrt{h})$ or decreases by a factor $d = u^{-1}$, where $h = T/n$ is the size of the time interval between two successive jumps and n is the number of time steps used in price computations. The probability of an up step is the risk-neutral probability $p = (\exp(rh) - d)/(u - d)$ and the probability of a down step is $q = 1 - p$. In the case of options with a time-varying barrier it is very difficult to derive a closed form solution for the option price that, in general, is computed by solving the dynamic programming equation

$$C(n, S(n)) = e^{-rh}[pC(n+1, uS(n))I_{(He^{(n+1)\delta h},\infty)} + qC(n+1, dS(n))I_{(He^{(n+1)\delta h},\infty)}] \tag{2}$$

where $C(n, S(n))$ is the option price after n time steps from inception and $I_{(He^{\delta\tau},\infty)}$ is the indicator function whose value is 1 if the underlying asset price is in the interval

TABLE 1
The performance of the standard binomial algorithm in evaluating barrier options.

n	$\delta = -0.05$	$\delta = -0.1$	$\delta = 0.05$	$\delta = 0.1$
25	8.849	8.979	7.013	6.750
50	7.447	7.787	6.918	6.355
75	7.249	7.745	6.178	5.724
100	7.609	7.917	6.517	5.837
150	6.974	7.488	6.255	5.676
200	7.365	7.681	5.887	5.335
300	6.849	7.366	5.993	5.444
400	6.926	7.323	6.037	5.398
500	6.862	7.329	5.851	5.317
DOC^c	6.466	6.896	5.485	4.928

$(He^{\delta\tau}, \infty)$ and 0 otherwise.

Table 1 below shows the numerical results of the standard binomial algorithm for pricing down-and-out European call options with an exponential boundary for different values of the coefficient δ. The last row illustrates the corresponding continuous time prices, DOC^c, computed using formula (1). The parameters involved in price computations are $S = 95$, $K = 100$, $H = 90$, $\sigma = 0.25$, $r = 0.1$, $T = 1$.

The performance of the standard binomial algorithm is very poor. Indeed, the prices obtained with the CRR method present a consistent bias with respect to continuous time values and moreover the algorithm converges very slowly to the corresponding continuous time values.

3. The extended binomial algorithm for pricing barrier options with a curved boundary. As shown in Table 1, the performance of the standard binomial algorithm for pricing European options with an exponential boundary is very poor. The reason was implicitly given in Boyle and Lau [2]. Indeed, as they pointed out, a naïve application of the CRR model, in general, produces consistently biased prices for barrier options.

In the case of barrier options with one flat boundary the way to overcome this drawback is to construct a binomial lattice with a set of nodes that mirrors the barrier evolution and moreover it is as close as possible to the barrier. The first condition is satisfied in the CRR model because it considers only sets of horizontal nodes while the second one is satisfied by choosing a suitable number of time steps in price computations.

Things are different if we consider barrier options with a time varying boundary. In the CRR framework it is not possible to build up a binomial tree with sets of nodes that mirror the barrier evolution. To do this we need to relax the usual condition $ud = 1$. The strategy is the following: first, we set the magnitude of a down step equal to $d = \exp(\delta h - \sigma\sqrt{h})$. Second, we select a number of steps for price computation such that, after a certain number, m, of successive down moves from inception, the underlying asset price reaches the barrier. Third, we define the size of an up step in a way that an entire set of nodes follows the barrier dynamics.

Let consider again the down-and-out European call option with the moving time barrier, $He^{\delta\tau}$, described in Section 2. The underlying asset price will touch the barrier after m consecutive down steps from inception if $Sd^m = He^{m\delta h}$, i.e.,

$$Se^{m(\delta h - \sigma\sqrt{h})} = He^{m\delta h}. \tag{3}$$

Recalling that $h = T/n$, we solve the above equation with respect to n and obtain,

$$n = \frac{m^2\sigma^2 T}{\log^2(H/S)}. \tag{4}$$

If we construct a binomial tree with n time-steps, after m successive down moves from inception, the underlying asset price reaches exactly the exponential boundary. The problem is that, in general, n is not an integer and so we consider a number of steps $n^* = [n]$, where $[x]$ denotes the largest integer smaller than or equal to x. It is worth to notice that the value of the parameter δ doesn't affect the number of steps to be used in price calculations and, as a consequence, it is computed following the same procedure proposed by Boyle and Lau in the case of options with one flat boundary.

The choice of n^* as stated above guarantees that there exists a node of the tree close to but just beyond the barrier. In order to define an entire set of nodes that follows the barrier evolution, it suffices to impose that an up move followed by a down move give rise to the same increment of the boundary value along a time interval of length $2h$. This is done by choosing the size of an up-step, u, such that

$$ud = e^{2\delta h},$$

i.e.,

$$u = e^{\delta h + \sigma\sqrt{h}}.$$

The transition probabilities to be used in price computations are the risk-neutral probabilities. In particular the probability of an up step is

$$p = \frac{e^{rh} - d}{u - d}$$

and, obviously, the probability of a down step is $q = 1 - p$.

In order to derive a closed-form formula for pricing the European down-and-out call option, we need to compute the number of trajectories that do not touch the exponential boundary. To do this, we may use the reflection principle of Desiré Andrè (see [6]) for a detailed description). The reflection principle is a combinatorial method that allows us to compute, in a random walk setting with one absorbing barrier, the number of trajectories of a particle that, after a certain number of time steps, doesn't touch the barrier. The reflection principle has been originally derived for a particle that follows a symmetrical random walk with one flat boundary to solve the ballot problem. The main feature of such a model is that the particle moves in a binomial tree with layers of horizontal nodes that exactly mimic the barrier evolution. The model presented here has the same structure because it considers a particle moving in a binomial tree with sets of nodes that follow the boundary dynamics. As a consequence, in this context the reflection principle can be applied to count the number of trajectories of the particle that end to any node of the tree without touching the exponential boundary.

Let $N_{n,j}$ be the number of trajectories of the underlying asset price with j up steps and $(n-j)$ down steps. This number is given by the binomial coefficient $\binom{n}{j}$. Let $N_{n,j}^t$ be the number of trajectories with j up steps and $(n-j)$ down steps that touch or cross the curved boundary. Let m be the number of successive down steps for the underlying asset price at inception to touch or cross the barrier. Applying the reflection principle we find that

$$N^t_{n,j} = \binom{n}{j+m}.$$

As a consequence, the number of trajectories with j up-steps and $n-j$ down-steps that do not touch or cross the exponential barrier is

$$N^{nt}_{n,j} = \binom{n}{j} - \binom{n}{j+m}.$$

The reflection principle allows us to derive the following closed form formula for pricing the down-and-out call option with the exponential boundary described before,

$$\text{DOC}^b = e^{-rT} \sum_{j=a}^{n} N^{nt}_{n,j} p^j q^{n-j} (Su^j d^{n-j} - K) \tag{5}$$

where a is the minimum number of up steps that the underlying asset price must make for the option to be in the money at maturity, and is equal to the smallest integer greater than

$$\frac{\log(K/Sd^n)}{\log(u/d)}.$$

The possibility of using the reflection principle in computing option prices is very important from a computational point of view. Indeed, it allows us to derive a closed form solution and, as a consequence, to consider only the nodes of the tree corresponding to the option values at maturity. The binomial algorithm described above generalizes the approach of Boyle and Lau [2]. Indeed, if we consider options with one flat boundary ($\delta = 0$), the evaluation formula (5) reduces to that of Cox, Ross and Rubinstein adapted to barrier options as proposed by Boyle and Lau.

Moreover, the extended binomial algorithm can be easily modified to price all kinds of knock-in and knock-out options with one exponential boundary.

4. The extended binomial algorithm for pricing double barrier options. In Section 3 we illustrated the extended binomial algorithm for pricing single barrier options with an exponential boundary. In this section we show that the extended binomial algorithm can be used to evaluate options with two parallel moving time barriers. A double barrier option is an option that is activated (knock-in) or expires (knock-out) if the underlying asset price reaches before maturity an upper boundary or a lower one.

We consider a double knock-out European call option with time to maturity T, written at time t on an asset with price $S(t) = S$. The upper exponential boundary is $Ue^{\delta\tau}$ ($U > S, \delta \in I\!R, 0 \le \tau \le T$) and the lower one is $He^{\delta\tau}$ ($H < S, \delta \in I\!R, 0 \le \tau \le T$). As before, K is the strike price, σ is the volatility and r is the continuously compounded risk-free interest rate.

In order to evaluate such option we build up a binomial tree with a set of nodes as close as possible to the lower barrier and a set of nodes as close as possible to the upper barrier. Moreover, the entire tree is made up of sets of nodes that follow the boundaries dynamics. To do this we first fix the magnitude of an up step, u, equal to $e^{\delta h+\sigma\sqrt{h}}$ and the magnitude of a down step, d, equal to $e^{\delta h-\sigma\sqrt{h}}$. A node of the tree will lie exactly on the upper barrier after m_1 consecutive up steps of the underlying asset price from inception if

$$Su^{m_1} = Ue^{m_1\delta h}.$$

Recalling that h is the size of each time step, the above condition requires to select a number of time steps for the binomial tree equal to

$$f(m_1) = \frac{m_1^2\sigma^2 T}{\log^2(U/S)}.$$

In general $f(m_1)$ is not an integer and so we consider $[f(m_1)]$, the largest integer smaller than $f(m_1)$. If we construct a binomial tree with $[f(m_1)]$ time steps, there will be a node of the tree just above the upper boundary. Following the same procedure, a node of the tree will lie on the lower boundary after m_2 consecutive down steps of the underlying asset price from inception if

$$Sd^{m_2} = He^{m_2\delta h}.$$

This means that, in order to have a tree with a node just below the lower boundary, we need to select the number of time steps equal to the largest integer smaller than

$$f(m_2) = \frac{m_1^2\sigma^2 T}{\log^2(H/S)}.$$

In general $[f(m_1)]$ and $[f(m_2)]$ can be expressed as functions of m_1 and m_2. Among the different values of m_1 and m_2, we need to select those such that $[f(m_1)] = [f(m_2)]$. The common value $[f(m_1)] = [f(m_2)]$ represents the number of steps, n^*, to be used in price computations.

As we will see later, it may happens that we cannot find values of m_1 and m_2 such that $[f(m_1)] = [f(m_2)]$. When this happens, we choose m_1 and m_2 in such a way that the absolute difference $|[f(m_1)] - [f(m_2)]|$ is as small as possible. After this, we set the number of steps for options evaluation equal to the minimum of $[f(m_1)]$ and $[f(m_2)]$. The numerical results show that this circumstance doesn't affect in a significant way the precision of the evaluation method.
Once we select the number of time steps to be used in price computations in such a way that there exists a node of the tree as close as possible to the upper barrier and a node of the tree as close as possible to the lower one, the condition $ud = e^{2\delta h}$ ensures that the binomial tree is made up of set of nodes that mirror the boundaries dynamics.

In order to develop a closed form formula to evaluate a double knock-out European call we need to count the number of trajectories, $N_{n,j}^{nt}$, with j up steps and $(n-j)$ down moves not touching neither the upper not the lower barrier. This can be done using repeatedly the reflection principle of Desiré Andrè (see [3] for a detailed description). Following this way, the pricing formula is

$$\text{DKOUT} = e^{-rT}\sum_{j=\alpha}^{\beta} N_{n,j}^{nt} p^j q^{n-j} \max[Su^j d^{n-j} - K, 0]$$

where α is the minimum number of up-steps that the underlying asset price must make to be above the lower boundary at maturity, and is equal to the smallest integer greater than

$$\frac{n}{2} + \frac{\log(H/S)}{2\sigma\sqrt{h}}.$$

In the same way β is the maximum number of up steps that the underlying asset price must make to be below the upper barrier at maturity, and is equal to the greatest integer smaller than

$$\frac{n}{2} + \frac{\log(U/S)}{2\sigma\sqrt{h}}.$$

5. Numerical results. In general, lattice models for options evaluation are defined in such a way that the first k moments of the discrete time distribution of the underlying asset price match the first k moments of the corresponding continuous time distribution of the underlying asset price. For example, in the case of binomial and trinomial models $k = 2$, in quadrinomial models $k = 3$ and so on.

In a risk-neutral environment, under the usual assumption of the Black-Scholes analysis, the underlying asset price is lognormally distributed and, after a time interval of length h, the first order moment is Se^{rh} and the second order moment is $S^2 e^{(2r+\sigma^2)h}$.

For the extended binomial algorithm, the matching of the first order moment gives

$$pu + qd = e^{rh}. \tag{6}$$

The choice of a risk-neutral probability measure guarantees that the condition (6) is satisfied. The matching of the second order moment gives

$$pu^2 + qd^2 = e^{(2r+\sigma^2)h}, \tag{7}$$

i.e.,

$$\frac{e^{rh} - e^{\delta h - \sigma\sqrt{h}}}{e^{\delta h + \sigma\sqrt{h}} - e^{\delta h - \sigma\sqrt{h}}} e^{2\delta h + 2\sigma\sqrt{h}} + \frac{e^{\delta h + \sigma\sqrt{h}} - e^{rh}}{e^{\delta h + \sigma\sqrt{h}} - e^{\delta h - \sigma\sqrt{h}}} e^{2\delta h - 2\sigma\sqrt{h}} = e^{(2r+\sigma^2)h} \tag{8}$$

Expanding the exponential in Taylor series, and ignoring terms of order h^2 and higher, we can easily check that the second order moment of the discrete distribution of the underlying asset price is $1 + 2rh + \sigma^2 h$ and it matches the second order moment of the lognormal random variable.

However, matching the first and the second order moments of the discrete time and continuous time distributions of the underlying asset price does not assure, in general, the convergence of the prices computed with the discrete time algorithm towards the corresponding values obtained in a continuous time setting. The convergence of the extended binomial algorithm can be easily proved following the same procedure illustrated in [4] to prove the convergence of the CRR algorithm to the continuous time values for a standard European option.

Numerical results of the extended binomial algorithm for pricing barrier options with an exponential boundary are shown in Table 2. We compute the prices of European down-and-out call options, DOC^b, with the same parameters specified in Section 2. At the beside of each price, the computation time (in milliseconds) is reported. In the last row the prices obtained with the continuous time formula (1) are shown.

TABLE 2
The extended binomial algorithm for option with an exponential boundary

		$\delta = -0.05$		$\delta = -0.1$		$\delta = 0.05$		$\delta = 0.1$	
m	n^*	DOC^b	time	DOC^b	time	DOC^b	time	DOC^b	time
1	21	6.5543	0.08	6.9444	0.07	5.5451	0.07	4.9667	0.07
2	85	6.4909	0.38	6.9199	0.34	5.5037	0.33	4.9402	0.32
3	192	6.4738	0.864	6.9040	0.814	5.4906	0.811	4.9322	0.798
4	342	6.4686	1.548	6.8996	1.537	5.4863	1.532	4.9283	1.514
5	534	6.4696	2.425	6.8978	2.228	5.4876	2.219	4.9296	2.197
6	769	6.4684	3.501	6.8984	3.477	5.4878	3.416	4.9295	3.869
7	1047	6.4676	4.784	6.8983	4.648	5.4871	4.227	4.9289	4.148
8	1368	6.4672	6.267	6.8971	6.124	5.4860	6.018	4.9282	6.001
9	1731	6.4671	7.934	6.8973	7.826	5.4866	7.759	4.9286	7.658
10	2138	6.4664	9.814	6.8961	9.629	5.4855	9.487	4.9277	9.382
15	4810	6.4662	22.121	6.8963	22.001	5.4856	21.916		
18	6927	6.4660	31.867	6.8963	31.436	5.4854	30.992		
20	8552	6.4660	39.442	6.8961	39.252				
26	14453	6.4660	66.781	6.8962	66.614				
31	20546	6.4659	95.062						
		DOC^c=6.4659		DOC^c=6.8962		DOC^c=5.4854		DOC^c=4.9277	

The computations were performed on a personal computer equipped with a 400MHz Pentium II processor and 128 MB of RAM.

As in Lyuu (1998), in order to evaluate the performance of the extended binomial algorithm, we compare the prices obtained in the binomial setting with those computed with the trinomial algorithm proposed by Ritchken (1995), illustrated in Table 3. We compute the prices, DOC^t, and the corresponding computation times for different values of the parameter m that, as in the binomial setting, represents the minimum number of successive down steps of the underlying asset price at inception to touch or cross the lower boundary.

Numerical results show that the extended binomial algorithm is clearly more efficient than the corresponding trinomial tree based algorithm. Indeed, for all the options considered, the extended binomial algorithm converges faster than the trinomial one to the corresponding analytical values. For example, for the down-and-out call option with the exponential barrier characterized by the parameter $\delta = 0.1$, the extended binomial algorithm reaches the analytical value of 4.9277 after 2138 time steps and 9.382 milliseconds while the trinomial algorithm after 500 time steps and 582.08 milliseconds still gives an error of 0.002%. The same considerations can be extended to all the options considered in Table 2 and in Table 3.

In Table 4 we illustrate the numerical results of the extended binomial algorithm in the case of European down-and-out call options when the underlying asset price at inception is near the boundary. It is well known that when the current underlying asset price is close to the boundary it is much more complicated for a discrete time algorithm to achieve the convergence to the corresponding continuous time values. Again, as for barrier options with a flat boundary, the extended binomial algorithm is very efficient and produces accurate prices. In, particular, we consider the same options illustrated in Table 2 with a lower barrier $He^{\delta\tau}$ and $H = 94.9$. For all the options considered, the extended binomial model reaches the corresponding analytical value after 56346 time steps and a maximum computation time of 261.522 milliseconds.

Table 5 illustrates the numerical results of the extended binomial algorithm for

TABLE 3
The trinomial algorithm for options with an exponential boundary

$\delta = -0.05$								
	$m = 1$		$m = 2$		$m = 3$		$m = 4$	
n	DOC^t	time	DOC^t	time	DOC^t	time	DOC^t	time
50	6.4645	1.684						
100	6.4579	9.671	6.4699	9.977				
200	6.4546	49.32	6.4666	50.25	6.4683	51.23		
300	6.4535	161.82	6.4655	164.53	6.4672	166.75		
400	6.4529	322.34	6.4650	326.71	6.4667	331.43	6.4671	335.69
500	6.4526	567.33	6.4647	571.44	6.4664	576.32	6.4668	580.25
$\delta = -0.1$								
	$m = 1$		$m = 2$		$m = 3$		$m = 4$	
n	DOC^t	time	DOC^t	time	DOC^t	time	DOC^t	time
50	6.8959	1.706						
100	6.8920	9.748	6.8994	9.986				
200	6.8900	49.93	6.8974	50.61	6.8978	51.74		
300	6.8894	162.33	6.8968	165.46	6.8972	169.04		
400	6.8891	324.13	6.8965	327.05	6.8968	331.96	6.8966	336.04
500	6.8889	568.29	6.8963	572.07	6.8967	576.83	6.8964	581.10
$\delta = 0.05$								
	$m = 1$		$m = 2$		$m = 3$		$m = 4$	
n	DOC^t	time	DOC^t	time	DOC^t	time	DOC^t	time
50	5.4833	1.786						
100	5.4795	9.766	5.4868	10.039				
200	5.4776	50.26	5.4850	50.84	5.4863	52.04		
300	5.4769	163.01	5.4843	165.84	5.4856	169.47		
400	5.4766	325.29	5.4840	327.91	5.4853	332.22	5.4858	337.46
500	5.4764	569.19	5.4838	573.53	5.4851	577.61	5.4856	581.51
$\delta = 0.1$								
	$m = 1$		$m = 2$		$m = 3$		$m = 4$	
n	DOC^t	time	DOC^t	time	DOC^t	time	DOC^t	time
50	4.9267	1.791						
100	4.9270	9.873	4.9273	10.065				
200	4.9271	50.61	4.9274	51.27	4.9275	52.34		
300	4.9272	163.81	4.9275	166.07	4.9275	170.24		
400	4.9272	326.29	4.9275	328.00	4.9275	332.95	4.9276	338.08
500	4.9272	569.71	4.9275	573.94	4.9276	577.96	4.9276	582.08

pricing double knock-out barrier options with parallel exponential boundaries. The parameters involved in price computations are $S = 95$, $K = 100$, $H = 70$, $U = 120$, $\sigma = 0.25$, $r = 0.1$, $T = 1$ and different values of the coefficient δ. For all the options considered the computation times were below 2 seconds. The last row illustrates the corresponding continuous time prices (ki) computed using the Kunimoto-Ikeda algorithm (1992). Again, numerical results show that the extended binomial algorithm is an efficient tool to evaluate complex barrier options.

6. Conclusions. We propose an extension of the standard CRR model for pricing complex barrier options with time-varying or multiple boundaries. The model is based on a binomial tree for the underlying asset dynamics with sets of nodes that exactly mirrors the barriers evolution. As a result, we define a very simple algorithm

TABLE 4
The extended binomial algorithm when the asset price is near the barrier

		$\delta=-0.05$		$\delta=-0.1$		$\delta=0.05$		$\delta=0.1$	
m	n^*	DOC^b	time	DOC^b	time	DOC^b	time	DOC^b	time
1	56346	0.1901	261.522	0.1708	261.047	0.1320	259.487	0.1126	259.371
		DOC^c=0.1901		DOC^c=0.1708		DOC^c=0.1320		DOC^c=0.1126	

TABLE 5
The extended binomial algorithm for double knockout barrier options

					$\delta=-0.05$	$\delta=-0.1$	$\delta=0.05$	$\delta=0.1$
m_1	m_2	$[f(m_1)]$	$[f(m_2)]$	n^*	DKOUT	DKOUT	DKOUT	DKOUT
13	17	193	193	193	0.3161	0.0796	1.4126	2.2352
16	21	293	295	293	0.3261	0.0909	1.4350	2.2621
23	30	605	603	603	0.3271	0.0835	1.4289	2.2596
26	34	774	774	774	0.3235	0.0846	1.4169	2.2512
36	47	1484	1480	1480	0.3268	0.0863	1.4268	2.2613
39	51	1741	1743	1741	0.3242	0.0849	1.4234	2.2558
52	68	3096	3098	3096	0.3256	0.0856	1.4231	2.2559
88	115	8868	8863	8863	0.3270	0.0863	1.4258	2.2583
127	166	18470	18467	18467	0.3262	0.0861	1.4244	2.2567
					ki=0.3262	ki=0.0861	ki=1.4242	ki=2.2564

that produces accurate prices with respect to the corresponding continuous time values. Moreover, numerical results show that the extended binomial algorithm is more efficient than the usual trinomial algorithm (Ritchken (1995)) employed to evaluate such options.

Acknowledgments. The author thanks Prof. Ivar Massabó for helpful comments and suggestions

REFERENCES

[1] F. BLACK AND M. SCHOLES, *The Pricing of Options and Corporate Liabilities*, Journal of Political Economy, 81 (1973), pp. 637–659.
[2] P. P. BOYLE AND S. H. LAU, *Bumping up Against the Barrier with the Binomial Method*, Journal of Derivatives, 1 (1994), pp. 6–14.
[3] M. COSTABILE, *A Discrete-Time Algorithm for Pricing Double Barrier Options*, Decisions in Economics and Finance, 24 (2001), pp. 49–59.
[4] J. C. COX, S. A. ROSS AND M. RUBINSTEIN, *Option Pricing: a Simplified Approach*, Journal of Financial Economics, 7 (1979), pp. 229-263.
[5] J. C. COX AND M. RUBINSTEIN, *Option Markets*, Englewood Cliffs, NJ: Prentice Hall, 1985.
[6] W. FELLER, *An Introduction to Probability Theory and its Application: Volume 1*, Academic Press, 1968.
[7] N. KUNIMOTO AND M. IKEDA, *Pricing Options with Curved Boundaries*, Mathematical Finance, 2 (1992), pp. 275–298.
[8] Y. D. LYUU, *Very Fast algorithm for Barrier Option Pricing and the Ballot Problem*, The Journal of Derivatives, 5 (1998), pp. 68–79.
[9] S. G. MOHANTY, *Lattice Path Counting*, Academic Press: New York, 1979.
[10] P. RITCHKEN, *On Pricing Barrier Options*, The Journal of Derivatives, 3 (1995), pp. 19–28.
[11] P. G. ZHANG, *Exotic Options: A Guide to Second Generation Options*, World Scientific, 1998.