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MIXED FINITE ELEMENT METHOD FOR NONLINEAR SECOND-ORDER ELLIPTIC PROBLEMS: RELAXATION SCHEME*

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Abstract. We consider a 2nd order nonlinear elliptic boundary value problem (BVP) in a bounded domain $\Omega \subset \mathbb{R}^N$, $N = 2, 3$ with a Dirichlet boundary condition. The mixed finite element method in lowest order Raviart-Thomas spaces is used. The exact solution is approximated via linear relaxation scheme. Error estimates are derived in $L_2(\Omega)$-norm.

Key words. nonlinear elliptic BVP, mixed finite element method, relaxation scheme

AMS subject classifications. 65N30, 35J65, 65N15

1. Introduction. Let $\Omega \subset \mathbb{R}^N$, $N = 2, 3$ be a bounded domain with a boundary $\partial\Omega \in C^2$. We consider the following study case:

$$\begin{aligned} -\Delta u + \alpha(u) &= f \qquad \text{in } \Omega \\ u &= g \qquad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where the data-functions satisfy

$$\begin{aligned} &\alpha(0) = 0, \quad 0 \le \alpha'(s) \le L, \\ &f \in L_2(\Omega), \\ &\text{there exists } \tilde{g} \in H^2(\Omega) \text{ with the trace } g \text{ on } \partial\Omega. \end{aligned} \tag{1.2}$$

The theory of monotone operators guarantees the existence and uniqueness of a weak solution $u \in H^1(\Omega)$ to the BVP (1.1). Applying [2, Th. 8.12] we get that $u \in H^2(\Omega)$.

Many useful physical models at steady state consist of nonlinear partial differential equations or systems in divergence form. The model problem (1.1) represents a simplified situation. We have skipped unnecessary coefficients and dependences in order to focus to the handling of the nonlinearity represented by the function α. This problem has been attacked by mixed finite element method in [8]. The error estimates have been derived assuming $u \in H^{\frac{5}{2}+\varepsilon}(\Omega)$, $0 < \varepsilon \ll 1$ and the function α was twice differentiable with bounded derivatives through second order. The Raviart-Thomas space of index $k = 0$ has been excluded from the analysis because of insufficient approximation properties. Mixed finite element methods for strongly nonlinear elliptic problems have been studied in [1, 3, 4, 5, 6, 7, 9] and [10].

The aim of this paper is to derive the error estimates for mixed finite element method in the lowest order Raviart-Thomas spaces (RT_0) using lower regularity assumptions for the function α (cf. (1.2)) and the exact solution ($u \in H^2(\Omega)$). We design a linear relaxation scheme (4.3) for the computation of (1.1). We show the convergence of the approximate solution to the exact one. The estimates are derived in a few steps. First, we introduce some temporarily BVPs, solutions of which represent in some sense approximations of u. Let us note, that the standard direct way for derivation of the error is not possible. We have to get rid of the relaxation parameter

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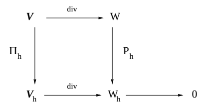
k and this is possible if and only if the right hand side does not contain any noise depending on h. Otherwise this can accumulate and blows up. The main result is stated in Theorem 4.3. At the end we present some numerical examples to demonstrate the efficiency of the proposed numerical scheme. We denote by $\mathcal{T}_h$ a regular triangulation of Ω consisting of elements of diameter not greater than h. Boundary elements can have one curved side. Let us define the following spaces

$$\boldsymbol{V} = \boldsymbol{H}\,(\mathrm{div}\,;\Omega) = \{\boldsymbol{u} \in L_2(\Omega)^N : \mathrm{div}\ \boldsymbol{u} \in L_2(\Omega)\}, \qquad W = L_2(\Omega). \tag{1.3}$$

Further, we introduce the lowest order Raviart-Thomas spaces $\boldsymbol{V}_h \times W_h$ (of index $k = 0$), the L_2-projector $P_h : W \to W_h$ and the Raviart-Thomas projection $\Pi_h : \boldsymbol{V} \to \boldsymbol{V}_h$, which have a useful commuting property shown in Figure 1.



Commutative diagram

The following approximation properties hold:

$$\|\boldsymbol{q} - \Pi_h \boldsymbol{q}\| \le Ch\,\|\boldsymbol{q}\|_1\,, \qquad \|p - P_h p\| \le Ch\,\|p\|_1\,, \tag{1.4}$$

where $\|\cdot\|$ denotes the L_2-norm and $\|\cdot\|_1$ stands for the norm in $H^1(\Omega)$. In that follows C, ε and C_ε denote generic positive constants depending only on the data, where ε is a small one and $C_\varepsilon = C\left(\frac{1}{\varepsilon}\right)$ is a large one.

2. Nonlocal BVP. As a first step we introduce the following BVP:

$$\begin{aligned} -\Delta w^h + \alpha(P_h w^h) &= f && \text{in } \Omega \\ w^h &= g && \text{on } \partial\Omega. \end{aligned} \tag{2.1}$$

This problem is non-standard. In fact, it is a nonlocal nonlinear BVP due to the operator P_h and the function α. One can see that (2.1) admits at most one solution. This follows from the monotonicity of α and the properties of P_h. We prove the existence of a solution in two steps. First, we consider a linear nonlocal problem and then we show that the solution to (2.1) can be obtained via a linear relaxation scheme.

2.1. Linear case. Let us consider the following nonlocal BVP:

$$\begin{aligned} -\Delta z^h + LP_h z^h &= \tilde{f} && \text{in } \Omega \\ z^h &= g && \text{on } \partial\Omega, \end{aligned} \tag{2.2}$$

where L is the Lipschitz constant of the function α and $\tilde{f} \in L_2(\Omega)$. The solution z^h to (2.2) will be obtained via a linear relaxation process, which is defined by

$$\begin{aligned} -\Delta z_k^h + Lz_k^h &= \tilde{f} + Lz_{k-1}^h - LP_h z_{k-1}^h && \text{in } \Omega \\ z_k^h &= g && \text{on } \partial\Omega. \end{aligned} \tag{2.3}$$

As a starting datum for the iterations we can take any $z_0^h \in H^1(\Omega)$. We recall that (2.3) is a standard linear BVP with the right-hand side from $L_2(\Omega)$. The well-posedness follows from the theory of linear elliptic equations (see [2, Th. 8.30]).

Lemma 2.1. *We assume (1.2). Then there exist* $0 < q < 1$ *and* $h_0 > 0$ *such that*

$$\left\|\nabla z_k^h - \nabla z^h\right\| \le q^k \left\|\nabla z_0^h - \nabla z^h\right\|,$$

which holds for any $k \in \mathbb{N}$ and any $h < h_0$.

Proof. The variational formulations of (2.2) and (2.3) take the form ($\varphi \in H_0^1(\Omega)$ and $z^h = z_k^h = g$ on $\partial\Omega$)

$$(\nabla z^h, \nabla\varphi) + L(z^h, \varphi) = \left(\tilde{f}, \varphi\right) + \left(Lz^h - LP_h z^h, \varphi\right) \tag{2.4}$$

$$(\nabla z_k^h, \nabla\varphi) + L(z_k^h, \varphi) = \left(\tilde{f}, \varphi\right) + \left(Lz_{k-1}^h - LP_h z_{k-1}^h, \varphi\right). \tag{2.5}$$

Subtracting(2.4) from (2.5) and setting $\varphi = z_k^h - z^h$ we get

$$\left\|\nabla z_k^h - \nabla z^h\right\|^2 + L\left\|z_k^h - z^h\right\|^2 = \left(L\left[z_{k-1}^h - z^h - P_h(z_{k-1}^h - z^h)\right], z_k^h - z^h\right).$$

Applying the Cauchy-Schwarz inequality, (1.4), Friedrichs' and Young's inequalities to the right-hand side, we deduce

$$\begin{aligned}\left\|\nabla z_k^h - \nabla z^h\right\|^2 + L\left\|z_k^h - z^h\right\|^2 &\leq L\left\|z_{k-1}^h - z^h - P_h(z_{k-1}^h - z^h)\right\|\left\|z_k^h - z^h\right\| \\ &\leq Ch\left\|z_{k-1}^h - z^h\right\|_1\left\|z_k^h - z^h\right\| \\ &\leq Ch\left\|\nabla z_{k-1}^h - \nabla z^h\right\|\left\|z_k^h - z^h\right\| \\ &\leq Ch^2\left\|\nabla z_{k-1}^h - \nabla z^h\right\|^2 + \frac{L}{2}\left\|z_k^h - z^h\right\|^2.\end{aligned}$$

Therefore, we can write $\left\|\nabla z_k^h - \nabla z^h\right\|^2 + \frac{L}{2}\left\|z_k^h - z^h\right\|^2 \leq Ch^2\left\|\nabla z_{k-1}^h - \nabla z^h\right\|^2$. From this we get the recursion formula for any $h < h_0$, any $k \in \mathbb{N}$ and some $0 < q < 1$ $\left\|\nabla z_k^h - \nabla z^h\right\| \leq q\left\|\nabla z_{k-1}^h - \nabla z^h\right\|$. The desired result is an immediate consequence of this recursion. □

We recall, that the estimate $\left\|\nabla z^h\right\| \leq C$ can be readily derived from (2.4) for $\varphi = z^h - \tilde{g}$. Moreover, the function z^h is unique.

2.2. Nonlinear case. We consider the linear relaxation scheme ($k \in \mathbb{N}$):

$$\begin{aligned} -\Delta w_k^h + LP_h w_k^h &= f + LP_h w_{k-1}^h - \alpha(P_h w_{k-1}^h) && \text{in } \Omega \\ w_k^h &= g && \text{on } \partial\Omega. \end{aligned} \tag{2.6}$$

As a starting datum for the iterations we can take any $w_0^h \in H^1(\Omega)$. The well-posedness of (2.6) follows from Section 2.1.

We show that w_k^h will approach w^h in the space $H^1(\Omega)$ as $k \to \infty$.

LEMMA 2.2. *We assume (1.2). Then there exist $C > 0$ and $\delta > 0$ such that*

$$\left\|\nabla w_k^h - \nabla w^h\right\|^2 + \left\|P_h w_k^h - P_h w^h\right\|^2 \leq C\left(\frac{L}{L+\delta}\right)^k\left\|P_h w_0^h - P_h w^h\right\|^2,$$

which holds for any $k \in \mathbb{N}$.

Proof. The variational formulations of (2.1) and (2.6) take the form ($\varphi \in H_0^1(\Omega)$ and $w^h = w_k^h = g$ on $\partial\Omega$)

$$(\nabla w^h, \nabla\varphi) + (\alpha(P_h w^h), \varphi) = (f, \varphi) \tag{2.7}$$

$$(\nabla w_k^h, \nabla\varphi) + L(P_h w_k^h, \varphi) = (f, \varphi) + \left(LP_h w_{k-1}^h - \alpha(P_h w_{k-1}^h), \varphi\right). \tag{2.8}$$

Now, we introduce the function $\beta : \mathbb{R} \to \mathbb{R}$ defined by $\beta(s) := Ls - \alpha(s)$. One can easily see that

$$L \geq \beta'(s) = L - \alpha'(s) \geq 0. \tag{2.9}$$

Subtracting (2.7) from (2.8) and setting $\varphi = w_k^h - w^h$ we have $\left\|\nabla w_k^h - \nabla w^h\right\|^2 + L\left\|P_h w_k^h - P_h w^h\right\|^2 = \left(\beta(P_h w_{k-1}^h) - \beta(P_h w^h), w_k^h - w^h\right) = (\beta(P_h w_{k-1}^h) - \beta(P_h w^h), P_h w_k^h - P_h w^h)$. We apply the Cauchy-Schwarz inequality, (2.9) and Young's inequality to the right-hand side, and deduce $\left\|\nabla w_k^h - \nabla w^h\right\|^2 + L\left\|P_h w_k^h - P_h w^h\right\|^2 \le \left\|\beta(P_h w_{k-1}^h) - \beta(P_h w^h)\right\| \left\|P_h w_k^h - P_h w^h\right\| \le L\left\|P_h w_{k-1}^h - P_h w^h\right\| \left\|P_h w_k^h - P_h w^h\right\| \le \frac{L}{2}\left(\left\|P_h w_{k-1}^h - P_h w^h\right\|^2 + \left\|P_h w_k^h - P_h w^h\right\|^2\right)$. Hence, we can write

$$\left\|\nabla w_k^h - \nabla w^h\right\|^2 + \frac{L}{2}\left\|P_h w_k^h - P_h w^h\right\|^2 \le \frac{L}{2}\left\|P_h w_{k-1}^h - P_h w^h\right\|^2 . \tag{2.10}$$

There exists a positive constant δ such that

$$\|\nabla z\| \ge C\|z\| \ge \sqrt{\delta}\,\|P_h z\| , \tag{2.11}$$

which holds for any $z \in H^1(\Omega)$. Thus, the relation (2.10) can be rewritten as

$$\frac{1}{2}\left\|\nabla w_k^h - \nabla w^h\right\|^2 + \frac{L+\delta}{2}\left\|P_h w_k^h - P_h w^h\right\|^2 \le \frac{L}{2}\left\|P_h w_{k-1}^h - P_h w^h\right\|^2 . \tag{2.12}$$

From this we successively deduce

$$\left\|P_h w_k^h - P_h w^h\right\|^2 \le \frac{L}{L+\delta}\left\|P_h w_{k-1}^h - P_h w^h\right\|^2 \le \ldots \le \left(\frac{L}{L+\delta}\right)^k \left\|P_h w_0^h - P_h w^h\right\|^2 .$$

Using the last inequality and (2.12) we conclude the proof. □

The estimate $\left\|\nabla w^h\right\| \le C$ can be easily derived from (2.7) for $\varphi = w^h - \tilde{g}$. This estimate and (2.11) imply $\left\|P_h w^h\right\| \le C$. Moreover, the function w^h is uniquely determined due to the monotonicity of α.

2.3. Error $\left\|\nabla w^h - \nabla u\right\|$. The variational formulation to the BVP (1.1) reads as ($\varphi \in H_0^1(\Omega)$ and $u = g$ on $\partial\Omega$)

$$(\nabla u, \nabla\varphi) + (\alpha(u), \varphi) = (f, \varphi) . \tag{2.13}$$

The following lemma derives the estimate for the error $\left\|\nabla w^h - \nabla u\right\|$.

Lemma 2.3. *We assume (1.2). Then there exists $C > 0$ and such that*

$$\left\|\nabla w^h - \nabla u\right\| \le Ch$$

holds for any $0 < h \le h_0$.

Proof. We subtract (2.13) from (2.7), we set $\varphi = w^h - u$ and we get $\left\|\nabla w^h - \nabla u\right\|^2 + \left(\alpha(w^h) - \alpha(u), w^h - u\right) = \left(\alpha(w^h) - \alpha(P_h w^h), w^h - u\right)$. The second term on the left is nonnegative due to the monotonicity of the function α. Applying the Cauchy-Schwarz inequality to the right-hand side, followed by Young's and Friedrichs' inequalities and (1.4), we deduce

$$\begin{aligned} \left\|\nabla w^h - \nabla u\right\|^2 &\le \left\|\alpha(w^h) - \alpha(P_h w^h)\right\| \left\|w^h - u\right\| \\ &\le C\left\|w^h - P_h w^h\right\| \left\|w^h - u\right\| \le \varepsilon\left\|w^h - u\right\|^2 + C_\varepsilon\left\|w^h - P_h w^h\right\|^2 \\ &\le \varepsilon\left\|\nabla w^h - \nabla u\right\|^2 + C_\varepsilon h^2\left\|w^h\right\|_1 \le \varepsilon\left\|\nabla w^h - \nabla u\right\|^2 + C_\varepsilon h^2. \end{aligned}$$

Choosing a sufficiently small positive ε we get $\left\|\nabla w^h - \nabla u\right\|^2 \le Ch^2$. Recall, that we have shown the existence of w^h for $h \le h_0$. □

3. Auxiliary problem. As the next step, we introduce the following sequence of nonlinear nonlocal BPVs ($k \in \mathbb{N}$):

$$\begin{aligned} -\Delta v_k^h + \alpha(P_h v_k^h) + \frac{1}{k} v_k^h &= f \qquad \text{in } \Omega \\ v_k^h &= g \qquad \text{on } \partial\Omega. \end{aligned} \tag{3.1}$$

The starting datum is any $v_0^h \in H^1(\Omega)$. The problem (3.1) for a given k is well-posed, as we have seen in Section 2.2. The variational formulation of (3.1) has the form ($\varphi \in H_0^1(\Omega)$ and $v_k^h = g$ on $\partial\Omega$, $k \in \mathbb{N}$)

$$\left(\nabla v_k^h, \nabla\varphi\right) + \left(\alpha(P_h v_k^h), \varphi\right) + \frac{1}{k}\left(v_k^h, \varphi\right) = (f, \varphi). \tag{3.2}$$

The following a priori estimates

$$\left\|\nabla v_k^h\right\| \leq C, \qquad \forall k \in \mathbb{N} \tag{3.3}$$

can be readily obtained from (3.2) by setting $\varphi = v_k^h - \tilde{g}$.

Next lemma derives the error estimate for $v_k^h - w^h$ in the Sobolev space $H^1(\Omega)$.

LEMMA 3.1. *We assume (1.2). Then there exists $C > 0$ such that*

$$\left\|\nabla v_k^h - \nabla w^h\right\| \leq C k^{-1}$$

holds for any $0 < h \leq h_0$.

Proof. We subtract (2.7) from (3.2), set $\varphi = w^h - v_k^h$ and get

$$\left\|\nabla w^h - \nabla v_k^h\right\|^2 + \left(\alpha(P_h w^h) - \alpha(P_h v_k^h), w^h - v_k^h\right) = k^{-1}\left(v_k^h, v_k^h - w^h\right).$$

We omit the second term on the left (it is nonnegative) and deduce using the Cauchy-Schwarz, Friedrichs' and Young's inequalities

$$\left\|\nabla w^h - \nabla v_k^h\right\|^2 \leq \frac{1}{k}\left\|v_k^h\right\|\left\|v_k^h - w^h\right\| \leq \frac{C}{k}\left\|\nabla w^h - \nabla v_k^h\right\| \leq \frac{C_\varepsilon}{k^2} + \varepsilon\left\|\nabla w^h - \nabla v_k^h\right\|^2.$$

Choosing a sufficiently small positive ε, we conclude the proof. □

The following lemma derives the estimate for $v_k^h - v_{k-1}^h$ in $H^1(\Omega)$.

LEMMA 3.2. *We assume (1.2). Then there exists $C > 0$ and such that*

$$\left\|\nabla v_k^h - \nabla v_{k-1}^h\right\| \leq C k^{-2}$$

holds for any $0 < h \leq h_0$.

Proof. We subtract (3.2) for $k = k-1$ from (3.2). We set $\varphi = v_k^h - v_{k-1}^h$ and get

$$\begin{aligned} \left\|\nabla v_k^h - \nabla v_{k-1}^h\right\|^2 + \left(\alpha(P_h v_k^h) - \alpha(P_h v_{k-1}^h), v_k^h - v_{k-1}^h\right) + \frac{1}{k}\left\|v_k^h - v_{k-1}^h\right\|^2 \\ = \left(\frac{1}{k-1} - \frac{1}{k}\right)\left(v_{k-1}^h, v_k^h - v_{k-1}^h\right). \end{aligned}$$

We omit the second and the third terms on the left (they are nonnegative) and we deduce in a standard way applying the Cauchy-Schwarz, Friedrichs' and Young's inequalities

$$\begin{aligned} \left\|\nabla w^h - \nabla v_k^h\right\|^2 &\leq \frac{C}{k^2}\left\|v_{k-1}^h\right\|\left\|v_k^h - v_{k-1}^h\right\| \\ &\leq \frac{C}{k^2}\left\|\nabla v_k^h - \nabla v_{k-1}^h\right\| \leq \frac{C_\varepsilon}{k^4} + \varepsilon\left\|\nabla v_k^h - \nabla v_{k-1}^h\right\|^2. \end{aligned}$$

Choosing a sufficiently small positive ε, we conclude the proof. □

4. Linear relaxation scheme for mixed finite elements. In Section 3 we have seen that the BVP (3.1) admits a unique weak solution $v_k^h \in H^1(\Omega)$. Applying the standard theory of elliptic equations [2, Theorem 8.12] we conclude that $v_k^h \in H^2(\Omega)$. Moreover, the uniform estimate $\left\|v_k^h\right\| \le C$, which holds for any $k \in \mathbb{N}$, yields

$$\left\|v_k^h\right\|_2 \le C, \tag{4.1}$$

where $\|\cdot\|_2$ denotes the norm in $H^2(\Omega)$.

Therefore, $\boldsymbol{q}_k^h = -\nabla v_k^h$ and v_k^h solve the following problem:

$$\begin{aligned} \left(\boldsymbol{q}_k^h, \boldsymbol{\phi}\right) - \left(v_k^h, \operatorname{div} \boldsymbol{\phi}\right) &= -\left(g, \boldsymbol{\phi}\cdot\boldsymbol{\nu}\right)_{\partial\Omega} & \forall \boldsymbol{\phi} \in \boldsymbol{V} \\ \left(\operatorname{div} \boldsymbol{q}_k^h, \psi\right) + \left(\alpha(P_h v_k^h), \psi\right) + \frac{1}{k}\left(v_k^h, \psi\right) &= (f, \psi) & \forall \psi \in W. \end{aligned} \tag{4.2}$$

Now, we are in a position to introduce the mixed formulation of the linear relaxation approximation scheme to the BVP (1.1) for $\boldsymbol{\phi}_h \in \boldsymbol{V}_h$ ad $\psi_h \in W_h$

$$\begin{gathered} \left(\boldsymbol{q}_{k,h}, \boldsymbol{\phi}_h\right) - \left(u_{k,h}, \operatorname{div} \boldsymbol{\phi}_h\right) = -\left(g, \boldsymbol{\phi}_h\cdot\boldsymbol{\nu}\right)_{\partial\Omega} \\ \left(\operatorname{div} \boldsymbol{q}_{k,h}, \psi_h\right) + \left(L + k^{-1}\right)\left(u_{k,h}, \psi_h\right) = (f, \psi_h) + \left(Lu_{k-1,h} - \alpha(u_{k-1,h}), \psi_h\right). \end{gathered} \tag{4.3}$$

The starting datum is any function $u_{0,h} \in L_2(\Omega)$. The problem (4.3) represents a standard linear mixed formulation, which, for a given $k \in \mathbb{N}$, admits a unique solution $(\boldsymbol{q}_{k,h}, u_{k,h}) \in \boldsymbol{V}_h \times W_h$. The following lemma plays an important role in the derivation of an error estimate for $u_{k,h} - P_h v_k^h$. The proof proceeds exactly in the same way as in [11, Lemma 4.1], thus we skip it.

Lemma 4.1 (algebraic). *Let a and $b \neq 1$ be positive real numbers. Assume that $\{y_k\}_{k=0}^{\infty}$ is a sequence of nonnegative real numbers obeying the following recursion formula*

$$y_k \le ak^{-2} + \left(1 - \frac{b}{k+b}\right) y_{k-1}, \qquad k \in \mathbb{N}.$$

Then there exists a constant $C = C(y_0, a, b) > 0$ such that $y_k \le Ck^{-\min\{b,1\}}$, $k \in \mathbb{N}$.

Our next step is to determine the error for $\boldsymbol{q}_{k,h} - \Pi_h \boldsymbol{q}_k^h$ and $u_{k,h} - P_h v_k^h$. We subtract (4.2) from (4.3) and get the mixed variational formulation for the error:

$$\begin{aligned} &\left(\boldsymbol{q}_{k,h} - \boldsymbol{q}_k^h, \boldsymbol{\phi}_h\right) - \left(u_{k,h} - v_k^h, \operatorname{div} \boldsymbol{\phi}_h\right) = 0 \\ &\left(\operatorname{div}\left(\boldsymbol{q}_{k,h} - \boldsymbol{q}_k^h\right), \psi_h\right) + \left(L + \frac{1}{k}\right)\left(u_{k,h} - v_k^h, \psi_h\right) \\ &= \left(Lu_{k-1,h} - \alpha(u_{k-1,h}), \psi_h\right) - \left(LP_h v_k^h - \alpha(P_h v_k^h), \psi_h\right), \end{aligned} \tag{4.4}$$

which holds for any $(\boldsymbol{\phi}_h, \psi_h) \in \boldsymbol{V}_h \times W_h$.

This variational identity is used in the proof of the following lemma.

Lemma 4.2. *We assume (1.2). Then there exists $C > 0$ such that*

$$\left\|u_{k,h} - P_h v_k^h\right\| + \left\|\boldsymbol{q}_{k,h} - \Pi_h \boldsymbol{q}_k^h\right\| \le Ck^{-\min\{1,\frac{1}{L}\}}$$

holds for any $k \in \mathbb{N}$.

Proof. We start from (4.4). We set $\boldsymbol{\phi}_h = \boldsymbol{q}_{k,h} - \Pi_h \boldsymbol{q}_k^h$, $\psi_h = u_{k,h} - P_h v_k^h$ and we sum both equations. We successively deduce

$$\left\|\boldsymbol{q}_{k,h} - \Pi_h \boldsymbol{q}_k^h\right\|^2 + \left(L + k^{-1}\right)\left\|u_{k,h} - P_h v_k^h\right\|^2$$

$$= \left(\beta(u_{k-1,h}) - \beta(P_h v_k^h), u_{k,h} - P_h v_k^h\right) \tag{4.5}$$

$$\leq \left\|\beta(u_{k-1,h}) - \beta(P_h v_k^h)\right\| \left\|u_{k,h} - P_h v_k^h\right\| \leq L \left\|u_{k-1,h} - P_h v_k^h\right\| \left\|u_{k,h} - P_h v_k^h\right\|.$$

We omit the first term on the left for a while. Using the triangle inequality and Lemma 3.2 we have

$$(L + k^{-1}) \left\|u_{k,h} - P_h v_k^h\right\| \leq L \left\|u_{k-1,h} - P_h v_k^h\right\|$$

$$\leq L \left\|u_{k-1,h} - P_h v_{k-1}^h\right\| + L \left\|P_h v_{k-1}^h - P_h v_k^h\right\|$$

$$\leq L \left\|u_{k-1,h} - P_h v_{k-1}^h\right\| + C \left\|v_{k-1}^h - v_k^h\right\| \leq L \left\|u_{k-1,h} - P_h v_{k-1}^h\right\| + \frac{C}{k^2}.$$

This can be rewritten into the following recursion formula

$$\left\|u_{k,h} - P_h v_k^h\right\| \leq \left(1 - \frac{\frac{1}{L}}{k + \frac{1}{L}}\right) \left\|u_{k-1,h} - P_h v_{k-1}^h\right\| + \frac{C}{k^2}.$$

An application of Lemma 4.1 implies $\left\|u_{k,h} - P_h v_k^h\right\| \leq C k^{-\min\{1, \frac{1}{L}\}}$, $\forall k \in \mathbb{N}$. The rest of the proof follows from (4.5). □

Now, we are in a position to state the main result of this paper.

THEOREM 4.3. *Suppose (1.2). Then there exist $h_0 > 0$, $C > 0$ such that*

$$\left\|u_{k,h} - u\right\| + \left\|\boldsymbol{q}_{k,h} + \nabla u\right\| \leq C \left(h + k^{-\min\{1, \frac{1}{L}\}}\right)$$

holds for any $k \in \mathbb{N}$.

Proof. The desired result follows from the triangle inequality, (1.4) and Lemmas 2.3, 3.1, 4.2. In fact, we can write

$$\left\|u_{k,h} - u\right\| \leq \left\|u_{k,h} - P_h v_k^h\right\| + \left\|P_h v_k^h - v_k^h\right\| + \left\|v_k^h - w^h\right\| + \left\|w^h - u\right\|$$

$$\leq C \left(k^{-\min\{1, \frac{1}{L}\}} + h \left\|v_k^h\right\|_1 + k^{-1} + h\right) \leq C \left(h + k^{-\min\{1, \frac{1}{L}\}}\right).$$

For fluxes we proceed analogously and get

$$\left\|\boldsymbol{q}_{k,h} + \nabla u\right\| \leq \left\|\boldsymbol{q}_{k,h} - \Pi_h \boldsymbol{q}_k^h\right\| + \left\|\Pi_h \boldsymbol{q}_k^h + \nabla v_k^h\right\| + \left\|\nabla w^h - \nabla v_k^h\right\| + \left\|\nabla u - \nabla w^h\right\|$$

$$\leq C \left(k^{-\min\{1, \frac{1}{L}\}} + h \left\|v_k^h\right\|_2 + k^{-1} + h\right) \leq C \left(h + k^{-\min\{1, \frac{1}{L}\}}\right).$$

We recall that $\boldsymbol{q}_k^h = -\nabla v_k^h$. □

Just proved Theorem 4.3 shows that the proposed linear relaxation scheme (4.3), for mixed finite element method in lowest order Raviart-Thomas spaces, is of order $\mathcal{O}(h)$. This result is valid for $u \in H^2(\Omega)$ and any monotonically increasing and Lipschitz continuous function α, which appears in the problem setting (1.1). In this sense, we have improved the results from up to now known results (e.g., [8]).

5. Numerical experiment. The aim of this section is to demonstrate the robustness and efficiency of the proposed linear relaxation scheme (4.3).

Let Ω be the unit square in $\mathbb{R}^2$. Consider a nonlinear function α given by

$$\alpha(s) = \begin{cases} \sqrt{20s+1} & \text{for } s > 0 \\ 1 & \text{elsewhere ,} \end{cases}$$

which is clearly Lipschitz continuous and monotonically increasing. We want to find a solution to the following nonlinear Dirichlet BVP

$$\begin{aligned} -\Delta u + \alpha(u) &= f \qquad \text{in} \quad \Omega \\ u &= g \qquad \text{on} \quad \partial\Omega. \end{aligned}$$

The data functions f and g are defined in such a way that the exact solution to this BVP is $u(x,y) = x^3 - y^2 + x + \sin(\pi x)\sin(\pi y)$. The solution u has been chosen in such a way that the range of u contains the point 0, at which the derivative α' is not continuous.

Let us introduce a random function ran whose range is uniformly distributed over $(-1,1)$. We present two computations. In the first case, we choose $u_{0,h}$ relatively close (up to 30% error) to the exact solution, i.e., $u_{0,h}(\boldsymbol{x}) = u(\boldsymbol{x})(1 + 0.3\; ran(\boldsymbol{x}))$. In the second event we begin with $u_{0,h}$, which is far away from the solution u, i.e., $u_{0,h}(\boldsymbol{x}) = 1000\; ran(\boldsymbol{x})$. Let us note that the random function ran has been evaluated once per a given triangle. We have used the linearization scheme (4.3) with $L = 10$ for computations. We have chosen a fixed uniform mesh consisting of 5000 triangles corresponding to $\Delta x = \Delta y = 0.02$, and we have computed 25 iterations. The results are depicted in Figures 5.1-5.3. The last Figure 5.4 shows logarithm of the error for

$u_{0,h}$ was close to u $\qquad$ $u_{0,h}$ was far from u

FIG. 5.1. *Logarithms of $L_2(\Omega)$-errors for $u_{k,h}$ versus iterations*

$u_{0,h}$ was close to u $\qquad$ $u_{0,h}$ was far from u

FIG. 5.2. *Logarithms of relative $L_2(\Omega)$-errors for $u_{k,h}$ versus iterations*

$u_{k,h}$ in the space $L_\infty(\Omega)$.

5.1. Conclusion. All graphs in Figures 5.1-5.3 are similar. The rapidly decreasing part at the beginning is followed by a more or less constant section. This is due to the fact that the linearization error was larger than the discretization one at the beginning of the iteration process, but later the opposite is true.

$u_{0,h}$ was close to u $\qquad$ $u_{0,h}$ was far from u

FIG. 5.3. *Logarithms of $L_2(\Omega)$-errors for $\boldsymbol{q}_{k,h}$ versus iterations*

$u_{0,h}$ was close to u $\qquad$ $u_{0,h}$ was far from u

FIG. 5.4. *Logarithms of $L_\infty(\Omega)$-errors for $u_{k,h}$ versus iterations*

The linearization scheme (4.3) is robust and the approximations converge towards the exact solutions independently of the fact, where the iteration process has started. Moreover, the numerical schemes is efficient. In particular, we needed $10-13$ iterations to get the best possible error for the given discretization, although $u_{0,h}$ was really badly chosen. In the instance of a good starting point $u_{0,h}$, it is enough to do $3-5$ iterations to achieve the discretization error.

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