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HETEROGENEOUS MULTISCALE METHOD IN EDDY CURRENTS MODELING*

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Abstract. The induction of eddy currents in a conductive piece is an electromagnetic phenomenon described by Maxwell's equations. For composite materials it is multiscale in its nature. We are looking for the macroscopic properties of the composite. We have applied the Heterogeneous Multiscale Method in order to avoid necessity of using full microscale solver.

Key words. eddy currents, homogenization, heterogeneous multiscale method, variational problem, finite elements

AMS subject classifications. 65M55, 65M60, 65Z05, 78M40, 78A48, 78A25, 78M10

1. Introduction. Induction of eddy currents is an electrical phenomenon when a circulating flow of electrons emerges inside of a conductor. This happens when the conductor is exposed to a changing magnetic field either due to a relative motion between the conductor and electromagnetic field or due to the variations of the field with time. These eddy currents create electromagnets with opposing magnetic fields, introducing power loss in electrical devices. Therefore understanding of this phenomenon is of a great importance.

Macroscopic behavior of a material is strongly dependent on its microscopic properties. Thus eddy currents problem for composite materials is multiscale in its nature. We use Heterogeneous Multiscale Method (HMM), a general framework introduced in [5] by E et al. to solve multiscale problems numerically.

Classical methods — multigrid method, domain decomposition, adaptive mesh refinement are general purpose solvers for the fine scale problem and their price is therefore the price of full microscale solver. Other problem of multigrid approach is in the abundance of information gained by the full microscale solver.

Modern methods like HMM, optimal prediction method or adaptive model refinement are trying to use special features of the problem such as scale separation to decrease the computational complexity of the model. For comparison of the recent methods see [8].

This paper is organized as follows. In Section 2 Heterogeneous Multiscale Method is introduced. In Section 3 eddy currents model is introduced and in the Section 4 numerical experiments supporting theoretical results are presented.

2. Heterogeneous Multiscale Method. HMM works as a macroscale solver which turns to the microscale model only if the macroscopic model is invalid or where constitutive relations are missing.

In order to be able to use the HMM we have to make following assumptions (see [5]):

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1. We have a microscopic variational problem

$$\min_u f(u), \tag{2.1}$$

where u is a state variable dependent on multiple scales, e.g. $u = u(x, x/\varepsilon)$, where $\varepsilon \ll 1$ is the dimension of the microscale. Here $u(x, y)$ is periodic in y with period 1.

2. There exists a macroscopic variational model

$$\min_U F(U), \tag{2.2}$$

which has a solution u^0, such that

$$u\left(x, \frac{x}{\varepsilon}\right) = u^0(x) + \mathcal{O}(\varepsilon). \tag{2.3}$$

3. We have compression operator $\mathcal{Q}$ mapping states from the microscopic scale to the macroscopic one and reconstruction $\mathcal{R}$ operator working in the opposite direction. It must hold that $\mathcal{QR} = \mathcal{I}$, where $\mathcal{I}$ is the identity operator.

To make the abstract framework of HMM more concrete consider the heat conductivity problem[1]

$$\nabla \cdot (A^\varepsilon \nabla u^\varepsilon) = g \quad \text{in } \Omega, \tag{2.4}$$

$$u^\varepsilon = 0 \quad \text{on } \partial\Omega, \tag{2.5}$$

where $A^\varepsilon(x) = A(x, x/\varepsilon)$. Let us remark, that there exist no explicit formulae in general for homogenized heat conductivity A^0 and it can be found only in some special cases.

A result of the homogenization theory [2, 3] is that for $\varepsilon \to 0$ converges u^ε in the weak sense to a $u^0(x) \in \boldsymbol{H}^1_0(\Omega)$, the solution of

$$\nabla \cdot (A^0 \nabla u^0) = g \quad \text{in } \Omega, \tag{2.6}$$

where A^0 is given by

$$a^0_{ij} = \int_Y a_{ij} + \sum_{k=1}^{3} a_{ik} \frac{\partial \chi^j}{\partial y_k}(x, y) dy, \quad i, j = 1, \ldots, 3. \tag{2.7}$$

Here, χ^j denote the solution to the cell problems

$$\nabla \cdot (A^\varepsilon \nabla \chi^j) = -\sum_{k=1}^{3} \frac{\partial A^\varepsilon_{kj}}{\partial y_k} \quad \text{in } \Omega, \quad j = 1, \ldots, 3. \tag{2.8}$$

In the case of fine scale problem (2.4)–(2.5) we opt for finite element space discretization, which is natural for variational problems. The homogenized u^0 is the macro state variable we are looking for. It solves (2.6). Thus finite elements are again a feasible choice.

[1] To consider eddy currents model itself would be unnecessary technical.

Let $\mathcal{T}_H$ be the macro triangulation of Ω. The diameter H of the triangulation is chosen to resolve well the macro scale properties of (2.4)–(2.5). We allow $H \gg \varepsilon$. Let the macro finite element space be defined as

$$P_0^k(\Omega, \mathcal{T}_H) := \left\{ U^H \in H_0^1 \;\middle|\; U^H|_K \in P^k(K) \; \forall K \in \mathcal{T}_H \right\}, \tag{2.9}$$

where $P^k(K)$ is the space of k-th order polynomials on the triangle $K \in \mathcal{T}_H$.

The variational formulation of (2.6) reads

$$(A^0 \nabla u^0, \nabla \varphi) = (g, \varphi) \quad \forall \varphi \in H_0^1(\Omega). \tag{2.10}$$

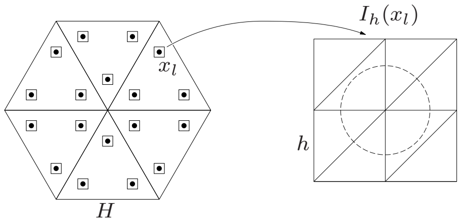


FIG. 2.1. *Illustration of HMM. H – diameter of macroelement, x_l – quadrature point, h – diameter of microelement, $I_h(x_l)$ – domain on which the microproblem is solved*

The macro data which have to be estimated using the fine scale problem (step 2 of the HMM) is the stiffness matrix corresponding to (2.10)

$$M_{ij}^H = (A^0 \nabla \Phi_i, \nabla \Phi_j), \tag{2.11}$$

where Φ_i are the basis functions of $P_0^k(\Omega, \mathcal{T}_H)$. Imagine, that we would know the effective conductivity A^0. Then we could simply evaluate M_{ij}^H by means of a numerical quadrature

$$M_{ij}^H \approx \sum_{K \in \mathcal{T}_H} |K| \sum_{x_l \in K} \omega_l A^0(x_l) \nabla \Phi_i(x_l) \nabla \Phi_j(x_l), \tag{2.12}$$

where x_l and ω_l are the quadrature points and weights, respectively. An idea of HMM is to approximate

$$f_{ij} = A^0(x_l) \nabla \Phi_i(x_l) \nabla \Phi_i(x_l) \tag{2.13}$$

by the solution to a fine scale problem on a small domain $I_h(x_l)$ around the quadrature point x_l (see Fig. 2.1). This idea is based on the homogenization theorem, see for example [3], Th. 6.1. Let $\mathcal{M}_Y(f)$ stand for the Y-cell average of a function f, i.e., $\mathcal{M}_Y(f) = 1/|Y| \int_Y f dy$. The matrix A^0 is given by

$$A^0 \lambda = \mathcal{M}_Y(A^\varepsilon \nabla w_\lambda) \quad \forall \lambda \in \mathbb{R}^n, \tag{2.14}$$

where w_λ is the solution to the next auxiliary cell problem.

PROBLEM 1. *Find w_λ such that $w_\lambda - \lambda \cdot y \in W_{per}(Y)$ and $a_Y(w_\lambda, v) = 0$ for all $v \in W_{per}(Y)$, where the space $W_{per}(Y)$ is defined as*

$$W_{per}(Y) := \left\{ v \in H_{per}^1(Y) \;\middle|\; \int_Y v = 0 \right\}. \tag{2.15}$$

Here $H^1_{per}(Y)$ is the closure of $C^\infty_{per}(Y)$, i.e. the subset of Y-periodic functions of $C^\infty(\mathbb{R}^N)$, in H^1-norm. The bilinear form $a_Y(u,v)$ is defined as

$$a_Y(u,v) := \int_Y A^\varepsilon \nabla u \nabla v \; dy \quad \forall u, v \in W_{per}(Y). \tag{2.16}$$

Moreover we have

$$\mathcal{M}_Y(w_Y - \lambda \cdot y) = 0. \tag{2.17}$$

On the basis of this characterization of A^0, particularly from (2.14) we can write

$$A^0(x_l)\nabla U(x_l) = \mathcal{M}_Y(A^\varepsilon \nabla u), \tag{2.18}$$

where $u = w_\lambda$ for $\lambda = \nabla U(x_l)$. Moreover the Y-periodicity of $w_Y - \lambda \cdot y$ yields

$$\nabla U(x_l) = \mathcal{M}_Y(\nabla u). \tag{2.19}$$

In this way, the homogenization theorem gives us the definition of the appropriate compression operator $\mathcal{Q}$. It is defined in the quadrature points by (2.19).

Consequently, the macro bilinear form can be defined by the HMM as

$$A(U^H, V^H) := \sum_{K \in \mathcal{T}_H} |K| \sum_{x_l \in K} \frac{\omega_l}{|I_h(x_l)|} (A^\varepsilon \nabla u, \nabla v)_{I_h(x_l)}, \tag{2.20}$$

where u, v are solutions to Problem 2 (see below) for $\Psi = U^H$, $\Psi = V^H$, respectively. The sublinear cost of the HMM stems from the fact that we solve the microproblem only on the small sampling domain $I_h(x_l)$ around the quadrature point x_l, not on the whole triangle K. Let us recall, that the convergence of the HMM in H is robust in a sense that it does not depend on ε, see [4].

PROBLEM 2. *Find u such that $u - \Psi \in W_{per}(I_h(x_l))$ and*

$$\int_{I_h(x_l)} \nabla u A^\varepsilon (\nabla v)^T = 0 \quad \forall v \in W_{per}(I_h(x_l)).$$

In [11] is studied how the choice of the boundary condition influences the accuracy of the solution. It appears that the periodic boundary condition is the best choice not only for periodic materials, but also for materials with stochastic nature. Only requirement is, that diameter of sampling domain $I_h(x_l)$ should be large enough to cover the whole microscopic structure.

3. Eddy currents problem. We will start from time-harmonic Maxwell's system (see e.g. [9])

$$\nabla \times \boldsymbol{E} = -i\omega\mu \boldsymbol{H}, \tag{3.1}$$

$$\nabla \times \boldsymbol{H} = \boldsymbol{J}_a + \sigma \boldsymbol{E}, \tag{3.2}$$

$$\nabla \cdot (\mu \boldsymbol{H}) = 0, \tag{3.3}$$

$$\nabla \cdot (\epsilon \boldsymbol{E}) = \varrho, \tag{3.4}$$

where $\boldsymbol{H}$ denotes the magnetic field, $\boldsymbol{J}_a$ is the current density, $\boldsymbol{E}$ is the electric field, ϱ is the electric charge density and σ, μ, ϵ are the matrices of conductivity, permeability and permittivity, respectively. Model (3.1)–(3.4) as stated here is known as *eddy*

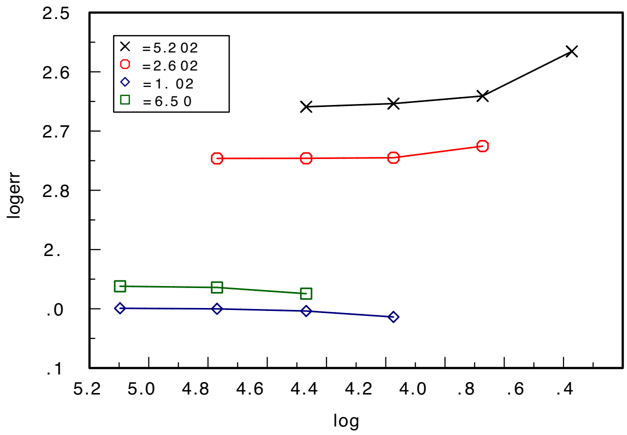


FIG. 4.1. *Convergence of the HMM*

currents model [1]. Applying the divergence to both sides of Faraday's law (3.1) we can see, that Gauss's magnetic law (3.3) is only a consequence of it. After elimination of $\boldsymbol{H}$ from Maxwell's equations (3.1)–(3.2) we arrive at

$$\nabla \times (\mu^{-1} \nabla \times \boldsymbol{E}) + i\omega\sigma \boldsymbol{E} = -i\omega \boldsymbol{J}_a \quad \text{in } \Omega. \tag{3.5}$$

Note, that the matrices σ, μ, ϵ can be in general dependent on both micro and macro scale and thus $\sigma = \sigma(x, y)$, $\mu = \mu(x, y)$ and $\epsilon = \epsilon(x, y)$.

4. Numerical experiments. To implement the eddy currents problem (3.5) we have used *feHMM* library. It is a C++ library being developed at our department. The main objective of this package is to work as a black-box solver, where user has to enter only geometry of the problem, right-hand side at the macro level, boundary conditions and the bilinear form of the microproblem. Since this library is build as a plug-in for Alberta finite element package [10] it possesses all strong features of Alberta as mesh generation based on bisectioning refinement method introduced in [7] and the usage of standard BLAS solvers. Another strong feature is, that code is automatically written for all space dimensions. Only geometry dependent issues like domain or boundary conditions have to be treated individually.

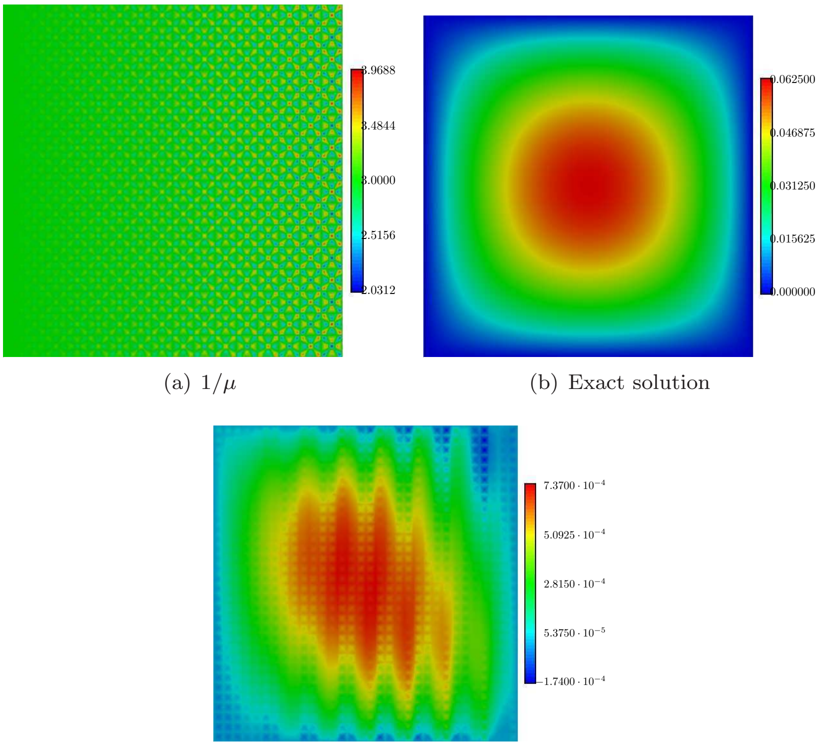
At present only Dirichlet boundary conditions are implemented in the *feHMM* library. The error estimate for Dirichlet boundary conditions as stated by E et al. in [6] is

$$\|u^0 - U^H\|_{L^2(\Omega)} \leq C\left(\frac{\varepsilon}{h} + h + H^2\right) \|f\|_{L^2(\Omega)}, \tag{4.1}$$

where u^0 is the homogenized solution and U^H its approximation obtained by the finite element HMM. H is the diameter of macro mesh, h is the diameter of micro mesh (for idea of this see Fig. 2.1). This estimate is valid under the assumption that microproblems are solved precisely.

On Fig. 4.1 is shown the convergence of HMM depending on size of the mesh. Logarithm of absolute error is shown. The dimension of microscale $\varepsilon = 0.001$.

The numerical experiments confirm the theoretical predictions. The error with respect to H decreases when H decreases until the term ε/h in (4.1) becomes dominant.



(a) $1/\mu$ (b) Exact solution

(c) Error

Fig. 4.2. *Testing problem and performance of the HMM*

For the biggest values of H the error decreases with respect to h when h decreases. Once the values of h become very small, microscopic details (around the quadrature points) are resolved with high precision. The error with respect to h does not improve any more.

On Fig. 4.2 is shown performance of the HMM method with $\epsilon = 1$, $\sigma = 1$ and $1/\mu = \sin(2\pi x/\varepsilon)x + 3$ with prescribed exact solution $E(x, y) = (x(x-1)y(y-1), x(x-1)y(y-1))$. The dimension of microscale $\varepsilon = 0.001$. Here $H = 0.006472$ and $h = 0.000034$.

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