

## COMPLEMENTARY FINITE VOLUME SCHEME FOR THE ANISOTROPIC SURFACE DIFFUSION FLOW\*

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**Abstract.** We study anisotropic surface diffusion flow of hypersurfaces in  $\mathbb{R}^n$ . We present a finite volume numerical scheme for the graph and the level-set formulation. The graph formulation is applied on surfaces in  $\mathbb{R}^3$  and with the level-set formulation we evolve curves in  $\mathbb{R}^2$ . The discretisation in time is done by mean of the method of lines which gives us a system of ordinary differential equations. To solve this system we apply the Merson modification of the fourth order Runge-Kutta method. We show several qualitative results for both, the graph and the level-set formulation.

**Key words.** surface diffusion flow, anisotropy, finite volume method, method of lines

**AMS subject classifications.** 35K35, 35K55, 53C44, 65M12, 65M20, 74S10

**1. Introduction.** In this article we study a motion of hypersurfaces in  $\mathbb{R}^n$ . We assume having a closed, compact, orientable moving  $C^1$ -hypersurface [10]  $\Gamma(t) \subset \mathbb{R}^n$  such that its outer unit normal  $\mathbf{n}$  is defined for all  $\mathbf{x} \in \Gamma(t)$ . The change of  $\Gamma(t)$  is given by a movement of each  $\mathbf{x} \in \Gamma(t)$  in the normal direction. The speed of this movement is so called normal velocity which we denote by  $V$ . To explain our notation of what will follow, we first assume having a function  $f \in C^2(\Gamma(t))$  defined on a hypersurface  $\Gamma(t)$  and its arbitrary  $C^2$ -extension i.e.  $\bar{f} \in C^2(\mathbb{R}^n)$  such that  $f = \bar{f}$  on  $\Gamma$  – see Kimura [10]. We define gradient of  $f$  on  $\Gamma(t)$  as

$$\nabla_{\Gamma(t)} f(\mathbf{x}) := \nabla \bar{f}(\mathbf{x}) - (\nabla \bar{f}(\mathbf{x}), \mathbf{n}(\mathbf{x})) \mathbf{n}(\mathbf{x}) \text{ for } \mathbf{x} \in \Gamma(t)$$

and the Laplace-Beltrami operator as

$$\Delta_{\Gamma(t)} f := \nabla_{\Gamma(t)} \cdot \nabla_{\Gamma(t)} \bar{f} \text{ for } \mathbf{x} \in \Gamma(t).$$

We consider an energy functional

$$E_\gamma(\Gamma(t)) = \int_{\Gamma(t)} \gamma(\mathbf{n}) \, dA \tag{1.1}$$

where  $\gamma$  is an *admissible anisotropy* i.e.  $\gamma$  is a function  $\gamma : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}^+$ ,  $\gamma \in C^3(\mathbb{R}^n \setminus \{0\})$  which is positively homogeneous of degree one and which is convex in the sense that there exists a constant  $c_0 > 0$  such that  $\mathbf{q}^T D^2(\gamma(\mathbf{p})) \mathbf{q} \geq c_0 |\mathbf{q}|^2$  for all  $\mathbf{p}, \mathbf{q} \in \mathbb{R}^n$  with  $\mathbf{p} \cdot \mathbf{q} = 0, |\mathbf{p}| = 1$  – see [4].  $H_\gamma$  is an anisotropic mean-curvature of  $\Gamma(t)$  determined by an admissible anisotropy  $\gamma$  and it is defined as the first variation of the surface energy

$$\mathcal{A}_\gamma := \int_{\Gamma(t)} \gamma(\mathbf{n}) \, dS.$$

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The anisotropic surface diffusion flow minimises the energy (1.1) while it preserves the volume of the interior of  $\Gamma(t)$  [10, 4]. It is a fourth order geometric problem defined as

$$V = -\Delta_{\Gamma(t)} H_\gamma \text{ on } \Gamma(t). \quad (1.2)$$

It is a modification of the isotropic surface diffusion flow for which some theoretical results were obtained mainly in the last decade. For example Elliott and Garcke [6] studied the surface diffusion flow of curves. They showed the local solution existence and global solution existence for initial curves close to circle. They also proved that the global solution converges to a circle. Escher, Mayer and Simonett [7] proved existence and uniqueness of the classical solution for immersed hypersurfaces in  $\mathbb{R}^n$ . They showed that "if the initial surface is embedded and close to a sphere... the solution exists globally and converges exponentially fast to a sphere".

Mayer and Simonett [11] gave proof that self-intersections can develop. Giga and Ito [8] showed that with smooth embedded initial data the evolved hypersurface  $\Gamma(t)$  may lose the property of being embedded.

In this article we study the graph and the level-set formulation.

**2. The graph formulation.** To evolve a surface  $\Gamma(t)$  given as a graph of some function  $u$  defined on a domain  $\Omega \in \mathbb{R}^{n-1}$ ,  $u \in C^4([0, T] \times \Omega)$  such that

$$\Gamma(t) \equiv \{(\mathbf{x}, u(\mathbf{x}, t)) \mid \mathbf{x} \in \Omega, t \in [0, T]\},$$

with respect to (1.2), we need to know what form  $V$ ,  $\Delta_{\Gamma(t)}$  and  $H_\gamma$  take. We refer to Deckelnick, Dziuk and Elliott [5] where it is shown that

$$V = \frac{u_t}{Q}, \quad H_\gamma = \nabla \cdot D_{\mathbf{p}} \gamma(\nabla u, -1), \quad \Delta_{\Gamma(t)} \phi = \nabla \cdot (\mathbb{P} \nabla \phi),$$

with  $Q = \sqrt{1 + |\nabla u|^2}$ ,  $D_{\mathbf{p}}$  denotes a differentiation w.r. to variable  $\mathbf{p}$  i.e.  $D_{\mathbf{p}} \gamma = (\gamma_{p_1}, \dots, \gamma_{p_{n-1}})^T$ . For example for the isotropic problem we set  $\gamma(\mathbf{p}, -1) = \sqrt{1 + |\mathbf{p}|^2}$  which gives  $D_{\mathbf{p}} \gamma = \mathbf{p} / \sqrt{1 + |\mathbf{p}|^2}$  and so  $H_\gamma = \nabla \cdot \frac{\nabla u}{Q}$ . Finally we set  $\mathbb{P} = Q\mathbb{I} - (\nabla u \otimes \nabla u) / Q$  where  $(\mathbf{v} \otimes \mathbf{w})_{ij} = \mathbf{v}_i \mathbf{w}_j$  for vectors  $\mathbf{v}, \mathbf{w} \in \mathbb{R}^{n-1}$ . With this notation in hand we may define the anisotropic surface diffusion flow of graphs:

DEFINITION 2.1. *The anisotropic surface diffusion flow of graphs is the fourth order parabolic partial differential problem defined as:*

$$u_t = -\nabla \cdot (\mathbb{P} \nabla H_\gamma) \text{ on } \Omega \times (0, T], \quad (2.1)$$

$$H_\gamma = \nabla \cdot D_{\mathbf{p}} \gamma(\nabla u, -1) \text{ on } \Omega \times [0, T], \quad (2.2)$$

$$u|_{t=0} = u_{ini} \text{ on } \Omega,$$

$$u = g, H_\gamma = 0 \text{ on } \partial\Omega \text{ (the Dirichlet b.c.)},$$

$$D_{\mathbf{p}} \gamma \cdot \nu = 0, (\mathbb{P} \nabla H_\gamma) \nu = 0, \text{ on } \partial\Omega \text{ (the Neumann b.c.)},$$

Let us now derive the weak formulation which will make clear the choice of the boundary conditions. We start with the equation (2.2) and multiply it by a test function  $\varphi \in H_0^1(\Omega)$  to get

$$\int_{\Omega} H_\gamma \varphi \, d\mathbf{x} = \int_{\Omega} \nabla \cdot D_{\mathbf{p}} \gamma \varphi \, d\mathbf{x} = \int_{\partial\Omega} D_{\mathbf{p}} \gamma \cdot \nu \varphi \, dS - \int_{\Omega} D_{\mathbf{p}} \gamma \cdot \nabla \varphi \, d\mathbf{x},$$

and the integral over  $\partial\Omega$  equals zero since  $\varphi$  vanishes at  $\partial\Omega$ . Such a choice of  $\varphi$  corresponds with the Dirichlet boundary conditions for  $H_\gamma$  i.e.  $H_\gamma = 0$  on  $\partial\Omega$ . If we do not want to fix  $H_\gamma$  at the boundaries, we take  $\varphi \in H^1(\Omega)$  and we set  $D_{\mathbf{p}}\gamma \cdot \nu = 0$  on  $\partial\Omega$  to eliminate the boundary integral. Taking now the equation (2.1) and multiply it by a test function  $\xi \in H_0^1(\Omega)$  we get

$$\int_{\Omega} u_t \xi \, d\mathbf{x} = - \int_{\Omega} \nabla \cdot (\mathbb{P}\nabla H_\gamma) \xi \, d\mathbf{x} = - \int_{\partial\Omega} (\mathbb{P}\nabla H_\gamma) \cdot \nu \xi \, dS + \int_{\Omega} (\mathbb{P}\nabla H_\gamma) \cdot \nabla \xi \, dS,$$

where the integral over the boundary of  $\Omega$  is zero since  $\xi$  vanishes on  $\partial\Omega$ . Such choice of  $\xi$  corresponds with the Dirichlet boundary conditions for  $u$  i.e.  $u = g$  on  $\partial\Omega$ . On the other hand if we take  $\xi \in H^1(\Omega)$  we need to have  $(\mathbb{P}\nabla H_\gamma) \cdot \nu = 0$ . Thus we get the Neumann boundary conditions for  $H_\gamma$  and also the weak formulation for the anisotropic surface diffusion flow of graphs

**DEFINITION 2.2.** *The weak solution of the graph formulation for the anisotropic surface diffusion flow with the Dirichlet boundary conditions*

$$\begin{aligned} u &= g && \text{on } \partial\Omega, \\ H_\gamma &= 0 && \text{on } \partial\Omega, \end{aligned}$$

is a couple  $u, H : (0, T) \rightarrow H_0^1(\Omega)$  which for each test function  $\varphi, \xi \in H_0^1(\Omega)$  and a.e. in  $(0, T)$  satisfies,

$$\int_{\Omega} \frac{u_t}{Q} \varphi \, d\mathbf{x} = \int_{\Omega} (\mathbb{P}\nabla H) \cdot \nabla \varphi \, d\mathbf{x} \quad (2.3)$$

$$\int_{\Omega} H_\gamma \xi \, d\mathbf{x} = - \int_{\Omega} D_{\mathbf{p}}\gamma \cdot \nabla \xi \, d\mathbf{x}. \quad (2.4)$$

with the initial condition

$$u|_{t=0} = u_{ini}. \quad (2.5)$$

The weak solution for the problem with the homogeneous Neumann boundary conditions

$$\begin{aligned} D_{\mathbf{p}}\gamma \cdot \nu &= 0 && \text{on } \partial\Omega, \\ (\mathbb{P}\nabla H_\gamma) \cdot \nu &= 0 && \text{on } \partial\Omega, \end{aligned}$$

is a couple  $u, H : (0, T) \rightarrow H^1(\Omega)$  which for each test function  $\varphi, \xi \in H^1(\Omega)$  and a.e. in  $(0, T)$  satisfies (2.3)-(2.4) and the initial condition (2.5).

**3. The level-set formulation.** In very similar way we may obtain the level-set formulation for the anisotropic surface diffusion flow. Now we assume that we have a domain  $\Omega \in \mathbb{R}^n$ , auxiliary level-set function  $u \in C^4(\Omega \times [0, T])$  and  $\Gamma(t)$  is given as

$$\Gamma(t) \equiv \{\mathbf{x} \in \Omega \mid u(\mathbf{x}, t) = 0\},$$

The normal to the hypersurface  $\Gamma(t)$  is now given by  $\nabla u / |\nabla u|$  instead of  $\frac{(\nabla u, -1)}{[(\nabla u, -1)]}$ . Adding the regularising parameter  $\epsilon$  to avoid dividing by zero in regions where the level-set function  $u$  is flat and  $\nabla u = \mathbf{0}$ , we get the level-set formulation.

DEFINITION 3.1. *The level-set formulation for the anisotropic surface diffusion flow is the fourth order parabolic partial differential equation defined as:*

$$\begin{aligned} u_t &= -\nabla \cdot (\mathbb{P}\nabla H_\gamma) \text{ on } \Omega \times (0, T], \\ H_\gamma &= \nabla \cdot D_{\mathbf{p}}\gamma(\nabla u) \text{ on } \Omega \times [0, T], \\ u|_{t=0} &= u_{ini} \text{ on } \Omega, \\ u &= g, H_\gamma = 0 \text{ on } \partial\Omega \text{ (the Dirichlet b.c.)}, \\ D_{\mathbf{p}}\gamma \cdot \nu &= 0, (\mathbb{P}\nabla H_\gamma) \cdot \nu = 0, \text{ on } \partial\Omega \text{ (the Neumann b.c.)}, \end{aligned}$$

where we denote

$$Q_\epsilon = \sqrt{\epsilon^2 + |\nabla u|^2}, \quad \mathbb{P} = Q_\epsilon \mathbb{I} - \frac{\nabla u \otimes \nabla u}{Q_\epsilon}. \quad (3.1)$$

In the same way we may also derive the weak solution for the level-set formulation. For more details about the level-set approach to the surface diffusion flow we refer to Smereka [12].

**4. Numerical scheme.** Let us now proceed to the numerical approximation of the anisotropic surface diffusion. Since the graph and the level-set formulation differ only in the choice of the term  $Q$  resp.  $Q_\epsilon$  we may try to use the same scheme for both problems. Before we present the complementary finite volume scheme for the anisotropic surface diffusion flow, we would like to mention some results of other researchers who studied numerical schemes for the surface diffusion. A scheme based on the finite element method for the surface diffusion of graphs together with an error analysis can be found in Bänsch, Morin and Nochetto [1], the anisotropic problem has been studied by Deckelnick, Dziuk and Elliott [5]. Finite element numerical scheme for the level-set formulation of the surface diffusion flow has been presented by Smereka [12], a scheme for the anisotropic problem was proposed by Clarenz, Hausser, Rumpf, Voigt and Weikard [3]. The method of complementary volumes combined with finite element method for the approximation of the mean-curvature flow has been introduced by Walkington in [14]. However, the method of the numerical approximation, we adopt in this article, originates rather in the article by Handlovičová, Mikula and Sgallari [9]. The same kind of scheme has been also applied for the approximation of the Willmore flow in Beneš, Mikula, Oberhuber and Ševčovič [2].

In this article we restrict ourselves only to a case when  $\Omega \subset \mathbb{R}^2$ . It means that we study either surfaces in  $\mathbb{R}^3$  given as a graph of some function  $u : \Omega \rightarrow \mathbb{R}$  or planar curves given as a zero level set of a real function defined also on the domain  $\Omega$ . Moreover, we assume that  $\Omega$  has rectangular shape i.e.  $\Omega \equiv \langle 0, L_1 \rangle \times \langle 0, L_2 \rangle$ . Let  $h_1, h_2$  be space steps such that  $h_1 = \frac{L_1}{N_1}$  and  $h_2 = \frac{L_2}{N_2}$  for some  $N_1, N_2 \in \mathbb{N}^+$ . We define a numerical grid, its closure and its boundary as

$$\begin{aligned} \omega_h &= \{(ih_1, jh_2) \mid i = 1 \cdots N_1 - 1, j = 1 \cdots N_2 - 1\}, \\ \bar{\omega}_h &= \{(ih_1, jh_2) \mid i = 0 \cdots N_1, j = 0 \cdots N_2\}, \\ \partial\omega_h &= \bar{\omega}_h \setminus \omega_h, \end{aligned}$$

for  $u \in C(\bar{\Omega})$  we define its piecewise constant approximation on  $\bar{\omega}$  as a grid function  $u^h$  defined as  $u^h(ih_1, jh_2) := u_{ij}^h := u(ih_1, jh_2)$ . We also need a dual mesh  $V_h$  defined

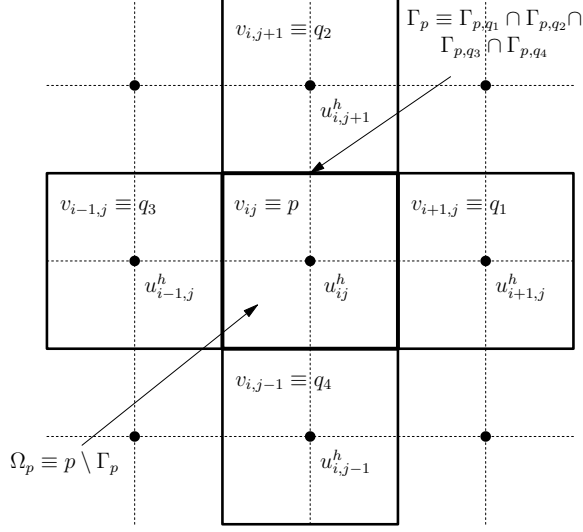


FIG. 4.1. Finite volumes on the dual mesh –  $p$  denotes finite volume we refer to (here  $v_{ij}$ ),  $\Omega_p$  its interior,  $\Gamma_p$  its boundary consisting of linear segments  $\Gamma_{p,q_1}, \Gamma_{p,q_2}, \Gamma_{p,q_3}$  and  $\Gamma_{p,q_4}$  where  $q_1, q_2, q_3$  and  $q_4$  are neighbouring finite volumes for which we also use a notation  $\mathcal{N}_p \equiv \{q_1, q_2, q_3, q_4\}$ .

as

$$V_h \equiv \left\{ v_{ij} = \left[ \left( i - \frac{1}{2} \right) h_1, \left( i + \frac{1}{2} \right) h_1 \right] \times \left[ \left( j - \frac{1}{2} \right) h_2, \left( j + \frac{1}{2} \right) h_2 \right] \mid i = 1 \cdots N_1 - 1, j = 1 \cdots N_2 - 1 \right\}. \quad (4.1)$$

For  $0 < i < N_1, 0 < j < N_2, i$  and  $j$  fixed, consider a volume  $p = v_{ij}$  of the dual mesh  $V_h$ , denote its interior as  $\Omega_p$ , its boundary as  $\Gamma_p$  and let  $\mu(\Omega_p)$  be the volume of  $\Omega_p$ . We also denote all neighbouring volumes of the volume  $p$  as  $\mathcal{N}_p$ . For all inner finite volumes  $p$  of the dual mesh  $V_h$ , the boundary  $\Gamma_p$  consists of four linear segments. We denote them as  $\Gamma_{pq}$  for  $q \in \mathcal{N}_p$ . It means that  $\Gamma_{pq}$  is a boundary between the finite volumes  $p$  and  $q$  – see the Figure 4.1. Let  $l_{pq}$  be the length of this part of  $\Gamma_p$ . To approximate the anisotropic mean-curvature  $H_\gamma$ , we take the equation (2.2), integrate it over the finite volume  $p$  and apply the Stokes theorem

$$\int_{\Omega_p} H_\gamma dx = \int_{\Omega_p} \nabla \cdot (D_{\mathbf{p}} \gamma) dx = \int_{\Gamma_p} D_{\mathbf{p}} \gamma \cdot \nu dS = \sum_{q \in \mathcal{N}_p} \int_{\Gamma_{pq}} D_{\mathbf{p}} \gamma \cdot \nu dS. \quad (4.2)$$

For the inner finite volume  $p \in V_h$ , there are four different neighbours  $q \in \mathcal{N}_p$ . Each boundary between  $p$  and  $q$  ( $\Gamma_{pq}$ ) is linear segment and so  $\nu_{pq}$  is constant there. Moreover we assume that the vector  $D_{\mathbf{p}} \gamma = D_{\mathbf{p}} \gamma_{pq}$  is constant on  $\Gamma_{pq}$  too. It gives

$$\sum_{q \in \mathcal{N}_p} \int_{\Gamma_{pq}} D_{\mathbf{p}} \gamma \cdot \nu dS \approx \sum_{q \in \mathcal{N}_p} l_{pq} D_{\mathbf{p}} \gamma_{pq} \cdot \nu_{pq}, \quad (4.3)$$

where the subindexes  $pq$  denote quantity evaluated on  $\Gamma_{pq}$ . Assuming that  $H_\gamma^h$  is

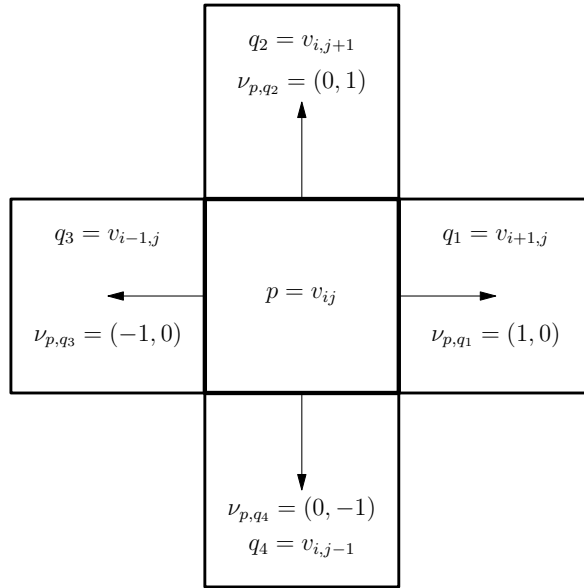


FIG. 4.2. This figure illustrates how the neighbouring finite volumes of  $p$  are determined by a form of the unit normal  $\nu_{pq} = \nu^{rs} = (r, s)$  for  $r, s \in \{-1, 1\}$  and  $|r| + |s| = 1$ .

constant on the finite volume  $p$  and equal to  $H_{\gamma,ij}^h$  we may write

$$\int_{\Omega} H_{\gamma}^h d\mathbf{x} = \mu(\Omega_p) H_{\gamma,ij}^h \quad (4.4)$$

where  $\mu(\Omega_p)$  denotes the measure of the finite volume  $p$ . Putting (4.3) and (4.4) together we get

$$H_{\gamma,p}^h = \frac{1}{\mu(\Omega_p)} \sum_{q \in \mathcal{N}_p} l_{pq} D_{\mathbf{P}} \gamma_{pq} \cdot \nu_{pq}. \quad (4.5)$$

For the dual mesh  $V_h$  given by (4.1), we may substitute  $\mu(\Omega_p) = h_1 h_2$ . For  $p, q$  such that  $\nu_{pq} = (\pm 1, 0)$ , we have  $l_{pq} = h_2$  and if  $\nu_{pq} = (0, \pm 1)$  then  $l_{pq} = h_1$ . It is easy to see, that for a fixed finite volume  $p$ , one of its neighbours is determined by the form of the normal  $\nu_{pq}$  of the boundary  $\Gamma_{pq}$ . For  $r, s \in \{-1, 1\}$  and  $|r| + |s| = 1$  the unit outer normal  $\nu_{pq}$  can take the values  $\nu_{pq} = \nu^{rs} = (r, s)$  when  $q = v_{i+r,j+s}$  – see the Figure 4.2. We can see that  $\Gamma_{pq}$  is uniquely determined by indexes  $i, j, r$  and  $s$ . We will therefore also denote the quantities evaluated on  $\Gamma_{pq}$  by subindexes  $ij$  and superindexes  $rs$ . By this mean we also write  $l_{pq} = l^{rs}$  (the boundary length does not depend on  $ij$  and so we omit these indexes).

We may now approximate  $D_{\mathbf{P}} \gamma$  on  $\Gamma_{pq}$  as

$$D_{\mathbf{P}} \gamma \approx D_{\mathbf{P}} \gamma_{pq} = D_{\mathbf{P}} \gamma_{ij}^{rs} = (\gamma_{p_1,ij}^{rs}, \gamma_{p_2,ij}^{rs})^T \text{ on } \Gamma_{pq}$$

we obtain

$$H_{\gamma,ij}^h = \frac{1}{h_1 h_2} \sum_{\substack{r,s \in \{-1,1\} \\ |r|+|s|=1}} l^{rs} D_{\mathbf{P}} \gamma_{ij}^{rs} \cdot \nu^{rs} = \frac{\gamma_{p_1,ij}^{1,0} - \gamma_{p_1,ij}^{-1,0}}{h_1} + \frac{\gamma_{p_2,ij}^{0,1} - \gamma_{p_2,ij}^{0,-1}}{h_2}$$

for  $i = 2, \dots, N_1 - 2$ ,  $j = 2, \dots, N_2 - 2$ . In the case of the Neumann boundary conditions, from  $D_{\mathbf{p}}\gamma \cdot \nu = 0$  we set:

$$\text{if } i = 1 \text{ then } \nu = (-1, 0) \Rightarrow \gamma_{p_1, 1, j}^{-1, 0} = 0, \quad (4.6)$$

$$\text{if } i = N_1 - 1 \text{ then } \nu = (1, 0) \Rightarrow \gamma_{p_1, N_1 - 1, j}^{1, 0} = 0, \quad (4.7)$$

$$\text{if } j = 1 \text{ then } \nu = (0, -1) \Rightarrow \gamma_{p_2, i, 1}^{0, -1} = 0, \quad (4.8)$$

$$\text{if } j = N_2 - 1 \text{ then } \nu = (0, 1) \Rightarrow \gamma_{p_2, i, N_2 - 1}^{0, 1} = 0. \quad (4.9)$$

The Dirichlet boundary conditions for  $H_\gamma$  are trivial to approximate.

We may now proceed to the approximation of the equation (2.1). Integrate it over  $\Omega_p$  and applying again the Stokes theorem we have

$$\int_{\Omega_p} u_t d\mathbf{x} = - \int_{\Gamma_p} \nabla \cdot (\mathbb{P} \nabla H_\gamma) d\mathbf{x},$$

which gives

$$\mu(\Omega_p) \frac{d}{dt} u_p^h = - \sum_{q \in \mathcal{N}_p} l_{pq} \left( Q_{pq} \mathbb{I} - \frac{\nabla u_{pq}^h \otimes \nabla u_{pq}^h}{Q_{pq}} \right) \nabla H_{\gamma_{pq}}^h \nu_{pq}. \quad (4.10)$$

The last relation may be also written as

$$\frac{d}{dt} u_{ij}^h = - \frac{1}{h_1 h_2} \sum_{\substack{r, s \in \{-1, 1\} \\ |r| + |s| = 1}} l^{rs} \left( Q_{ij}^{rs} \mathbb{I} - \frac{\nabla u_{ij}^{h, rs} \otimes \nabla u_{ij}^{h, rs}}{Q_{ij}^{rs}} \right) \nabla H_{\gamma_{ij}}^{h, rs} \nu^{rs}. \quad (4.11)$$

To approximate  $Q_{ij}^{rs}$  we set

$$Q_{ij}^{rs} = \sqrt{1 + |\nabla u_{ij}^{h, rs}|^2}, \text{ resp. } Q_{\epsilon, ij}^{rs} = \sqrt{\epsilon^2 + |\nabla u_{ij}^{h, rs}|^2},$$

for

$$\nabla u_{ij}^{h, r, 0} = \left( r \frac{u_{i+r, j}^h - u_{ij}^h}{h_1}, \frac{u_{ij}^{r, 1} - u_{ij}^{r, -1}}{h_2} \right), \nabla u_{ij}^{h, 0, s} = \left( \frac{u_{ij}^{1, s} - u_{ij}^{-1, s}}{h_1}, s \frac{u_{i, j+s}^h - u_{ij}^h}{h_2} \right), \quad (4.12)$$

where  $u_{ij}^{r, s}$  is an average defined as (see the Figure 4.3):

$$u_{ij}^{r, s} = \frac{1}{4} (u_{ij}^h + u_{i+r, j}^h + u_{i, j+s}^h + u_{i+r, j+s}^h).$$

The discretisation of (4.11) in time is done by mean of the method of lines and resulting system of ODE's is solved using the fourth order Runge-Kutta-Merson method with adaptive choice of the time step – see Vitásek [13]. The approximation of the Dirichlet boundary conditions is trivial. The Neumann boundary conditions are more difficult to discretise because of their implicit nature. The boundary conditions  $D_{\mathbf{p}}\gamma \cdot \nu = 0$  might be discretised by the following linear system

$$\frac{\gamma_{11, i - \frac{1}{2}, j}^{*, n-1}}{h_1} (u_{ij}^h - u_{i-1, j}^*) + \frac{\gamma_{12, i - \frac{1}{2}, j}^{*, n-1}}{4h_1 h_2} (u_{i, j+1}^h + u_{i-1, j+1}^* - u_{i, j-1}^h - u_{i-1, j-1}^*) = 0 \text{ for } i = 1, \quad (4.13)$$

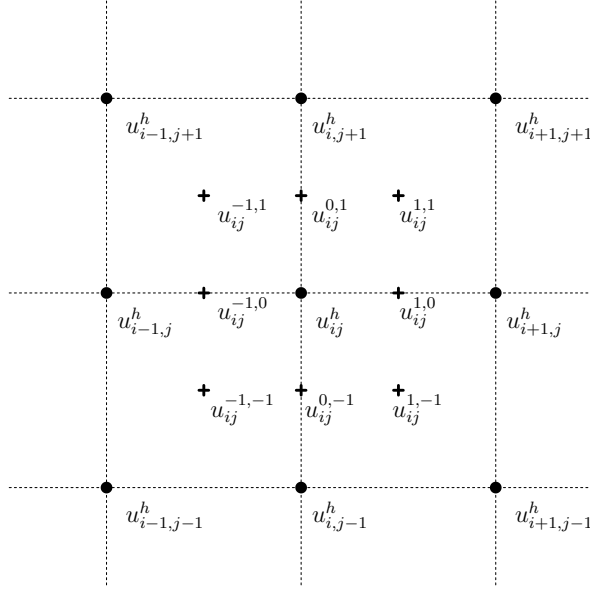


FIG. 4.3. Figure demonstrates a notation used for the grid functions.

$$\frac{\gamma_{11,i+\frac{1}{2},j}^{*,n-1}}{h_1} (u_{i+1,j}^* - u_{i,j}^h) + \frac{\gamma_{12,i+\frac{1}{2},j}^{*,n-1}}{4h_1h_2} (u_{i+1,j+1}^* + u_{i,j+1}^h - u_{i+1,j-1}^* - u_{i,j-1}^h) = 0 \text{ for } i = N_1 - 1, \quad (4.14)$$

$$\frac{\gamma_{21,i,j-\frac{1}{2}}^{*,n-1}}{4h_1h_2} (u_{i+1,j-1}^* + u_{i+1,j}^h - u_{i-1,j-1}^* - u_{i-1,j}^h) + \frac{\gamma_{22,i,j-\frac{1}{2}}^{*,n-1}}{h_2} (u_{ij}^h - u_{i,j-1}^*) = 0 \text{ for } j = 1, \quad (4.15)$$

$$\frac{\gamma_{21,i,j+\frac{1}{2}}^{*,n-1}}{4h_1h_2} (u_{i+1,j+1}^* + u_{i+1,j}^h - u_{i-1,j+1}^* - u_{i-1,j}^h) + \frac{\gamma_{22,i,j+\frac{1}{2}}^{*,n-1}}{h_2} (u_{i,j+1}^* - u_{ij}^h) = 0 \text{ for } j = N_2 - 1, \quad (4.16)$$

where  $u_{ij}^*$  are known values representing the boundary values of  $u_{ij}^h$ . The same holds for the boundary condition  $(\mathbb{P}_{ij}^{rs} \nabla H_{\gamma,ij}^h) \nu = 0$  on  $\partial\Omega$  where we obtain the following



system

$$\frac{\mathbb{P}_{11,i-\frac{1}{2},j}}{h_1} (H_{\gamma,i,j}^h - H_{\gamma,i-1,j}^*) + \quad (4.17)$$

$$\frac{\mathbb{P}_{12,i-\frac{1}{2},j}}{4h_1h_2} (H_{\gamma,i,j+1}^h + H_{\gamma,i-1,j+1}^* - H_{\gamma,i,j-1}^h - H_{\gamma,i-1,j-1}^*) = 0 \text{ for } i = 1,$$

$$\frac{\mathbb{P}_{11,i+\frac{1}{2},j}}{h_1} (H_{\gamma,i+1,j}^* + H_{\gamma,i,j}^h) + \quad (4.18)$$

$$\frac{\mathbb{P}_{12,i+\frac{1}{2},j}}{4h_1h_2} (H_{i,j+1}^h + H_{i+1,j+1}^* - H_{i,j-1}^h - H_{i+1,j-1}^*) = 0 \text{ for } i = N_1 - 1,$$

$$\frac{\mathbb{P}_{21,i,j-\frac{1}{2}}}{4h_1h_2} (H_{\gamma,i+1,j}^h + H_{\gamma,i+1,j-1}^* - H_{\gamma,i-1,j}^h - H_{i-1,j-1}^*) +$$

$$\frac{\mathbb{P}_{22,i,j-\frac{1}{2}}}{h_2} (H_{\gamma,i,j}^h - H_{\gamma,i,j-1}^*) = 0 \text{ for } j = 1, \quad (4.19)$$

$$\frac{\mathbb{P}_{21,i,j+\frac{1}{2}}}{4h_1h_2} (H_{\gamma,i+1,j}^h + H_{\gamma,i+1,j+1}^* - H_{\gamma,i-1,j}^h - H_{\gamma,i-1,j+1}^*) + \quad (4.20)$$

$$\frac{\mathbb{P}_{22,i,j+\frac{1}{2}}}{h_2} (H_{\gamma,i,j+1}^* - H_{\gamma,i,j}^h) = 0 \text{ for } j = N_2 - 1,$$

solution of which gives us an extension  $H_{\gamma,i,j}^*$  of  $H_{\gamma,i,j}^h$  on  $\partial\omega_h$ .

An advantage of the explicit solver is in fact that during the evaluation of the right-hand side of (4.11) we always meet the same terms as the left-hand sides of (4.13)–(4.20) which equal zero. Therefore we do not need to know  $u_{ij}^h$  and  $H_{\gamma,i,j}^h$  on  $\partial\omega_h$  to be able to construct correct approximation of  $u$  on  $\omega_h$ . If we do not need these quantities explicitly, we may simplify the system (4.13)–(4.20) to  $\nabla u_{ij}^h \cdot \nu = 0$  and  $\nabla H_{\gamma,i,j}^h \cdot \nu = 0$  on  $\partial\omega_h$ . In fact, this is how we handled the boundary conditions because the explicit scheme leads to small time step and it would require a lot of CPU time to solve linear systems (4.13)–(4.20) at each iteration. In the case of the level-set formulation we impose simplified Neumann boundary conditions in a form  $\nabla u_{ij}^h \cdot \nu = 1$  and  $\nabla H_{\gamma,i,j}^h \cdot \nu = 0$  on  $\partial\omega_h$  which better reflect the shape of the signed distance function.

**5. Computational studies.** For the numerical experiments we chose an anisotropy induced by a quadratic form  $G : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ ,  $G(\mathbf{p}_1, \mathbf{p}_2) = \mathbf{p}_1^T \mathbb{G} \mathbf{p}_2$  given by a positive definite matrix  $\mathbb{G} \in \mathbb{R}^{2,2}$ . This anisotropy, which might be also understood as a weighted Euclidean norm, is defined as

$$\gamma_{\mathbb{G}}(\mathbf{p}, -1) := \sqrt{1 + \mathbf{p}^T \mathbb{G} \mathbf{p}}, \text{ resp. } \gamma_{\mathbb{G}}(\mathbf{p}) := \sqrt{\epsilon^2 + \mathbf{p}^T \mathbb{G} \mathbf{p}}. \quad (5.1)$$

The Figure 5.1 shows an evolution of an initial surface given as a graph of function  $u_{ini}(x, y) := \sin(2\pi x) \sin(2\pi y)$  on the domain  $\Omega \equiv [0, 1]^2$ . The matrix  $\mathbb{G}$  inducing the anisotropy (5.1) takes a form  $\mathbb{G} := \begin{pmatrix} 10 & 0 \\ 0 & 1 \end{pmatrix}$ . We set the zero Neumann boundary conditions and space step  $h_1 = h_2 = 0.01$ . The Figure 5.2 demonstrates the level-set formulation for the surface diffusion flow. As an initial condition we took a square given by  $u_{ini} := (|x| - 0.75)(|y| - 0.75) = 0$  on the domain  $\Omega \equiv [-2, 2]^2$  and we let it evolve until the time  $t = 0.2$ . We set the same anisotropy function as in the previous numerical experiment but induced by a matrix  $\mathbb{G} := \begin{pmatrix} 10 & 8 \\ 8 & 10 \end{pmatrix}$ . The regularising

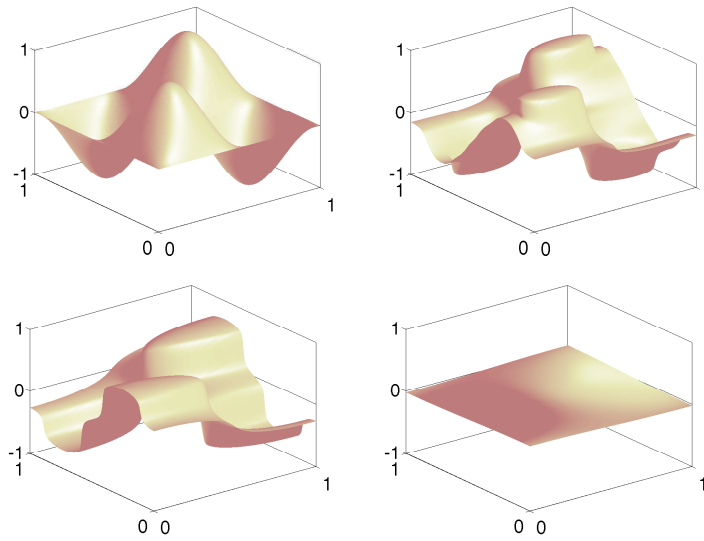


FIG. 5.1. The evolution obtained using the initial condition  $u_{ini}(x, y) := \sin(2\pi x) \sin(2\pi y)$  and an anisotropy induced by (5.1) at times  $t = 0$ ,  $t = 3.2 \cdot 10^{-5}$ ,  $t = 2.56 \cdot 10^{-4}$  and  $t = 5.6 \cdot 10^{-3}$ . The Neumann boundary conditions were imposed.

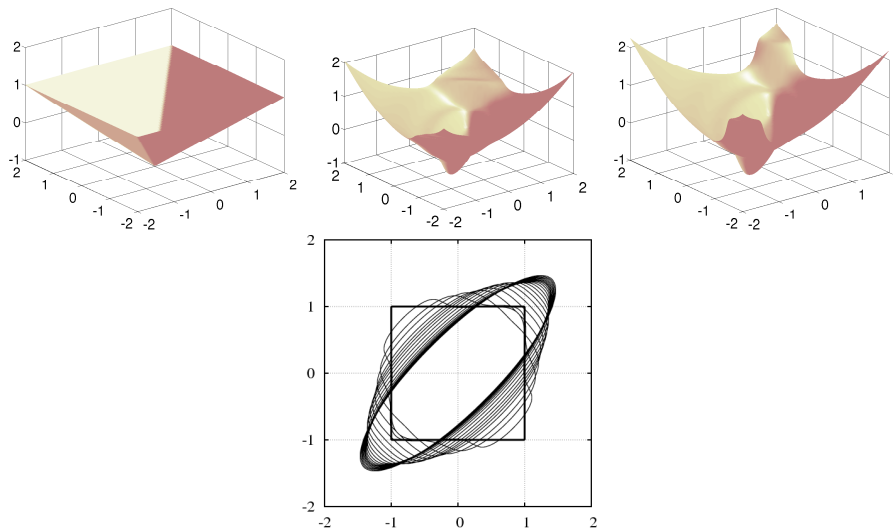


FIG. 5.2. Evolution of a square obtained by the level-set method – graphs of the level-set function at times  $t = 0$ ,  $t = 0.1$ ,  $t = 0.2$  and evolution of the initial curve until the time  $t = 0.2$ .

parameter was equal to  $\epsilon = 0.01$ . We set the space step  $h_1 = h_2 = 0.04$ . The last Figure 5.3 shows evolution of an astroid given as  $x^{2/3} + y^{2/3} = 0.75^{2/3}$  on the domain  $\Omega \equiv [-1, 1]^2$  with  $h_1 = h_2 = 0.016$ , regularisation  $\epsilon = 0.01$  and anisotropy indices by  $\mathbb{G} := \begin{pmatrix} 10 & 0 \\ 0 & 1 \end{pmatrix}$ .

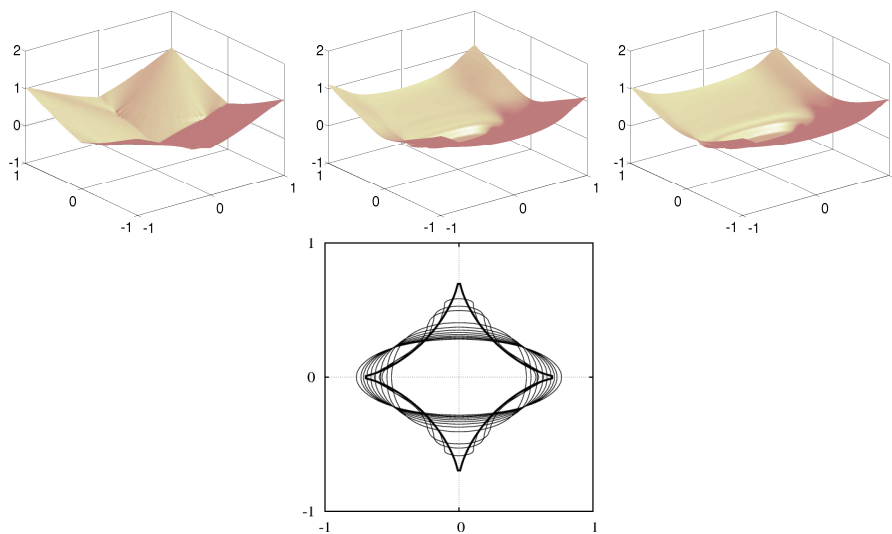


FIG. 5.3. Evolution of an astroid obtained by the level-set method – graphs of the level-set function at times  $t = 0$ ,  $t = 0.025$ ,  $t = 0.05$  and evolution of the initial curve until the time  $t = 0.05$ .

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#### REFERENCES

- [1] E. BÄNSCH, P. MORIN, AND R. NOCHETTO, *Surface diffusion of graphs: variational formulation, error analysis and simulation*, SIAM J. Numer. Anal., 42 (2004), pp. 773–799.
- [2] M. BENEŠ, K. MIKULA, T. OBERHUBER, AND D. ŠEVČOVIČ, *Comparison study for level set and direct Lagrangian methods for computing Willmore flow of closed planar curves*, Computing and Visualization in Science, (2008).
- [3] U. CLARENZ, F. HAUSSER, M. RUMPF, A. VOIGT, AND U. WEIKARD, *Multiscale modelling of epitaxial growth*, vol. 149, Birkhäuser, Basel, 2005, ch. On level set formulations for anisotropic mean curvature flow and surface diffusion, p. 227237.
- [4] K. DECKELNICK, G. DZIUK, AND C. ELLIOTT, *Computation of geometric partial differential equations and mean curvature flow*, Acta Numer., (2005), pp. 139–232.
- [5] K. DECKELNICK, G. DZIUK, AND C. ELLIOTT, *Fully discrete finite element approximation for anisotropic surface diffusion of graphs*, SIAM J. Numer. Anal., (2005), pp. 1112–1138.
- [6] C. M. ELLIOTT, H. GARCKE, *Existence results for geometric interface models for surface diffusion*, in Adv. Math. Sci. Appl, 7 (1997), pp. 467–490.
- [7] J. ESCHER, U. F. MAYER, G. SIMONETT, *The surface diffusion flow for immersed hypersurfaces*, in SIAM J. Math. Anal., 29, (1998), pp. 1419–1433.
- [8] Y. GIGA AND K. ITO, *On pinching of curves moved by surface diffusion*, Communications in Applied Analysis, 2 (1998), pp. 393–406.
- [9] A. HANDLOVIČOVÁ, K. MIKULA, AND F. SGALLARI, *Semi-implicit complementary volume scheme for solving level set like equations in image processing and curve evolution*, Numerische Mathematik, 93 (2003), pp. 675–695.
- [10] M. KIMURA, *Topics in mathematical modeling*, vol. 4, Matfyzpress, Prague, Czech Republic, 2008, lecture notes Geometry of hypersurfaces and moving hypersurfaces in  $R^m$  for the study of moving boundary problems, pp. 39–94.
- [11] U. F. MAYER, G. SIMONETT, *Self-intersections for the surface diffusion and the volume-preserving mean curvature flow*, Differential and Integral Equations, 13 (2000), pp. 1189–1199.
- [12] P. SMEREKA, *Semi-implicit level set methods for curvature and surface diffusion motion*, Journal of Scientific Computing, 19 (2003), pp. 439–456.

- [13] E. VITÁSEK, *Numerické metody*, SNTL, Nakladatelství technické literatury, 1987.
- [14] J. WALKINGTON, N., *Algorithms for computing motion by mean curvature*, SIAM J. Numer. Anal., 33 (1996), pp. 2215–2238.