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# NUMERICAL SOLUTION OF FLUID-STRUCTURE INTERACTION PROBLEMS BY FINITE ELEMENT METHOD *

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**Abstract.** This paper is devoted to the topic of mathematical modelling and numerical simulation of the interaction of two dimensional incompressible viscous flow and a vibrating structure. A solid airfoil with two degrees of freedom is considered. The numerical simulation consists of the finite element solution of the Navier-Stokes equations coupled with the system of ordinary differential equations describing the airfoil motion. The time dependent computational domain and a moving grid are taken into account with the aid of the Arbitrary Lagrangian-Eulerian (ALE) formulation of the Navier-Stokes equations. High Reynolds numbers up to $10^6$ require the application of a suitable stabilization of the finite element discretization. Here, the modified Streamline-Upwind/Petrov-Galerkin(SUPG) together with Pressure Stabilizing/Petrov-Galerkin(PSPG) stabilization is applied and modified within the context of ALE formulation of Navier-Stokes system of equations. The fluid model is coupled with the nonlinear structure model for the solid airfoil. The method is applied on several technical problems.

**Key words.** aeroelasticity, finite element method, Arbitrary Lagrangian-Eulerian method

**AMS subject classifications.**

**1. Introduction.** In many technical disciplines the interaction of fluid flow and an elastic structure plays an important role. The research in aeroelasticity or hydroelasticity focuses on the interaction between moving fluids and vibrating structures [see, e.g., [6], [16]]. Usually only special problems of aeroelasticity or hydroelasticity are solved, mainly limited to linearized models. The nonlinear postcritical limit states usually had not been considered, as the appearance of any aerodynamic instability is not admissible in normal flight regimes. Recently, the modelling of post-flutter behaviour began to be more important.

Nonlinear fluid/structure interaction problems arise in many engineering and scientific applications. During last years, significant advances have been made in the development and use of computational methods for fluid flows with structural interactions. The more efficient computational techniques were reached with the increasing computational power, see for example [1]. As the valuable information coming from fluid-structure interaction analysis need to be performed in many fields of industry (automobile, airplane) as well as in biomedicine, the analysis of fluid-structure interaction problems become more effective and more general. Since there is a need for effective fluid-structure interaction analysis procedures, various approaches have been proposed. In current simulations, arbitrary Lagrangian-Eulerian (ALE) formulations are now widely used. The ALE method is straightforward; however there is a number of important computational issues, cf. [7], [13], [9], [2].

In this paper, attention is paid step-by-step to the following aspects: second order time discretization and space finite element discretization of the Reynolds Averaged Navier-Stokes equations, GLS stabilization of the FEM, the choice of stabilization

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parameters, discretization of the structural model, numerical realization of the non-linear discrete problem including the coupling of the fluid flow and airfoil motion. The developed sufficiently accurate and robust method is applied to a technically relevant case of flow-induced airfoil vibrations.

**2. Fluid model.** In order to take into account the deformations of the computational domain, we start with a short introduction of Arbitrary Lagrangian-Eulerian (ALE) method, see also [18], [17], [9].

**2.1. Arbitrary Lagrangian-Eulerian method.** Let us assume that there exists a mapping $\Phi = \Phi(\xi, t)$ defined for any $\xi \in \Omega_0$ and $t \in [0, T]$ such that for any $t$ the mapping $\Phi(\cdot, t)$ is a one-to-one transformation of $\Omega_0$ onto $\Omega_t$. Let us denote $\mathcal{A}_t = \Phi(\cdot, t)$. The mapping $\mathcal{A}_t$ is called arbitrary Eulerian-Lagrangian mapping (ALE mapping). We assume that for any $t \in I$ the mapping $\mathcal{A}_t$ denotes $C^1$ continuous bijective mapping from the reference (original) configuration $\Omega_0$ onto the domain $\Omega_t$ at time $t$ (the current configuration).

The time derivative of the ALE mapping $\mathcal{A}_t$ yields the *domain velocity* $\mathbf{w}_D = \mathbf{w}_D(x, t)$ for $x \in \Omega_t$ and $t \in [0, T]$.

$$\mathbf{w}_D(x, t) = \frac{\partial}{\partial t}\Phi(\xi, t), \qquad \mathcal{A}_t(\xi) = x, \quad \xi \in \Omega_0. \tag{2.1}$$

Furthermore, by $D^{\mathcal{A}}/Dt$ the *ALE derivative* is denoted (derivative with respect to a fixed point $\xi$ in the reference domain $\xi \in \Omega_0$). The ALE derivative is related to the time and spatial derivatives as

$$\frac{D^{\mathcal{A}} f}{Dt}(x, t) = \frac{\partial f}{\partial t}(x, t) + \mathbf{w}_D(x, t) \cdot \nabla f(x, t), \tag{2.2}$$

for any $x \in \Omega_t$ and $t \in (0, T)$.

**2.2. Reynolds equations.** We consider the following ALE form of Reynolds equations in $\Omega_t$

$$\begin{aligned} \frac{D^{\mathcal{A}}\mathbf{v}}{Dt} - \nabla \cdot (\nu_{eff}\mathbf{S}(\mathbf{v})) + ((\mathbf{v} - \mathbf{w}_D) \cdot \nabla)\mathbf{v} + \nabla p &= 0, \\ \nabla \cdot \mathbf{v} &= 0, \end{aligned} \tag{2.3}$$

where $\mathbf{S}(\mathbf{v}) = \left(\nabla \mathbf{v} + (\nabla \mathbf{v})^T\right)$, $\nu_{eff} = (\nu + \nu_T)$, $\mathbf{v}$ denotes the vector of mean part of the velocity, $p$ denotes the mean part of the kinematic pressure, $\nu$ denotes the kinematic viscosity, and $\nu_T$ denotes the turbulent viscosity.

The system is equipped with the boundary conditions prescribed on the mutually disjoint parts of the boundary $\partial\Omega_t$:

$$\begin{aligned} &\text{a) } \mathbf{v}(x, t) = \mathbf{v}_D(x), && x \in \Gamma_D, \\ &\text{b) } \mathbf{v}(x, t) = \mathbf{w}_D(x, t), && x \in \Gamma_{Wt}, \\ &\text{c) } -\nu_{eff}\mathbf{S}(\mathbf{v}) \cdot \mathbf{n} + p\,\mathbf{n} = 0, && x \in \Gamma_O. \end{aligned} \tag{2.4}$$

Finally, we prescribe the initial condition

$$\mathbf{v}(x, 0) = \mathbf{v}_0(x), \qquad x \in \Omega_0.$$

**2.3. Time discretization.** We consider a partition $0 = t_0 < t_1 < \cdots < T,\ t_k = k\tau$, with a time step $\tau > 0$, of the time interval $[0, T]$ and approximate the solution $\mathbf{v}(\cdot, t_n)$ and $p(\cdot, t_n)$ (defined in $\Omega_{t_n}$) at time $t_n$ by $\mathbf{v}^n$ and $p^n$, respectively. For the time discretization we employ a second-order two-step scheme using the computed approximate solution $\mathbf{v}^{n-1}$ in $\Omega_{t_{n-1}}$ and $\mathbf{v}^n$ in $\Omega_{t_n}$ for the calculation of $\mathbf{v}^{n+1}$ in the domain $\Omega_{t_{n+1}} = \Omega_{n+1}$.

We define for a fixed time $t = t_{n+1}$ the function spaces $\mathcal{W}, \mathcal{X}$ by

$$\mathcal{W} = \mathbf{H}^1(\Omega_{t_{n+1}}), \qquad \mathcal{X} = \left\{\mathbf{z} \in \mathcal{W} \,:\, \mathbf{z} = 0 \text{ on } \Gamma_D \cup \Gamma_{W t_{n+1}}\right\},$$

and space $\mathcal{Q} = L^2(\Omega_{t_{n+1}})$. Furthermore, the ALE velocity $\mathbf{w}_D(t_{n+1})$ is approximated by $\mathbf{w}_D^{n+1}$ and we set $\widehat{\mathbf{v}}^i = \mathbf{v}^i \circ \mathcal{A}_{t_i} \circ \mathcal{A}_{t_{n+1}}^{-1}$. The vector-valued functions $\widehat{\mathbf{v}}^i$ are defined in the domain $\Omega_{t_{n+1}}$.

The second-order two-step ALE time discretization on each time level $t_{n+1}$ yields the problem of finding unknown functions $\mathbf{v}^{n+1} : \Omega_{t_{n+1}} \to R^2$ and $p^{n+1} : \Omega_{t_{n+1}} \to R$ satisfying the equations

$$\frac{3\mathbf{v}^{n+1} - 4\widehat{\mathbf{v}}^n + \widehat{\mathbf{v}}^{n-1}}{2\tau} + \left(\overline{\mathbf{w}}^{n+1} \cdot \nabla\right) \mathbf{v}^{n+1} - \nabla \cdot (\nu_{eff} \mathbf{S}(\mathbf{v})) + \nabla p^{n+1} = 0, \quad (2.5)$$
$$\operatorname{div} \mathbf{v}^{n+1} = 0,$$

in $\Omega_{t_{n+1}}$, $\overline{\mathbf{w}}^{n+1} = \mathbf{v}^{n+1} - \mathbf{w}_D^{n+1}$, and the boundary conditions (2.4 a,b). The problem (2.5) is then weakly formulated. For the weak formulation we shall make use of the forms

$$a(U^*; U, V) = \left(\frac{3\mathbf{v}}{2\tau}, \mathbf{z}\right)_{\Omega_{n+1}} + \int_{\Omega_{n+1}} (\overline{\mathbf{w}} \cdot \nabla \mathbf{v}) \cdot \mathbf{z} dx$$
$$+ (\nu_{eff} \mathbf{S}(\mathbf{v}), \nabla \mathbf{z})_{\Omega_{n+1}} - (p, \nabla \cdot \mathbf{z})_{\Omega_{n+1}} + (\nabla \cdot \mathbf{v}, q)_{\Omega_{n+1}}$$
$$f(V) = \int_\Omega \frac{4\widehat{\mathbf{v}}^n - \widehat{\mathbf{v}}^{n-1}}{2\tau} \cdot \mathbf{z} dx - \int_{\Gamma_O} p_{\mathrm{ref}} \mathbf{z} \cdot \mathbf{n} \, \mathrm{dS}, \quad (2.6)$$

where $U = (\mathbf{v}, p)$, $V = (\mathbf{z}, q)$, $U^* = (\mathbf{v}^*, p)$, and where $\Omega = \Omega_{n+1}$, $\overline{\mathbf{w}} = \mathbf{v}^* - \mathbf{w}_D^{n+1}$.

PROBLEM 2.1 (Weak formulation of Navier-Stokes in ALE form). *Find* $U = (\mathbf{v}, p)$ *such that*

$$a(U; U, V) = f(V), \quad (2.7)$$

*holds for all* $V = (\mathbf{z}, q) \in \mathcal{X} \times \mathcal{Q}$, *and* $\mathbf{v}$ *satisfies conditions (2.4 a,b).*

**2.4. Stabilized finite element method.** In order to apply the Galerkin FEM, we approximate the spaces $\mathcal{W}$, $\mathcal{X}$, $\mathcal{Q}$ from the weak formulation by finite dimensional subspaces $\mathcal{W}_\triangle$, $\mathcal{X}_\triangle$, $\mathcal{Q}_\triangle$, $\triangle \in (0, \triangle_0)$, $\triangle_0 > 0$, $\mathcal{X}_\triangle = \{\mathbf{v}_\triangle \in \mathcal{W}_\triangle; \mathbf{v}_\triangle|_{\Gamma_D \cap \Gamma_{Wt}} = 0\}$.

The couple $(\mathcal{X}_\triangle, \mathcal{Q}_\triangle)$ of the finite element spaces should either satisfy the Babuška–Brezzi (BB) condition, cf. [11], [12] or [19], or the BB condition can be violated provided additional stabilization is applied, cf. [15]. In practical computations we assume that the domain $\Omega_{n+1}$ is a polygonal approximation of the region occupied by the fluid at time $t_{n+1}$. The spaces $W_\triangle$, $\mathcal{X}_\triangle$, $\mathcal{Q}_\triangle$ are defined over a triangulation $\mathcal{T}_\triangle$ of the domain $\Omega_{n+1}$, formed by a finite number of closed triangles $K \in \mathcal{T}_\triangle$. We use the standard assumptions on the system of triangulation, cf. [4]. Here $\triangle$ denotes the size of the mesh $\mathcal{T}_\triangle$. In this paper the non-conforming equal order finite elements are used, i.e for $k=1$ or $k=2$ we define

$$\begin{aligned} \mathcal{H}_\triangle &= \{v \in C(\overline{\Omega_{n+1}}); v|_K \in P_k(K) \text{ for each } K \in \mathcal{T}_\triangle\}, \\ \mathcal{W}_\triangle &= [\mathcal{H}_\triangle]^d, \qquad \mathcal{X}_\triangle = \mathcal{W}_\triangle \cap \mathcal{X}, \\ \mathcal{Q}_\triangle &= \{v \in C(\overline{\Omega_{n+1}}); v|_K \in P_k(K) \text{ for each } K \in \mathcal{T}_\triangle\}. \end{aligned} \tag{2.8}$$

As mentioned above the Galerkin approximation of the weak formulation suffer from two sources of instabilities. One instability is caused by the incompatibility of the pressure and velocity pair of finite elements. It can be overcome by the use of pressure stabilizing terms. Further, the dominating convection requires to introduce some stabilization of the finite element scheme, as, e.g. upwinding or streamline-diffusion method. In order to overcome both difficulties, modified Galerkin Least Squares method is applied, cf. [10]. We start with the definition of the local element rezidual terms $\mathcal{R}_K^a$ and $\mathcal{R}_K^f$ defined by

$$\mathcal{R}_K^a(\overline{\mathbf{w}}; \mathbf{v}, p) = \frac{3\mathbf{v}}{2\Delta t} - \nabla \cdot \left((\nu + \nu_T)\left(\nabla \mathbf{v} + (\nabla \mathbf{v})^T\right)\right) + (\overline{\mathbf{w}} \cdot \nabla)\,\mathbf{v} + \nabla p \tag{2.9}$$

and

$$\mathcal{R}_K^f(\hat{\mathbf{v}}_n, \hat{\mathbf{v}}_{n-1}) = \frac{1}{2\Delta t}(4\hat{\mathbf{v}}_n - \hat{\mathbf{v}}_{n-1}). \tag{2.10}$$

The GLS stabilizing terms are then defined

$$\begin{aligned} \mathcal{L}_\triangle(U_\triangle^*; U_\triangle, V_\triangle) &= \sum_{K \in \mathcal{T}_\triangle} \delta_K \Big(\mathcal{R}_K^a(\overline{\mathbf{w}}^{n+1}; \mathbf{v}, p), \left(\overline{\mathbf{w}}^{n+1} \cdot \nabla\right)\mathbf{z} + \nabla q\Big)_K, \qquad (2.11) \\ \mathcal{F}_\triangle(V_\triangle) &= \sum_{K \in \mathcal{T}_\triangle} \delta_K \Big(\mathcal{R}_K^f(\hat{\mathbf{v}}_n, \hat{\mathbf{v}}_{n-1}), \left(\overline{\mathbf{w}}^{n+1} \cdot \nabla\right)\mathbf{z} + \nabla q\Big)_K, \end{aligned}$$

where the local element rezidual terms $\mathcal{R}_K^a(\,\cdot\,;\,\cdot,\,\cdot)$ and $\mathcal{R}_K^f(\cdot,\cdot)$ are defined by equations (2.9) and (2.10), and where the function $\overline{\mathbf{w}}^{n+1} = \mathbf{v}^* - \mathbf{w}_D^{n+1}$.

Further, the div-div stabilization is introduced

$$\mathcal{P}_\triangle(U_\triangle, V_\triangle) = \sum_{K \in \mathcal{T}_\triangle} \tau_K (\nabla \cdot \mathbf{v}, \nabla \cdot \mathbf{z})_K, \tag{2.12}$$

$$U = (\mathbf{v}, p), \quad V = (\mathbf{z}, q).$$

The following choice of parameters $\tau_K, \delta_K$ is used

$$\tau_K = \nu_K \left(1 + Re^{loc} + \frac{h_K^2}{\nu_K\, \Delta t}\right), \qquad \delta_K = \frac{h_K^2}{\tau_K},$$

where $\nu_K = |\nu + \nu_T|_{0,2,K}$, $h_K$ denotes the local element size and the local Reynolds number $Re^{loc}$ is defined as

$$Re^{loc} = \frac{h_K \|\mathbf{v}\|_K}{2\nu_K}.$$

The *stabilized GLS scheme* then reads: Find $U_\triangle = (\mathbf{v}, p) \in \mathcal{W}_\triangle \times \mathcal{Q}_\triangle$ such that $\mathbf{v}$ satisfies approximately the Dirichlet boundary conditions (2.4 a,b) and the equation

$$\begin{aligned} a(U_\triangle; U_\triangle, V_\triangle) + \mathcal{L}(U_\triangle; U_\triangle, V_\triangle) + \mathcal{P}_\triangle(U_\triangle, V_\triangle) \\ = f(V_\triangle) + \mathcal{F}(V_\triangle), \end{aligned} \tag{2.13}$$

holds for all $V_\triangle = (\mathbf{z}, q) \in \mathcal{X}_\triangle \times \mathcal{Q}_\triangle$.

**3. Spalart-Allmaras turbulence model.** The system of equations (2.3) is coupled with the nonlinear partial differential equation for an additional quantity $\widetilde{\nu}$. The one equation turbulence model reads

$$\frac{\partial \widetilde{\nu}}{\partial t} + \mathbf{v} \cdot \nabla \widetilde{\nu} = \frac{1}{\beta} \left[ \sum_{i=1}^{2} \frac{\partial}{\partial x_i} \left( (\nu + \widetilde{\nu}) \frac{\partial \widetilde{\nu}}{\partial x_i} \right) + c_{b_2} \left( \nabla \widetilde{\nu} \right)^2 \right] + G(\widetilde{\nu}) - Y(\widetilde{\nu}), \qquad (3.1)$$

where the functions $G(\widetilde{\nu})$ and $Y(\widetilde{\nu})$ are functions of the tensor $(\omega_{ij})_{ij}$ of rotation of the mean velocity and of the wall distance $y$. Here, the components of the rotation tensor are defined by $\omega_{ij} = \frac{1}{2} \left( \frac{\partial v_j}{\partial x_j} - \frac{\partial v_j}{\partial x_i} \right)$. The turbulent viscosity $\nu_T$ is defined by

$$\nu_T = \widetilde{\nu} \frac{\chi^3}{\chi^3 + c_v^3}, \qquad \chi = \frac{\widetilde{\nu}}{\nu}. \qquad (3.2)$$

Furthermore we use the following relations (see also [20])

$$G(\widetilde{\nu}) = c_{b_1} \widetilde{S} \widetilde{\nu}, \qquad Y(\widetilde{\nu}) = c_{w_1} \frac{\widetilde{\nu}^2}{y^2} \left( \frac{1 + c_{w_3}^6}{1 + c_{w_3}^6 / g^6} \right)^{\frac{1}{6}}, \qquad \widetilde{S} = \left( S + \frac{\widetilde{\nu}}{\kappa^2 y^2} f_{v_2} \right),$$

$$f_{v_2} = 1 - \frac{\chi}{1 + \chi f_{v_1}}, \qquad g = r + c_{w_2}(r^6 - r), \qquad r = \frac{\widetilde{\nu}}{\widetilde{S} \kappa^2 y^2}, \qquad S = \sqrt{2 \sum_{i,j} \omega_{ij}^2},$$

where $y$ denotes the distance from a wall. The following choice of constants is used $c_{b_1} = 0.1355$, $c_{b_2} = 0.622$, $\beta = \frac{2}{3}$, $c_v = 7.1$, $c_{w_2} = 0.3$, $c_{w_3} = 2.0$, $\kappa = 0.41$, $c_{w_1} = c_{b_1}/\kappa^2 + (1 + c_{b_2})/\beta$.

**3.1. Time discretization.** The equation (3.1) is time discretized with the aid of $\theta$-stepping scheme, cf. [14]. We choose the parameter $\theta \in \langle 0, 1 \rangle$ and at every time step $t_n$ we approximate $\widetilde{\nu}(t_n) \approx \widetilde{\nu}^{(n)}$. We define

$$\widetilde{\nu}^{(n+\theta)} = (1 - \theta) \widetilde{\nu}^{(n)} + \theta \widetilde{\nu}^{(n+1)}$$

and for every $n = 0, 1, \ldots$ solve the nonlinear equation of Spalart-Allmaras turbulence model coupled with the Reynolds equations (2.3), i.e.

$$\frac{\widetilde{\nu}^{(n+\theta)} - \widetilde{\nu}^{(n)}}{\theta \Delta t} + (\mathbf{v} \cdot \nabla) \widetilde{\nu}^{(n+\theta)} = \frac{1}{\beta} \left[ \nabla \cdot \left( (\nu + \widetilde{\nu}^{(n+\theta)}) \nabla \widetilde{\nu}^{(n+\theta)} \right) + c_{b_2} \left( \nabla \widetilde{\nu}^{(n+\theta)} \right)^2 \right]$$
$$+ c_{b_1} \tilde{S} \widetilde{\nu}^{(n+\theta)} - c_{w_1} \frac{\left( \widetilde{\nu}^{(n+\theta)} \right)^2}{y^2} \left( \frac{1 + c_{w_3}^6}{1 + c_{w_3}^6 / g^6} \right)^{\frac{1}{6}}. \qquad (3.3)$$

Once $\widetilde{\nu}^{(n+\theta)}$ has been computed, the value $\widetilde{\nu}^{(n+1)}$ is obtained by

$$\widetilde{\nu}^{(n+1)} = \frac{1}{\theta} \widetilde{\nu}^{(n+\theta)} + \left( 1 - \frac{1}{\theta} \right) \widetilde{\nu}^{(n)}.$$

**3.2. Weak formulation and linearization.** The numerical solution of the Spalart-Allmaras problem is performed with the aid of finite element method. The turbulence model is described by one partial differential equation of the convection-diffusion-reaction character with dominating convection, strong nonlinear behaviour and with abrupt changes of the source functions.

Now, we choose the space $\mathcal{V} = H_0^1(\Omega)$, take a test function $\varphi \in \mathcal{V}$, multiply equation (3.3) by $\varphi$, integrate over $\Omega$ and apply Green's theorem. Thus we get the weak formulation of the problem: Find $\widetilde{\nu} \in \mathcal{V}$ such that $B_{sa}(\widetilde{\nu}, \varphi) = L_{sa}(\varphi)$ for all $\varphi \in \mathcal{V}$, where

$$\begin{aligned} B_{sa}(\widetilde{\nu}, \varphi) &= (\varepsilon \nabla \widetilde{\nu}, \nabla \varphi)_\Omega \\ &\quad + \left( \frac{\widetilde{\nu}}{\theta \Delta t} + (\mathbf{v} \cdot \nabla)\, \widetilde{\nu} + s\, (\widetilde{\nu})^2 - \frac{c_{b_2}}{\beta} (\nabla \widetilde{\nu})^2 - c_{b_1} \tilde{S} \widetilde{\nu}, \varphi \right)_\Omega, \\ L_{sa}(\varphi) &= \left( \frac{\widetilde{\nu}^{(n)}}{\theta \Delta t}, \varphi \right)_\Omega \end{aligned}$$

and we set

$$s = c_{w_1} \frac{1}{y^2} \left( \frac{1 + c_{w_3}^6}{1 + c_{w_3}^6 / g^6} \right)^{\frac{1}{6}}, \qquad \varepsilon = \varepsilon(\widetilde{\nu}) = \frac{\nu + \widetilde{\nu}}{\beta}. \tag{3.4}$$

The weak formulation of Spalart-Allmaras turbulence model is nonlinear and requires application of a linearization procedure. We use the following linearization:

$$s \left( \widetilde{\nu}^{(n+\theta)} \right)^2 \approx s\, (\widetilde{\nu}^n)^2 + 2s\, \widetilde{\nu}^n \, (\widetilde{\nu}^{(n+\theta)} - \widetilde{\nu}^n), \qquad \left( \nabla \widetilde{\nu}^{(n+\theta)} \right)^2 \approx \nabla \widetilde{\nu}^{(n)} \cdot \nabla \widetilde{\nu}^{(n+\theta)}.$$

Thus the linearized problem reads: Find $\widetilde{\nu} \in \mathcal{V}$ such that

$$B(\widetilde{\nu}, \varphi) = L(\varphi), \tag{3.5}$$

for all $\varphi \in \mathcal{V}$, where

$$\begin{aligned} B(\widetilde{\nu}, \varphi) &= \left( \frac{\widetilde{\nu}}{\theta \Delta t} + \left( \mathbf{v} - \frac{c_{b_2}}{\beta} \nabla \widetilde{\nu}^{(n)} \right) \cdot \nabla \widetilde{\nu} + 2s\, \widetilde{\nu}^{(n)}\, \widetilde{\nu}, \varphi \right)_\Omega \\ &\quad + (\varepsilon\, \nabla \widetilde{\nu}, \nabla \varphi)_\Omega, \end{aligned} \tag{3.6}$$

$$L(\varphi) = \left( s\, \widetilde{\nu}^n\, \widetilde{\nu}^n + \frac{\widetilde{\nu}^{(n)}}{\theta \Delta t} + c_{b_1} \tilde{S}\, \widetilde{\nu}^{(n)}, \varphi \right)_\Omega. \tag{3.7}$$

The viscosity parameter $\varepsilon$ is taken as $\varepsilon = \varepsilon(\widetilde{\nu}^n) = \frac{\nu + \widetilde{\nu}^n}{\beta}$.

**3.3. Space discretization of the turbulence model.** In order to aproximate the problem (3.5) the space $\mathcal{V}$ is approximated by the finite element subspace $\mathcal{V}_\Delta \subset \mathcal{V}$

$$\mathcal{V}_\Delta = \left\{ \varphi \in C_0(\overline{\Omega}) \, : \varphi|_K \in P_1(K) \ \forall K \in \mathcal{T}_\Delta \right\},$$

and Galerkin approximations are sought by solution of the problem: Find $\widetilde{\nu}_\Delta \in \mathcal{V}_\Delta$ such that (3.5) holds for any $\varphi_\Delta \in \mathcal{V}_\Delta$.

Due to large mesh Péclet numbers the Galerkin approximations are unstable. In order to obtain admissible solution, the SUPG stabilization is applied:

$$\begin{aligned} B_{SUPG}(\widetilde{\nu}, \varphi) &= B(\widetilde{\nu}, \varphi) + \\ &\quad + \sum_{K \in \mathcal{T}_\Delta} \delta_K \Big( \frac{\widetilde{\nu}}{\theta \Delta t} + \mathbf{b} \cdot \nabla \widetilde{\nu} + 2s\, \widetilde{\nu}^{(n)} \widetilde{\nu} + \nabla \cdot (\varepsilon\, \nabla \widetilde{\nu})\, , (\mathbf{b} \cdot \nabla) \varphi \Big)_K \end{aligned} \tag{3.8}$$

$$L_{SUPG}(\varphi) = L(\varphi) + \sum_{K \in \mathcal{T}_\Delta} \delta_K \Big( s\, \widetilde{\nu}^n\, \widetilde{\nu}^n + \frac{\widetilde{\nu}^{(n)}}{\theta \Delta t} + c_{b_1} \tilde{S}, (\mathbf{b} \cdot \nabla) \varphi \Big)_K, \tag{3.9}$$

where the vector $\mathbf{b}$ is defined locally on every element $K \in \mathcal{T}_\Delta$ as $\mathbf{b} = \left(\mathbf{v} - \frac{c_{b_2}}{\beta}\nabla\widetilde{\nu}^{(n)}\right)$, and the parameters $\delta_K$ are defined by

$$\delta_K = \left(\frac{4|\varepsilon|_{0,K}}{h_K^2} + \frac{2|\mathbf{b}|_{0,K}}{h_K} + |s|_{0,K}\right)^{-1}.$$

Nevertheless, the use of SUPG/GLS stabilization still does not avoid local oscillations near sharp layers, which can lead to patological situations with negative viscosity. In order to solve this problem we shall make use of the discontinuity capturing techniques (or shock capturing techniques). These stabilization techniques introduce additional dissipation in *crosswind* direction, cf. [15], [5]. The nonlinear stabilization problem reads: Find $\widetilde{\nu} \in \mathcal{V}_\Delta$ such that

$$B_{DC}(\widetilde{\nu}, \varphi) = L_{SUPG}(\varphi), \qquad \forall \varphi \in \mathcal{V}_\Delta, \tag{3.10}$$

where

$$B_{DC}(\widetilde{\nu}, \varphi) = B_{SUPG}(\widetilde{\nu}, \varphi) + \sum_{K \in \mathcal{T}_\Delta} \int_K \alpha_K \nabla\widetilde{\nu} \cdot \nabla\varphi dx +$$
$$+ \sum_{K \in \mathcal{T}_\Delta} \int_K \left(\max(\alpha_K - \alpha'_K, 0) - \alpha_K\right) \nabla\widetilde{\nu} \cdot \left(\frac{\mathbf{b} \otimes \mathbf{b}}{\|\mathbf{b}\|^2_{0,\infty,K}}\right) \nabla\varphi dx.$$

Here $\alpha'_K$ is the additional diffusion from SUPG terms

$$\alpha'_K = \delta_K \|\mathbf{b}\|_{0,\infty,K}$$

and $\alpha_K$ is the diffusion of the shock capturing method, see [5], [14]. The additional diffusion is based on the local element reziduals

$$rez(\widetilde{\nu}) = \frac{\widetilde{\nu}}{\theta\Delta t} + \mathbf{b} \cdot \nabla\widetilde{\nu} + 2s\,\widetilde{\nu}^{(n)}\widetilde{\nu} + \nabla \cdot (\varepsilon\nabla\widetilde{\nu}) - s\,\widetilde{\nu}^n\,\widetilde{\nu}^n - \frac{\widetilde{\nu}^{(n)}}{\theta\Delta t} - c_{b_1}\tilde{S}.$$

We set

$$\alpha_K = \begin{cases} \frac{1}{2}A_K h_K \frac{\|rez(\widetilde{\nu})\|_{0,2,K}}{\|\nabla\widetilde{\nu}\|_{0,2,K}} & \text{if } \|\nabla\widetilde{\nu}\|_{0,2,K} \neq 0, \\ 0 & \text{elsewhere,} \end{cases}$$

where $A_K = \max\left(0, 0.7 - \frac{2\varepsilon}{\|\mathbf{a}_1\|_{0,2,K} h_K}\right)$, $\mathbf{a}_1 = \frac{rez(\widetilde{\nu})}{\|\nabla\widetilde{\nu}\|^2_{0,2,K}}\nabla\widetilde{\nu}$, and $h_K$ is the characteristic length of the element $K$

**4. Structure model.** Here, a solid flexibly supported airfoil is considered. The airfoil can be vertically displaced and rotated. Figure 4.1 shows the elastic support of the airfoil on translational and rotational springs. The pressure and viscous forces acting on the vibrating airfoil immersed in fluid result in the lift force $L(t)$ and the torsional moment $M(t)$. The governing nonlinear equations are written in the form (see [6], [8])

$$\begin{aligned} m\ddot{h} + S_\alpha\,\ddot{\alpha}\,\cos\alpha - S_\alpha\dot{\alpha}^2\sin\alpha + d_{hh}\dot{h} + k_{hh}h &= -L(t), \\ S_\alpha\ddot{h}\cos\alpha + I_\alpha\ddot{\alpha} + d_{\alpha\alpha}\dot{\alpha} + k_{\alpha\alpha}\alpha &= M(t), \end{aligned} \tag{4.1}$$

where $k_{hh}$ and $k_{\alpha\alpha}$ are the bending stiffness and torsional stiffness, respectively, and $m$ is the mass of the airfoil, $S_\alpha$ is the static moment around the elastic axis EA, $I_\alpha$ is the inertia moment around the elastic axis EA.

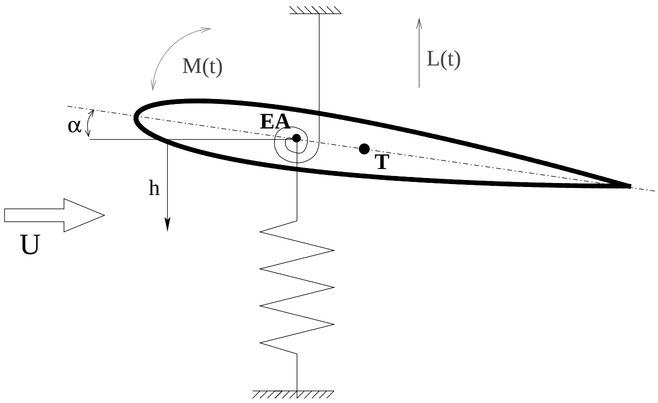


FIG. 4.1. *The elastic support of the airfoil on translational and rotational springs.*

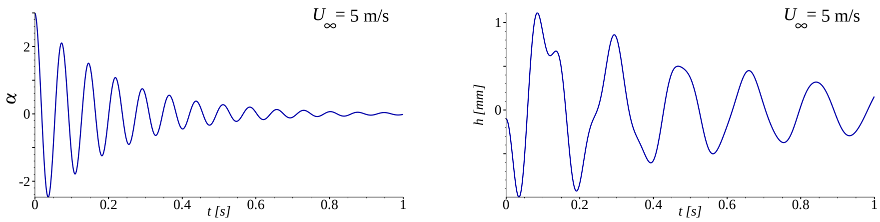


FIG. 5.1. *The aeroelastic response $(h, \alpha)$ for the far field velocity $U_\infty = 5m/s$.*

**5. Numerical results.** For the aeroelastic simulations we compare the presented approach to the results of the method published previously in [8], where the laminar incompressible Navier-Stokes equations were employed and discretized by the conforming finite element method (Taylor-Hood family of finite elements) together with SUPG and grad-div stabilization. We use the modified parameter values taken from [3], where the critical velocity determined by NASTRAN was 30.4 m/s. We present comparison of laminar and turbulence results for the far field velocity in the range $U_\infty = 10 - 32m/s$. The parameters of the structural model was set as

$$m = 0.086622 \text{ kg}, \quad S_\alpha = 0.000779673 \text{ kg m}, \quad I_\alpha = 0.000487291 \text{ kg m}^2,$$
$$k_{hh} = 105.109 \text{ N m}^{-1}, \quad k_{\alpha\alpha} = 3.695582 \text{ N m rad}^{-1}, \quad l = 0.05 \text{ m}, \quad c = 0.3 \text{ m}.$$

The elastic axis is located at 40% of the airfoil, $\rho = 1.225$ kg m$^{-3}$, $\nu = 1.5 \cdot 10^{-5}$ m s$^{-2}$. The numerical computations were performed for airfoils NACA 0012. Figures 5.1, 5.2, 5.3 shows the comparison of the aeroelastic response for different values of the inlet velocity.

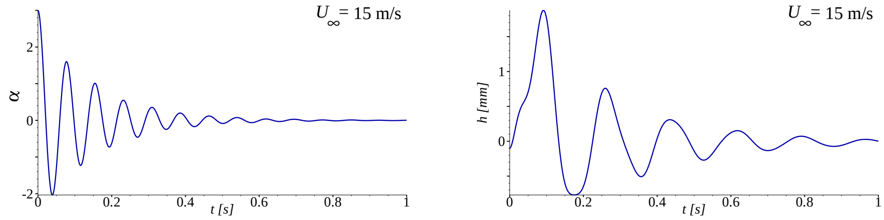


FIG. 5.2. *The aeroelastic response* $(h, \alpha)$ *for the far field velocity* $U_\infty = 15m/s$.

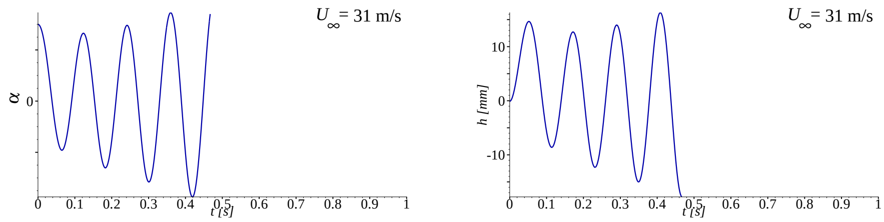


FIG. 5.3. *The aeroelastic response* $(h, \alpha)$ *for the far field velocity* $U_\infty = 31m/s$.

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