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SECOND ORDER TIME DISCONTINUOUS GALERKIN METHOD FOR NONLINEAR CONVECTION-DIFFUSION PROBLEMS*

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Abstract. We deal with a numerical solution of a scalar nonstationary convection-diffusion equation with a nonlinear convection and a linear diffusion. We carry out the space semi-discretization with the aid of the nonsymmetric interior penalty Galerkin (NIPG) method and the time discretization by the time discontinuous Galerkin method linearized by extrapolation from previous time interval. The resulting scheme is unconditionally stable, has a high order of accuracy with respect to space and time coordinates and requires only solutions of linear algebraic problems at each time step. We derive a priori error estimate in the L^2-norm.

Key words. a priori error estimate, time discontinuous Galerkin method

1. Introduction. We numerically solve a nonstationary nonlinear convection-diffusion equation, which represents a model problem for the system of the compressible Navier-Stokes equations. The class of *discontinuous Galerkin* (DG) methods seems to be one of the most promising candidates to construct high order accurate schemes for solving convection-diffusion problems. For a survey about DG methods, see [1] or [3]. An analysis of DG methods was presented in many papers, see, e.g. [6], [8], [11], [10].

In [8] we carried out the space semi-discretization of the scalar convection-diffusion equation with the aid of the *discontinuous Galerkin finite element* method and derived a priori error estimates. Within this contribution, we deal with the time discretization of the resulting system of ordinary differential equations. In contrary to [9], where we used the so-called backward difference formulae (BDF) approach, here we employ the time discontinuous Galerkin method, see [15]. Since this scheme is implicit and we would like to avoid solving strongly nonlinear problem at each time level, we recomend linearization in a similar way as it was done before for BDF in [9]. We present a formulation of the arbitrary order linearized time discontinuous Galerkin scheme and derive a priori error estimate for the second order scheme.

2. Continuous problem. Let $\Omega \subset R^d$ ($d = 2$ or 3) be a bounded polyhedral domain and $T > 0$. We set $Q_T = \Omega \times (0, T)$. By $\overline{\Omega}$ and $\partial\Omega$ we denote the closure and boundary of Ω, respectively. Let us consider the following *initial-boundary value problem*: Find $u : Q_T \to R$ such that

$$\frac{\partial u}{\partial t} + \nabla \cdot \vec{f}(u) = \Delta u + g \quad \text{in } Q_T, \tag{1}$$

$$u\big|_{\partial\Omega\times(0,T)} = u_D, \tag{2}$$

$$u(x, 0) = u^0(x), \quad x \in \Omega. \tag{3}$$

In (1) – (3), $\vec{f} = (f_1, \dots, f_d)$, $f_s \in C^2(R)$, $f_s(0) = 0$, $s = 1, \dots, d$ represents convective terms, $g \in C([0, T]; L^2(\Omega))$ represents volume sources. The Dirich-

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let boundary condition is given over $\partial\Omega$ by u_D, which is the trace of some $u^* \in C([0,T];H^1(\Omega)) \cap L^\infty(Q_T)$ and $u^0 \in L^2(\Omega)$ is an initial condition. We use the standard notation for Lebesgue, Sobolev and Bochner function spaces (see, e. g. [12]).

In order to introduce the concept of a weak solution, we define

$$\begin{aligned}(u,w) &= \int_\Omega uw\,\mathrm{d}x, \quad u,\, w \in L^2(\Omega),\\ a(u,w) &= \int_\Omega \nabla u\cdot\nabla w\,\mathrm{d}x, \quad u,\, w \in H^1(\Omega),\\ b(u,w) &= \int_\Omega \nabla\cdot\vec{f}(u)\,w\,\mathrm{d}x, \quad u\in H^1(\Omega)\cap L^\infty(\Omega),\ w\in L^2(\Omega),\end{aligned}$$

Definition 2.1. *We say that a function u is a* weak solution *of (1) – (3) if the following conditions are satisfied*

$$\begin{aligned}&\textit{a)}\quad u-u^*\in L^2(0,T;H_0^1(\Omega)), \quad u\in L^\infty(Q_T),\\ &\textit{b)}\quad \frac{d}{dt}(u(t),w)+b(u(t),w)+a(u(t),w)=(g(t),w)\\ &\qquad\qquad \textit{for all } w\in H_0^1(\Omega) \textit{ in the sense of distributions on } (0,T),\\ &\textit{c)}\quad u(0)=u^0 \quad \textit{in } \Omega.\end{aligned} \tag{4}$$

By $u(t)$ we denote the function on Ω such that $u(t)(x)=u(x,t)$, $x\in\Omega$.

With the aid of techniques from [13] and [14], it is possible to prove that there exists a unique weak solution. We shall assume that the weak solution u is sufficiently regular, namely,

$$u\in W^{1,\infty}(0,T;H^{p+1}(\Omega))\cap W^{2,\infty}(0,T;H^1(\Omega)), \tag{5}$$

an integer $p\ge 1$ will denote a given degree of polynomial approximations. Such a solution satisfies problem (1) – (3) pointwise.

3. Space semi-discretization. We discretize problem (4) in space with the aid of the *discontinuous Galerkin finite element method* with *nonsymmetric treatment of stabilization terms* and *interior and boundary penalties*. This approach is called the NIPG variant of the DGFE method, see [1]. We derived the space discretization of (1) – (3) by the NIPG variant of DGFE method in [8] hence here we present only the final expressions.

3.1. Triangulation. Let $\mathcal{T}_h$ $(h>0)$ be a partition of the domain Ω into a finite number of closed d-dimensional mutually disjoint simplices K i.e., $\overline{\Omega}=\bigcup_{K\in\mathcal{T}_h}K$. By ∂K we denote the boundary of element $K\in\mathcal{T}_h$ and set $h_K=\mathrm{diam}(K)$, $h=\max_{K\in\mathcal{T}_h}h_K$. We set Γ the faces of $\mathcal{T}_h$ $(\Gamma=\bigcup_{K\in\mathcal{T}_h}\partial K)$. By ρ_K we denote the radius of the largest d-dimensional ball inscribed into K and by $|K|$ we denote the d-dimensional Lebesgue measure of K.

Furthermore, we use the following notation: $\boldsymbol{n}=(n_1,\dots,n_d)$ – a normal vector to Γ which is well defined almost everywhere (on $\partial\Omega$ we use outer normal, inside of Ω we use one (arbitrary but fixed) direction at every point of Γ).

3.2. Broken Sobolev spaces. We define the so-called *broken Sobolev space* in the following way

$$H^s(\Omega,\mathcal{T}_h)=\{w;w|_K\in H^s(K)\ \forall\, K\in\mathcal{T}_h\} \tag{6}$$

and define there the norm

$$\|w\|_{H^s(\Omega,\mathcal{T}_h)} = \left(\sum_{K\in\mathcal{T}_h} \|w\|^2_{H^s(K)} \right)^{1/2} \tag{7}$$

and the seminorm

$$|w|_{H^s(\Omega,\mathcal{T}_h)} = \left(\sum_{K\in\mathcal{T}_h} |w|^2_{H^s(K)} \right)^{1/2}. \tag{8}$$

For $w \in H^1(\Omega, \mathcal{T}_h)$, we introduce the following notation on $\Gamma \setminus \partial\Omega$:

$$\begin{aligned} w_R(x) &= \lim_{\delta\to 0+} w(x+\delta\boldsymbol{n}) \\ w_L(x) &= \lim_{\delta\to 0-} w(x+\delta\boldsymbol{n}) \\ \langle w\rangle &= \frac{1}{2}\left(w_R + w_L\right), \\ [w] &= w_L - w_R \end{aligned} \tag{9}$$

and on $\partial\Omega$ we put

$$\begin{aligned} w_L(x) &= \lim_{\delta\to 0-} w(x+\delta\boldsymbol{n}) \\ \langle w\rangle &= w_L, \\ [w] &= w_L \end{aligned} \tag{10}$$

3.3. Space discretization. For $u, w \in H^2(\Omega, \mathcal{T}_h)$ we set

$$\begin{aligned} A_h(u,w) = \sum_{K\in\mathcal{T}_h} \int_K \nabla u\cdot\nabla w \,\mathrm{d}x - \int_\Gamma \Big(\langle\nabla u\rangle\cdot\boldsymbol{n}[w] - \langle\nabla w\rangle\cdot\boldsymbol{n}[u]\Big)\,\mathrm{d}S \\ + \int_\Gamma \sigma[u]\,[w]\,\mathrm{d}S \end{aligned} \tag{11}$$

$$\begin{aligned} b_h(u,w) = \int_{\Gamma\setminus\partial\Omega} H(u_L,u_R,\boldsymbol{n})\,[w]\,\mathrm{d}S + \int_{\partial\Omega} H(u_L,u_D,\boldsymbol{n})\,w_L\,\mathrm{d}S \\ - \sum_{K\in\mathcal{T}_h}\int_K \vec{f}(u)\cdot\nabla w\,\mathrm{d}x, \quad u,\, w\in H^1(\Omega,\mathcal{T}_h), \quad u\in L^\infty(\Omega) \end{aligned} \tag{12}$$

$$\ell_h(w)\,(t) = (g(t),w) - \int_{\partial\Omega} \left(\nabla w\cdot\boldsymbol{n}\,u_D(t) - \sigma u_D(t)\,w\right)\,\mathrm{d}S. \tag{13}$$

The penalty parameter function σ in (11) and (13) along the face $e \subset \Gamma$ is defined by $\sigma|_e = 1/(h_K + h_{\tilde{K}})$, $e = K \cap \tilde{K}$. The function $H(\cdot,\cdot,\cdot)$ in the face integrals in (12) is called the *numerical flux*, well-known from the finite volume method and it approximates the terms $\vec{f}(u)\cdot\boldsymbol{n}$. Now we define the space of discontinuous piecewise polynomial functions

$$S_h = S^{p,-1}(\Omega,\mathcal{T}_h) = \{w; w|_K \in P_p(K)\ \forall\, K\in\mathcal{T}_h\}, \tag{14}$$

where $P_p(K)$ denotes the space of all polynomials on K of degree $\le p$, where the integer $p \ge 1$ is a given degree of approximation.

We find that the exact solution of (4) with property (5) satisfies the identity

$$\left(\frac{\partial u}{\partial t}(t), w_h\right) + A_h(u(t), w_h) + b_h(u(t), w_h) = \ell_h(w_h)\,(t) \tag{15}$$

for all $w_h \in S_h$ and all $t \in (0, T)$.

The (semi)-discrete problem (15) represents a system of ordinary differential equations (ODEs) which is solved by a suitable solver in the next section.

4. Time discretization. Since problem (15) is stiff, it is necessary to solve it with a method having a large stability domain. In [9] we employ the well known backward difference formulae (BDF). Within this contribution, we present a new approach based on the time discontinuous Galerkin method. Since this method is implicit, we introduce linearization from previous time interval in such a way that linear part of the problem is treated implicitly and the nonlinear one explicitly. Therefore, the resulting semi-implicit scheme leads to a highly stable method which requires only a solution of the linear algebraic problem at each time step.

We consider a partition $0 = t_0 < t_1 < \ldots < t_r = T$ consisting of time intervals $I_m = (t_{m-1}, t_m],\ m = 1, \ldots, r$ of the length $|I_m| = \tau_m$ and $\tau = \max_{m=1,\ldots,r}\tau_m$. We set the space–time fully discrete space

$$S_{h,\tau} \equiv \{w \in L^2(Q_T) : w|_{I_m} = \sum_{s=0}^{q} t^s z_s,\ z_s \in S_h\}. \tag{16}$$

For simplicity we set

$$w^m_\pm = \lim_{\delta \to 0\pm} w(t_m + \delta), \tag{17}$$

$$\{w\}_m = w^m_+ - w^m_-. \tag{18}$$

To avoid the nonlinearity in our problem we define $\hat{w}|_{I_m}$ as prolongation of w from previous time interval I_{m-1}, it means there exists $z \in P_q(I_{m-1} \cup I_m)$ such that

$$\begin{aligned} z(t) &= w|_{I_{m-1}}(t), \quad \forall t \in I_{m-1}, \\ \hat{w}|_{I_m}(t) &= z(t), \quad \forall t \in I_m. \end{aligned} \tag{19}$$

Now we are able to define fully discrete solution $U \in S_{h,\tau}$.

Definition 4.1. *We define the* approximate solution *of problem from Definition (2.1) a)–c) as functions $U \in S_{h,\tau}$ satisfying the conditions*

$$\begin{aligned} &a)\int_{I_m} (U', w) + A_h(U, w) + b_h(\hat{U}, w)dt + (\{U\}_{m-1}, w^{m-1}_+) = \int_{I_m} \ell_h(w)dt \\ &\qquad\qquad\qquad\qquad\qquad\qquad \forall\, w \in S_{h,\tau},\ m = 2, \ldots, r, \\ &b)(U^0_-, w) = (u^0, w) \quad \forall\, w \in S_h. \end{aligned} \tag{20}$$

We should remark on some strange character of this scheme. This scheme is one–step method as it is usual for time discontinuous Galerkin, but on the other hand we need to use the information from the whole previous time interval for construction of the prolongation, so this method is not self–started.

5. Error estimates. Our goal is to analyse the error estimates of the approximate solution U obtained by the method (20) for the piecewise linear approximation in time ($q = 1$). In the next we set

$$\|\!|w|\!\|^2 = A_h(w,w)\ \forall w \in H^2(\Omega, \mathcal{T}_h). \tag{21}$$

5.1. Parabolic projection. For the next purpose we need projection

$$\pi : L^2(Q_T) \to S_{h,\tau}$$

well known for the time discontinuous Galerkin. This projection satisfies

$$\begin{aligned} &\int_{I_m} (\pi v - v, t^j w) dt = 0 \qquad \forall w \in S_h,\ j = 0, \dots, q-1, \\ &\pi v_-^m = \Pi v_-^m, \end{aligned} \tag{22}$$

where Π be the standard $L^2(\Omega) \to S_h$ orthogonal projection. About the projection π it is possible to prove following estimates

Lemma 5.1. *Let $w \in W^{q+1,\infty}(I_m, S_h)$, then*

$$\begin{aligned} &\|w(t) - \pi w(t)\|_{L^2(\Omega)} \le C\tau_m^{q+1}, \quad \forall t \in I_m, \\ &\|\!|w(t) - \pi w(t)|\!\| \le C\tau_m^{q+1}, \quad \forall t \in I_m, \end{aligned} \tag{23}$$

where C depends on time derivatives of w.

Proof. The lemma can be prooved by standard scaling argument. □

Since we have simplicial mesh we obtain standard estimates for projection Π under sufficient regularity of function w

$$\begin{aligned} &\|w - \Pi w\|_{L^2(\Omega)} \le Ch^{p+1} \\ &\|\!|w - \Pi w|\!\| \le Ch^p, \end{aligned} \tag{24}$$

we can estimate projection error of arbitrary function $w \in W^{q+1,\infty}(I_m, H^1(\Omega)) \cap W^{1,\infty}(I_m, H^{p+1}(\Omega))$ by

$$\begin{aligned} &\|w(t) - \pi w(t)\|_{L^2(\Omega)} \le C(h^{p+1} + \tau_m^{q+1}), \quad \forall t \in I_m, \\ &\|\!|w(t) - \pi w(t)|\!\| \le C(h^p + \tau_m^{q+1}), \quad \forall t \in I_m. \end{aligned} \tag{25}$$

5.2. Extrapolation. We set $\hat{w}$ as extrapolation from previous time interval. For $q = 1$ we get

$$\hat{w}(t) = (1 + \frac{t - t_{m-1}}{\tau_{m-1}}) w_-^{m-1} - \frac{t - t_{m-1}}{\tau_{m-1}} w_+^{m-2} \quad \forall t \in I_m. \tag{26}$$

As standard result from interpolation theory with additional assumption

$$\frac{\tau_m}{\tau_{m-1}} \le C \tag{27}$$

and under sufficient regularity $w \in W^{2,\infty}(0, T, H^1(\Omega))$ we get

$$\begin{aligned} &\|w - \hat{w}\|_{L^2(\Omega)} \le C\tau^2, \\ &\|\!|w - \hat{w}|\!\| \le C\tau^2, \end{aligned} \tag{28}$$

where constant C depends on time derivatives of w. This can be prooved by standard scaling argument.

5.3. Properties of the forms A_h and b_h. Here we want to summarize the properties of the forms A_h and b_h.

LEMMA 5.2. *Let $u \in W^{1,\infty}(0,T,H^{p+1}(\Omega))$. Then it holds that*

$$\begin{aligned} |A_h(u(t)-\Pi u(t),w)| &\le Ch^p |\!|\!| w |\!|\!| \quad \forall w \in S_h, \\ |A_h(v,w)| &\le C |\!|\!| v |\!|\!| \, |\!|\!| w |\!|\!| \quad \forall v,w \in S_h. \\ |A_h(u(t)-\pi u(t),w)| &\le C(h^p+\tau^{q+1}) |\!|\!| w |\!|\!| \quad \forall w \in S_h \end{aligned}$$

Proof. The proof can be found in [8] in Lemma 9. □

LEMMA 5.3. *Let $u \in W^{1,\infty}(0,T,H^{p+1}(\Omega))$. Let the numerical fluxes H be Lipschitz continuous, conservative and consistent. Then it holds that*

$$\begin{aligned} &|b_h(v,w)-b_h(\bar v,w)| \le C(\|v-\bar v\|_{L^2(\Omega)} + |\!|\!| v-\bar v |\!|\!|)\, |\!|\!| w |\!|\!| \quad \forall v,\bar v \in L^2(\Omega),\ w \in S_h, \\ &|b_h(u,w)-b_h(\Pi u,w)| \le Ch^{p+1} |\!|\!| w |\!|\!| \quad \forall w \in S_h, \\ &|b_h(v,w)-b_h(\bar v,w)| \le C\|v-\bar v\|_{L^2(\Omega)} \, |\!|\!| w |\!|\!| \quad \forall v,\bar v,w \in S_h, \\ &|b_h(u,w)-b_h(\hat u,w)| \le C\tau^2 |\!|\!| w |\!|\!| \quad w \in S_h. \end{aligned}$$

Proof. The proof can be found in [8] in Lemma 5 and Lemma 6. □

5.4. Main result. In the sequel we use the notation $U-u = U-\pi u+\pi u-u = \xi+\eta$.

THEOREM 5.4. *Let u be the exact solution of problem (4) satisfying (5). Let the mesh be regular ($h_K/\rho_K \le C$), assumption (27) be satisfied and the numerical fluxes H be Lipschitz continuous, conservative and consistent. Let U be the approximate solution defined by (20). Then*

$$\max_{s=0,\dots,r} \|U_-^s - u(t_s)\|^2 \le C(h^{2p}+\tau^4+\|e_-^1\|^2_{L^2(\Omega)}+\|e_-^0\|^2_{L^2(\Omega)})e^{TC} \tag{29}$$

Proof. Since $U_-^s - u(t_s) = \xi_-^s + \eta^s$, in virtue of (25), it is only sufficient to estimate $\|\xi_-^s\|$. Let us integrate (4) over I_m and substract this equation from (20). Then we have

$$\begin{aligned} \int_{I_m} (\xi',w) + A_h(\xi,w)dt + (\{\xi\}_{m-1}, w_+^{m-1}) = -\left(\int_{I_m} (\eta',w)dt + (\{\eta\}_{m-1}, w_+^{m-1})\right) \\ - \int_{I_m} A_h(\eta,w)dt + \int_{I_m} b_h(u,w) - b_h(\hat U,w)dt \quad w \in S_{h,\tau}. \end{aligned} \tag{30}$$

Now let us have a look at all the parts of (30).

$$\begin{aligned} &\int_{I_m} (\eta',w)dt + (\{\eta\}_{m-1}, w_+^{m-1}) \\ &= (\eta_-^m, w_-^m) - (\eta_+^{m-1}, w_+^{m-1}) - \int_{I_m} (\eta, v')dt + (\{\eta\}_{m-1}, w_+^{m-1}) \\ &= (\eta_-^m, w_-^m) - (\eta_-^{m-1}, w_+^{m-1}) - \int_{I_m} (\eta, w')dt \end{aligned} \tag{31}$$

Using properties (22) of π we obtain

$$(\eta_-^m, w_-^m) - (\eta_-^{m-1}, w_+^{m-1}) - \int_{I_m} (\eta, w')dt = 0. \tag{32}$$

Estimating linear form A_h using Lemma 5.2 we obtain

$$-\int_{I_m} A_h(\eta, w)dt \le \frac{1}{4}\int_{I_m} |||w|||^2 dt + \tau_m C(h^{2p} + \tau^4). \tag{33}$$

Now we should estimate nonlinear form b_h.

$$\begin{aligned} |b_h(u,v) - b_h(\hat{U}, w)| &\le |b_h(u,w) - b_h(\hat{u}, w)| \\ +|b_h(\hat{u}, w) - b_h(\Pi\hat{u}, w)| &+ |b_h(\Pi\hat{u}, w) - b_h(\hat{U}, w)| \end{aligned} \tag{34}$$

We estimate the individual terms using Lemma 5.3.

$$\begin{aligned} |b_h(u,w) - b_h(\hat{u}, w)| &\le C\tau^2 |||w||| \\ |b_h(\hat{u}, w) - b_h(\Pi\hat{u}, w)| &\le Ch^{p+1} |||w||| \\ |b_h(\Pi\hat{u}, w) - b_h(\hat{U}, w)| &\le C\|\Pi\hat{u} - \hat{U}\|_{L^2(\Omega)}\ |||w||| \end{aligned} \tag{35}$$

With the aid of the definition of extrapolation of $\hat{u}$ and $\hat{U}$ we get

$$\begin{aligned} &\|\Pi\hat{u} - \hat{U}\|_{L^2(\Omega)} \\ &= \|\left(1 + \frac{t - t_{m-1}}{\tau_{m-1}}\right)(\Pi u^{m-1} - U_-^{m-1}) - \frac{t - t_{m-1}}{\tau_{m-1}}(\Pi u^{m-2} - U_+^{m-2})\|_{L^2(\Omega)} \\ &\le \left(1 + \frac{t - t_{n-1}}{\tau_{n-1}}\right)\|\xi_-^{m-1}\|_{L^2(\Omega)} + \frac{t - t_{n-1}}{\tau_{n-1}}\|\Pi u^{m-2} - U_+^{m-2}\|_{L^2(\Omega)}. \end{aligned} \tag{36}$$

We estimate last norm by (24) and (25)

$$\begin{aligned} &\|\Pi u^{m-2} - U_+^{m-2}\|_{L^2(\Omega)} \\ &\le \|\Pi u^{m-2} - u^{m-2}\|_{L^2(\Omega)} + \|u^{m-2} - \pi u_+^{m-2}\|_{L^2(\Omega)} + \|\pi u_+^{m-2} - U_+^{m-2}\|_{L^2(\Omega)} \\ &\le Ch^{p+1} + C(h^{p+1} + \tau^2) + \|\xi_+^{m-2}\|_{L^2(\Omega)} \\ &\le C(h^{p+1} + \tau^2) + \|\{\xi\}_{m-2}\|_{L^2(\Omega)} + \|\xi_-^{m-2}\|_{L^2(\Omega)}. \end{aligned} \tag{37}$$

When we use assumption (27), we could complete estimate for form b_h

$$\begin{aligned} \int_{I_m} b_h(u,w) - b_h(\hat{U}, w)dt &\le \frac{1}{4}\int_{I_m} |||w|||^2 dt + \tau_m C(h^{2p+2} + \tau^4) \\ &+\tau_m C(\|\xi_-^{m-1}\|^2_{L^2(\Omega)} + \|\{\xi\}_{m-2}\|^2_{L^2(\Omega)} + \|\xi_-^{m-2}\|^2_{L^2(\Omega)}). \end{aligned} \tag{38}$$

When we apply $w = 2\xi$ to the left-hand side of (30), we obtain

$$\begin{aligned} &\int_{I_m} 2(\xi', \xi) + 2|||\xi|||^2 dt + 2(\{\xi\}_{m-1}, \xi_+^{m-1}) \\ &= \|\xi_-^m\|^2_{L^2(\Omega)} - \|\xi_+^{m-1}\|^2_{L^2(\Omega)} + 2\|\xi_+^{m-1}\|^2_{L^2(\Omega)} - 2(\xi_-^{m-1}, \xi_+^{m-1}) + 2\int_{I_m} |||\xi|||^2 dt \\ &= \|\xi_-^m\|^2_{L^2(\Omega)} - \|\xi_-^{m-1}\|^2_{L^2(\Omega)} + \|\{\xi\}_{m-1}\|^2_{L^2(\Omega)} + 2\int_{I_m} |||\xi|||^2 dt. \end{aligned} \tag{39}$$

If we apply all these estimates with $w = 2\xi$ together we gain

$$\begin{aligned} &\|\xi_-^m\|^2_{L^2(\Omega)} - \|\xi_-^{m-1}\|^2_{L^2(\Omega)} + \|\{\xi\}_{m-1}\|^2_{L^2(\Omega)} \\ &\le \tau_m q(h,\tau) + \tau_m C(\|\xi_-^{m-1}\|^2_{L^2(\Omega)} + \|\{\xi\}_{m-2}\|^2_{L^2(\Omega)} + \|\xi_-^{m-2}\|^2_{L^2(\Omega)}), \end{aligned} \tag{40}$$

where $q(h,\tau) = O(h^{2p} + \tau^4)$. Summing over $m = 2, \ldots, k \le r$ and using (27), we gain

$$\begin{aligned}
(41) \qquad & \|\xi_-^k\|^2_{L^2(\Omega)} - \|\xi_-^1\|^2_{L^2(\Omega)} + \|\{\xi\}_{k-1}\|^2_{L^2(\Omega)} \\
&\le Tq(h,\tau) + C\sum_{s=1}^{k-1}\tau_{s+1}\left(\|\xi_-^s\|^2_{L^2(\Omega)} + \|\{\xi\}_{s-1}\|^2_{L^2(\Omega)} + \|\xi_-^{s-1}\|^2_{L^2(\Omega)}\right) \\
&\le Tq(h,\tau) + \tau_2 C\|\xi_-^0\|^2_{L^2(\Omega)} + C\sum_{s=1}^{k-1}(\tau_{s+1}+\tau_{s+2})\|\xi_-^s\|^2_{L^2(\Omega)} + \tau_{s+1}\|\{\xi\}_{s-1}\|^2_{L^2(\Omega)} \\
&\le Tq(h,\tau) + \tau_2 C\|\xi_-^0\|^2_{L^2(\Omega)} + C\sum_{s=1}^{k-1}\tau_{s+1}\|\xi_-^s\|^2_{L^2(\Omega)} + \tau_{s+1}\|\{\xi\}_{s-1}\|^2_{L^2(\Omega)}.
\end{aligned}$$

Applying Gronwall's lemma we obtain

$$\begin{aligned}
(42) \quad \|\xi_-^k\|^2_{L^2(\Omega)} + \|\{\xi\}_{k-1}\|^2_{L^2(\Omega)} &\le \left(C\|\xi_-^0\|^2_{L^2(\Omega)} + \|\xi_-^1\|^2_{L^2(\Omega)} + Tq\right)\prod_{s=2}^{k}(1+\tau_s C) \\
&\le \left(C\|\xi_-^0\|^2_{L^2(\Omega)} + \|\xi_-^1\|^2_{L^2(\Omega)} + Tq\right)e^{TC},
\end{aligned}$$

which proves our theorem. □

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