EXISTENCE AND A PRIORI ESTIMATES FOR SEMILINEAR ELLIPTIC SYSTEMS OF HARDY TYPE

J. PAČUTA

Abstract. We study semilinear elliptic systems of Hardy type on bounded domains. We look for conditions guaranteeing the existence and uniform boundedness of very weak solutions satisfying homogeneous Dirichlet boundary conditions.

1. Introduction

Consider the problem

$$
\begin{cases}
-\Delta u = a(x)|x|^{-\kappa} v^q & x \in \Omega, \\
-\Delta v = b(x)|x|^{-\lambda} u^p & x \in \Omega, \\
u = v = 0, & x \in \partial\Omega,
\end{cases} \tag{1}
$$

where

$$
\begin{cases}
\Omega \text{ is a bounded domain in } \mathbb{R}^n \ (n \geq 2) \text{ of the class } C^{2+\gamma} \\
\text{for some } \gamma \in (0,1),\ 0 \in \partial\Omega, \quad p, q > 0, \quad pq > 1, \\
a, b \in L^\infty(\Omega), \quad a, b \geq 0, \quad a, b \not\equiv 0, \quad \kappa, \lambda \in (0,2).
\end{cases} \tag{2}
$$

Received November 14, 2013.
2010 *Mathematics Subject Classification.* Primary 35J66, 35B45; Secondary 35B65.
Key words and phrases. elliptic systems; very weak solutions; a priori estimates.
The author was supported in part by VEGA grant 1/0711/12.

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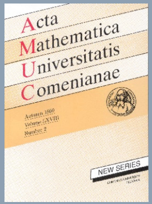

In this paper, we study boundedness and existence of nonnegative very weak solutions of problem (1). We say that (u, v) is a very weak solution of (1) if $u, v \in L^1(\Omega)$, the right-hand sides in (1) belong to the weighted Lebesgue space $L^1(\Omega; \operatorname{dist}(x, \partial\Omega)\,\mathrm{d}x)$ and

$$-\int_\Omega u\Delta\varphi\,\mathrm{d}x = \int_\Omega a(x)|x|^{-\kappa}v^q\varphi\,\mathrm{d}x, \qquad -\int_\Omega v\Delta\varphi\,\mathrm{d}x = \int_\Omega b(x)|x|^{-\lambda}u^p\varphi\,\mathrm{d}x$$

for every $\varphi \in C^2(\overline{\Omega})$, $\varphi = 0$ on $\partial\Omega$.

Problem (1) with $\kappa = \lambda = 0$ has been widely studied. Concerning very weak solutions, necessary and sufficient conditions for their boundedness were found in [3], [11] and [13]. In those papers the existence of very weak solution was studied as well.

Problem (1) with $a = b \equiv 1$, $0 \in \Omega$ and general $\kappa, \lambda \in \mathbb{R}$ has been studied by several authors, who were mainly interested in the existence of classical solutions (if $\max\{\kappa, \lambda\} \le 0$) or solutions of the class $C^2(\Omega \setminus \{0\}) \cap C(\Omega)$ (if $\max\{\kappa, \lambda\} > 0$). If $\max\{\kappa, \lambda\} \ge 2$, then (1) has no positive solution in this class for any domain Ω containing the origin; see [1]. If $\max\{\kappa, \lambda\} < 2$, Ω is a bounded starshaped domain and some additional assumptions are satisfied, then (1) has a positive solution if and only if the following condition is satisfied

$$\frac{n-\kappa}{1+q} + \frac{n-\lambda}{1+p} > n-2; \tag{3}$$

see, e.g., [4], [5], [7], [9] for details. If $\max\{\kappa, \lambda\} < 2$ and $\Omega = \mathbb{R}^n$, $n \ge 3$, then (1) has no positive radial solution if and only if (3) is true. The conjecture is that if (3) holds, (1) has no positive nonradial solution for $\Omega = \mathbb{R}^n$; see [2]. This conjecture has been partially proved in, e.g., [10].

We will assume (2) and we will deal with the problem

$$
(4)\qquad \begin{cases} -\Delta u = a(x)|x|^{-\kappa} v^q + t(u+\varphi_1), & x \in \Omega, \\ -\Delta v = b(x)|x|^{-\lambda} u^p, & x \in \Omega, \\ \quad u = v = 0, & x \in \partial\Omega \end{cases}
$$

if $q \geq 1$, $p > 0$ and with problem

$$
(5)\qquad \begin{cases} -\Delta u = a(x)|x|^{-\kappa} v^q, & x \in \Omega, \\ -\Delta v = b(x)|x|^{-\lambda} u^p + t(v+\varphi_1), & x \in \Omega, \\ \quad u = v = 0, & x \in \partial\Omega \end{cases}
$$

if $q < 1$, $p > 1$. In both cases we will assume $t \geq 0$. The terms $t(u+\varphi_1)$ in (4) or $t(v+\varphi_1)$ in (5) are needed to use the topological degree in the proof of the existence of solutions of (1). Denote

$$
(6)\qquad \alpha := \frac{(2-\lambda)q + 2 - \kappa}{pq - 1}, \qquad \beta := \frac{(2-\kappa)p + 2 - \lambda}{pq - 1}.
$$

We have the following results.

Theorem 1.1. *Assume (2) and* $\max\{\alpha, \beta\} > n - 1$. *If* $q \geq 1$, $p > 0$, *then for every nonnegative very weak solution of problem (4) with* $t \geq 0$, *we have* $u, v \in L^\infty(\Omega)$ *and there exists constant* $C(\Omega, a, b, p, q, \kappa, \lambda) > 0$ *such that*

$$
t + \|u\|_\infty + \|v\|_\infty \leq C(\Omega, a, b, p, q, \kappa, \lambda).
$$

If $q < 1$, $p > 1$, *then the same result holds for nonnegative very weak solutions of problem (5) with* $t \geq 0$.

Theorem 1.2. *Assume (2) and* $\max\{\alpha, \beta\} > n - 1$. *Then there exists a positive bounded very weak solution of problem (1).*

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Theorem 1.3. *Assume* (2) *and* $\max\{\alpha,\beta\} < n-1$. *Then there exist functions* $a, b \in L^\infty(\Omega)$, $a, b \geq 0$, $a, b \not\equiv 0$ *and a positive very weak solution* (u, v) *of problem* (1) *such that* $u, v \notin L^\infty(\Omega)$.

Theorem 1.1 will be proved by a bootstrap method in weighted Lebesgue spaces used in [**3**], [**11**], for example. Although [**11**, Theorem 2.1] also implies the assertion of Theorem 1.1, the corresponding assumptions on p, q, κ, λ are more restrictive than our condition $\max\{\alpha,\beta\} > n-1$. Theorem 1.3 is based on a modification of the proof in [**13**].

Analogous results to the above theorems are true in the case of the scalar problem

$$\begin{cases} -\Delta u = a(x)|x|^{-\kappa}u^p, & x \in \Omega, \\ \quad u = 0, & x \in \partial\Omega. \end{cases} \tag{7}$$

The condition $\max\{\alpha,\beta\} > n-1$ or $\max\{\alpha,\beta\} < n-1$ is then replaced by $\frac{2-\kappa}{p-1} > n-1$ or $\frac{2-\kappa}{p-1} < n-1$, respectively. The proofs of such assertions are simpler than those of Theorems 1.1–1.3.

The case $\max\{\alpha,\beta\} = n-1$ seems to be open in the vector case. The existence of unbounded solutions of problem (7) with $\kappa = 0$ for $\frac{2}{p-1} = n-1$ was proved in [**6**].

2. Preliminaries

Denote

$$\delta(x) = \operatorname{dist}(x, \partial\Omega) \qquad \text{for} \quad x \in \Omega,$$

and for $1 \leq p \leq \infty$ define the weighted Lebesgue spaces $L^p_\delta = L^p_\delta(\Omega) := L^p(\Omega; \delta(x)\,\mathrm{d}x)$. If $1 \leq p < \infty$, then the norm in L^p_δ is defined by

$$\|u\|_{p,\delta} = \left(\int_\Omega |u(x)|^p \delta(x)\,\mathrm{d}x\right)^{1/p}.$$

Recall that $L^\infty_\delta = L^\infty(\Omega; \mathrm{d}x)$ with $\|u\|_{\infty,\delta} = \|u\|_\infty$. We will use the notation $\|\cdot\|_p$ for the norm in $L^p(\Omega)$ for $p \in [1, \infty)$ as well.

In the proofs we use the following lemmas.

Lemma 2.1. ([12, Theorem 49.1, Theorem 49.2(i)]) *Let Ω be a bounded domain of class $C^{2+\gamma}$ for some $\gamma \in (0, 1)$. Assume that $1 \le p \le q \le \infty$ satisfy*

$$\frac{1}{p} - \frac{1}{q} < \frac{2}{n+1}.$$

Let $f \in L^1_\delta(\Omega)$. Then there exists a unique very weak solution u of

$$\begin{cases} -\Delta u = f, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{8}$$

If $f \in L^p_\delta(\Omega)$, then $u \in L^q_\delta(\Omega)$ and

$$\|u\|_{q,\delta} \le C(p, q, \Omega)\|f\|_{p,\delta}.$$

Lemma 2.2. ([12, Remark 49.12(i)]) *Let $f \in L^1_\delta(\Omega)$ satisfy $f \ge 0$ a.e. Then the very weak solution of (8) satisfies*

$$u(x) \ge C(\Omega)\|f\|_{1,\delta}\delta(x), \quad x \in \Omega.$$

For $F\colon \mathbb{R} \to \mathbb{R}$ and $x \in \mathbb{R}$ we denote $F^{(0)}(x) = x$ and $F^{(j)}(x) = F(F^{(j-1)}(x))$ ($j \in \mathbb{N}$), the j-th iteration of F.

Lemma 2.3. *Let $F\colon [a, b) \to \mathbb{R}$ be a continuous function ($b \le \infty$) and*

$$F(x) > x \qquad \text{for all} \quad x \in [a, b). \tag{9}$$

Then, for all $Q \in (a, b)$ there exists $j \in \mathbb{N}$, that $F^{(j)}(a) > Q$.

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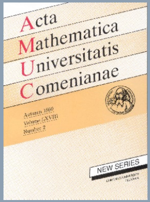


Proof of Lemma 2.3. The function F is continuous on the compact interval $[a, Q]$. The inequality (9) implies the existence of $\mu = \mu(Q) > 0$ such that for every $x \in [a, Q]$, we have

$$F(x) \geq \mu + x.$$

This implies $F^{(j)}(a) \geq j\mu + a$ for all $j \in \mathbb{N}$ such that $F^{(j-1)}(a) \leq Q$. □

Lemma 2.4 ([13])**.** *Let $n \geq 2$ and let Ω be a bounded domain of the class C^2. Assume that $0 \in \partial\Omega$. Let $-2 < \gamma < n-1$. Then there exist $R > 0$ and a revolution cone Σ_1 of the vertex 0 with $\Sigma := \Sigma_1 \cap \{x \in \mathbb{R}^n;\ |x| < R\} \subset \Omega \cup \{0\}$ such that the function*

$$\phi := |x|^{-(\gamma+2)} \chi_\Sigma$$

belongs to $L^1_\delta(\Omega)$ and the very weak solution $u > 0$ of the problem

$$\begin{cases} -\Delta u = \phi, & x \in \Omega, \\ \quad\ u = 0, & x \in \partial\Omega \end{cases}$$

satisfies the estimate

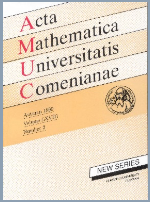
$$u \geq C|x|^{-\gamma} \chi_\Sigma.$$

3. Proofs of theorems

Proof of Theorem 1.1. In the proof, we use C or C' to denote constants which can vary from step to step.

Observe that α, β defined by (6) satisfy

$$\begin{aligned} \alpha p + \lambda &= \beta + 2, \\ \beta q + \kappa &= \alpha + 2. \end{aligned} \tag{10}$$



Suppose first $\alpha \geq \beta$, so $\alpha > n-1$. Using these conditions and (10), we obtain

$$p < \frac{n+1-\lambda}{n-1}, \qquad q > 1. \tag{11}$$

Thus we will deal with system (4) in the following. The case $\beta \geq \alpha$ can be treated similarly to dealing with system (5).

Denote $f(x,v) = a(x)|x|^{-\kappa}v^q + t(u+\varphi_1)$, $g(x,u) = b(x)|x|^{-\lambda}u^p$. Let (u,v) be a very weak solution of (4), $u, v \geq 0$. By definition of a very weak solution we have $u, v \in L^1(\Omega)$, $f, g \in L^1_\delta(\Omega)$ and for $\varphi = \varphi_1$, it holds

$$\begin{aligned} \lambda_1 \int_\Omega u\varphi_1 \,\mathrm{d}x &= \int_\Omega u(-\Delta\varphi_1)\,\mathrm{d}x = \int_\Omega f\varphi_1\,\mathrm{d}x, \\ \lambda_1 \int_\Omega v\varphi_1\,\mathrm{d}x &= \int_\Omega g\varphi_1\,\mathrm{d}x, \end{aligned} \tag{12}$$

where λ_1 is the first eigenvalue of the problem

$$\begin{cases} -\Delta\phi = \lambda\phi, & x \in \Omega, \\ \phi = 0, & x \in \partial\Omega \end{cases}$$

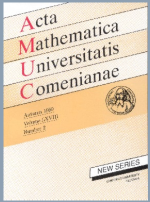
and φ_1 is the corresponding positive eigenfunction satisfying $\|\varphi_1\|_2 = 1$. Using (12), we have

$$(\lambda_1 - t)\int_\Omega u\varphi_1\,\mathrm{d}x = \int_\Omega a|x|^{-\kappa}v^q\varphi_1\,\mathrm{d}x + t \geq 0, \tag{13}$$

therefore, $t \leq \lambda_1$ for $u \not\equiv 0$. The equality in (13) further implies that $(0,v)$ is not a solution of problem (4) for any nonnegative $v \in L^1(\Omega)$ and $t > 0$. Hence, in both cases we have $t \leq C(\Omega)$.

Using (12) and

$$C(\Omega)\delta(x) \leq \varphi_1(x) \leq C'(\Omega)\delta(x) \qquad \text{for all } x \in \Omega,$$



we get

$$C(\Omega)\|f\|_{1,\delta} \le \|u\|_{1,\delta} \le C'(\Omega)\|f\|_{1,\delta}, \\ C(\Omega)\|g\|_{1,\delta} \le \|v\|_{1,\delta} \le C'(\Omega)\|g\|_{1,\delta}. \tag{14}$$

In this part of the proof, we estimate $\int_\Omega f^r\delta \,\mathrm{d}x$, $\int_\Omega g^s\delta \,\mathrm{d}x$ for $r,s \ge 1$. Let (u,v) be a very weak solution of (4), $u \in L^k_\delta(\Omega)$, $v \in L^l_\delta(\Omega)$ for $k,l \ge 1$, $u,v \ge 0$. Then it holds

$$\begin{aligned} \int_\Omega f^r\delta \,\mathrm{d}x &\le C(r)\left(\int_\Omega a^r|x|^{-\kappa r}v^{qr}\delta \,\mathrm{d}x + \int_\Omega ((t\,u)^r + (t\,\varphi_1)^r)\delta \,\mathrm{d}x\right) \\ &\le C(\Omega,a,r,\theta_1)\left(1 + \int_\Omega |x|^{-\frac{\kappa r}{\theta_1}+1}\,\mathrm{d}x + \int_\Omega (v^{\frac{qr}{1-\theta_1}} + u^r)\delta \,\mathrm{d}x\right) \end{aligned} \tag{15}$$

for all $\theta_1 \in (0,1)$, where we have successively used boundedness of function a, the Young inequality, boundedness of t and the assumption $0 \in \partial\Omega$ (then it holds $\delta(x) \le |x|$). Similarly it holds

$$\int_\Omega g^s\delta \,\mathrm{d}x \quad\le\quad C(\Omega,b,s,\theta_2)\left(\int_\Omega |x|^{-\frac{\lambda s}{\theta_2}+1}\,\mathrm{d}x + \int_\Omega u^{\frac{ps}{1-\theta_2}}\delta \,\mathrm{d}x\right) \tag{16}$$

for all $\theta_2 \in (0,1)$. We will show that if k,l are large enough, then the right-hand sides in (15), (16) can be estimated by $\|u\|_{k,\delta}$, $\|v\|_{l,\delta}$ for some $r,s \ge 1$.

Now we determine the dependence r,s on k,l. If

$$r < \widetilde{r}(l) := \frac{(n+1)l}{\kappa l + (n+1)q},$$

then there exists $\theta_1 \in (0,1)$ such that

$$-\frac{\kappa r}{\theta_1} + 1 > -n, \qquad \frac{qr}{1-\theta_1} \le l.$$

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If moreover $r \le k$, then estimate (15) implies $f \in L^r_\delta(\Omega)$. Thus

$$\|f\|_{r,\delta} \le C(\Omega, a, \kappa, q, r, \|u\|_{k,\delta}, \|v\|_{l,\delta}) \qquad \text{if } r < \min\{\widetilde{r}(l), k\}. \tag{17}$$

Similarly,

$$s < \widetilde{s}(k) := \frac{(n+1)k}{\lambda k + (n+1)p}$$

implies the existence of $\theta_2 \in (0,1)$ such that

$$-\frac{\lambda s}{\theta_2} + 1 > -n, \qquad \frac{ps}{1-\theta_2} \le k.$$

Then estimate (16) implies $g \in L^s_\delta(\Omega)$. Thus

$$\|g\|_{s,\delta} \le C(\Omega, b, \lambda, s, p, \|u\|_{k,\delta}) \qquad \text{if } s < \widetilde{s}(k). \tag{18}$$

On the other hand, Lemma 2.1 gives us estimates for $\|u\|_{k,\delta}, \|v\|_{l,\delta}$, $k, l \ge 1$. If $f \in L^r_\delta(\Omega)$, then $u \in L^k_\delta(\Omega)$ and it holds

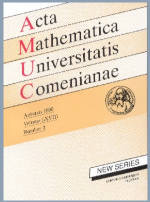
$$\|u\|_{k,\delta} \le C(\Omega, k, r)\|f\|_{r,\delta}, \tag{19}$$

where $1 \le r \le k \le \infty$ satisfy $\frac{1}{r} - \frac{1}{k} < \frac{2}{n+1}$. In particular, we can take

$$k < \widetilde{k}(r) := \frac{(n+1)r}{n+1-2r} \quad \text{if } r \in \left[1, \frac{n+1}{2}\right).$$

If $r = \frac{n+1}{2}$, $1 \le k < \infty$ can be chosen arbitrarily and if $r > \frac{n+1}{2}$, then we can take $k = \infty$. Similarly, if $g \in L^s_\delta(\Omega)$, then $v \in L^l_\delta(\Omega)$ and it holds

$$\|v\|_{l,\delta} \le C(\Omega, l, s)\|g\|_{s,\delta}, \tag{20}$$



where $1 \le s \le l \le \infty$ satisfy

$$l < \widetilde{l}(s) := \frac{(n+1)s}{n+1-2s} \qquad \text{if } s \in \left[1, \frac{n+1}{2}\right).$$

If $s = \frac{n+1}{2}$, $1 \le l < \infty$ can be chosen arbitrarily and if $s > \frac{n+1}{2}$, then we can take $l = \infty$.

We know that $f \in L^1_\delta(\Omega)$. Estimate (19) implies $u \in L^k_\delta(\Omega)$ for $1 < k < k_0$ where $k_0 := \frac{n+1}{n-1} = \widetilde{k}(1)$. Given $s < \widetilde{s}(k_0) = \frac{n+1}{\lambda+(n-1)p}$, the continuity and the monotonicity of $\widetilde{s}$ assures existence of $k < k_0$ such that $s < \widetilde{s}(k) < \widetilde{s}(k_0)$. Hence $g \in L^s_\delta(\Omega)$ for $s \in \left(1, \frac{n+1}{\lambda+(n-1)p}\right)$ (inequality (11) implies $\frac{n+1}{\lambda+(n-1)p} > 1$). If $p > \frac{2-\lambda}{n-1}$, then $v \in L^l_\delta(\Omega)$ for $l < l_0 := \widetilde{l}(\widetilde{s}(k_0)) = \frac{n+1}{\lambda-2+(n-1)p}$. Finally we have $f \in L^r_\delta(\Omega)$ for $r < \min\left\{\widetilde{r}\left(\frac{n+1}{\lambda-2+(n-1)p}\right), k_0\right\} = \min\left\{\frac{n+1}{\kappa+(\lambda+(n-1)p-2)q}, \frac{n+1}{n-1}\right\} =: r_0$. Then $r_0 > 1$ due to the assumption $\alpha > n-1$. If $p \le \frac{2-\lambda}{n-1}$, then $\frac{n+1}{\lambda+(n-1)p} \ge \frac{n+1}{2}$ and due to the continuity and the monotonicity of $\widetilde{l}$ we have $v \in L^l_\delta(\Omega)$ for all $l < \infty$. Thus $f \in L^r_\delta(\Omega)$ for $r < \min\left\{\frac{n+1}{\kappa}, \frac{n+1}{n-1}\right\} =: r_0'$. The preceding computations show that if $k \le k_0$ ($l \le l_0$) is close enough to k_0 (l_0) or larger, then the right-hand sides in (15), (16) can be estimated by $\|u\|_{k,\delta}, \|v\|_{l,\delta}$ for some $r, s \ge 1$.

We have shown that if $f \in L^1_\delta(\Omega)$, then $f \in L^r_\delta(\Omega)$ for $r < r_0$ ($r < r_0'$) if $p > \frac{2-\lambda}{n-1}$ ($p \le \frac{2-\lambda}{n-1}$). We claim that it holds

$$\text{if } f \in L^r_\delta(\Omega) \text{ for some } r \in \left[1, \frac{n+1}{\kappa}\right) \text{ then } f \in L^{F(r)}_\delta(\Omega) \tag{21}$$

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for some continuous function $F\colon \left[1, \frac{n+1}{\kappa}\right) \to \mathbb{R}$ satisfying (9). In the following we give expression of such function F. For $p > \frac{2-\lambda}{n-1}$, denote

$$\widetilde{F}(r) := \begin{cases} \min\{\widetilde{r}(\widetilde{l}(\widetilde{s}(\widetilde{k}(r)))), \widetilde{k}(r)\} \\ = \min\left\{\frac{n+1}{\kappa+\left(\lambda+\left(\frac{n+1}{r}-2\right)p-2\right)q}, \frac{(n+1)r}{n+1-2r}\right\}, & r \in \left[1, \frac{(n+1)p}{2p+2-\lambda}\right), \\ \min\left\{\frac{n+1}{\kappa}, \frac{(n+1)r}{n+1-2r}\right\}, & r \in \left[\frac{(n+1)p}{2p+2-\lambda}, \frac{n+1}{2}\right), \\ \frac{n+1}{\kappa}, & r \in \left[\frac{n+1}{2}, \frac{n+1}{\kappa}\right) \end{cases}$$

(for such p, $\frac{(n+1)p}{2p+2-\lambda} > 1$ holds). For $p \le \frac{2-\lambda}{n-1}$, denote

$$\widetilde{F}(r) := \begin{cases} \frac{(n+1)r}{n+1-2r}, & r \in \left[1, \frac{n+1}{2+\kappa}\right) \text{ if } \frac{n+1}{2+\kappa} > 1, \\ \frac{n+1}{\kappa}, & r \in \left[\max\left\{1, \frac{n+1}{2+\kappa}\right\}, \frac{n+1}{\kappa}\right). \end{cases}$$

Function $\widetilde{F}\colon \left[1, \frac{n+1}{\kappa}\right) \to \mathbb{R}$ is continuous and due to the assumption $\alpha > n-1$, (9) holds. Define $F(r) := \frac{\widetilde{F}(r)+r}{2}$. Then $r < F(r) < \widetilde{F}(r)$ for all $r \in \left[1, \frac{n+1}{\kappa}\right)$. Observe that $\widetilde{F}(1) = r_0$ ($\widetilde{F}(1) = r_0'$) for $p > \frac{2-\lambda}{n-1}$ ($p \le \frac{2-\lambda}{n-1}$), hence claim (21) has already been proved for $r = 1$. For $r > 1$ fixed, the same monotonicity and continuity argument is used. If $p > \frac{2-\lambda}{n-1}$ and $r < \frac{(n+1)p}{2p+2-\lambda}$, then $u \in L^k_\delta(\Omega)$ for $k < \widetilde{k}(r)$ due to (19). Consequently from (18), we get $g \in L^s_\delta(\Omega)$ for $s < \widetilde{s}(\widetilde{k}(r))$ and then (20) implies $v \in L^l_\delta(\Omega)$ for $l < \widetilde{l}(\widetilde{s}(\widetilde{k}(r)))$. Finally, (17) implies $f \in L^{r'}_\delta(\Omega)$ for $r' < \min\{\widetilde{r}(\widetilde{l}(\widetilde{s}(\widetilde{k}(r)))), \widetilde{k}(r)\} = \widetilde{F}(r)$, hence $f \in L^{F(r)}_\delta(\Omega)$. Claim (21) in the remaining cases can be proved similarly.

The assumptions of Lemma 2.3 are satisfied for F, hence there exists $\overline{j} \in \mathbb{N}$ such that

$$F^{(\overline{j})}(1) > \frac{n+1}{2} + \varepsilon \tag{22}$$

for $\varepsilon > 0$ small. Using (21) $\overline{j}$-times we get $f \in L_\delta^{F^{(\overline{j})}(1)}(\Omega)$, thus $f \in L_\delta^{\frac{n+1}{2}+\varepsilon}(\Omega)$ from (22). Lemma 2.1 then implies $u \in L^\infty(\Omega)$. From (18) we get $g \in L_\delta^{\frac{n+1}{2}+\varepsilon}(\Omega)$ and consequently, $v \in L^\infty(\Omega)$.

Now we prove

$$\|u\|_\infty + \|v\|_\infty \leq C(\Omega, p, q, \kappa, \lambda, a, b, \|u\|_{1,\delta}, \|v\|_{1,\delta}). \tag{23}$$

Using (17), (18), (19), (20), we have

$$\|f\|_{F(r),\delta} \leq C(\Omega, a, b, \kappa, \lambda, p, q, k, l, r, s, \|f\|_{r,\delta}, \|g\|_{s,\delta}). \tag{24}$$

Iterating (24) $\overline{j}$-times and using (22), (14), we have

$$\|f\|_{\frac{n+1}{2}+\varepsilon,\delta} \leq C(\Omega)\|f\|_{F^{(\overline{j})}(1),\delta} \leq C(\Omega, a, b, \kappa, \lambda, p, q, \|u\|_{1,\delta}, \|v\|_{1,\delta}).$$

Lemma 2.1 and (18) then imply assertion (23).

Now we turn to prove uniform boundedness of $\|u\|_{1,\delta}$ and $\|v\|_{1,\delta}$. Due to Lemma 2.2,

$$u \geq C(\Omega)\,\delta \int_\Omega a|x|^{-\kappa} v^q \delta + t(u+\varphi_1)\delta \,\mathrm{d}x,$$
$$v \geq C(\Omega)\,\delta \int_\Omega b|x|^{-\lambda} u^p \delta \,\mathrm{d}x.$$

holds. This implies

$$\begin{aligned}\int_\Omega a|x|^{-\kappa}v^q\delta + t(u+\varphi_1)\delta \,\mathrm{d}x &\geq C(\Omega,q)\int_\Omega a|x|^{-\kappa}\delta^{q+1}\,\mathrm{d}x\left(\int_\Omega b|x|^{-\lambda}u^p\delta\,\mathrm{d}x\right)^q \\ &\geq C(\Omega,q,a,\kappa)\left(\int_\Omega b|x|^{-\lambda}u^p\delta\,\mathrm{d}x\right)^q \end{aligned}\tag{25}$$

and

$$\int_\Omega b|x|^{-\lambda}u^p\delta\,\mathrm{d}x \geq C(\Omega,p,b,\lambda)\left(\int_\Omega a|x|^{-\kappa}v^q\delta + t(u+\varphi_1)\delta\,\mathrm{d}x\right)^p.\tag{26}$$

Using (25), (26) and the assumption $pq > 1$, we get

$$\|f\|_{1,\delta} + \|g\|_{1,\delta} \leq C(\Omega,p,q,a,b,\kappa,\lambda).$$

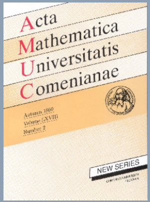
The estimate $\|u\|_{1,\delta} + \|v\|_{1,\delta} \leq C(\Omega,p,q,a,b,\kappa,\lambda)$ then follows from (14). Inequality (23) then implies the last assertion of the theorem. □

Proof of Theorem 1.2. Suppose first $\alpha \geq \beta$. As in proof of Theorem 1.1, it is enough to deal with system (4) in the following. Again, the case $\beta \geq \alpha$ can be treated similarly dealing with system (5).

Denote now $f(x,v) = a(x)|x|^{-\kappa}|v|^q$, $g(x,u) = b(x)|x|^{-\lambda}|u|^p$. Set $X := L^\infty(\Omega)\times L^\infty(\Omega)$. Given $(u,v)\in X$ and $t\geq 0$, let $S_t(u,v) = (w,w')$ be the unique solution of the linear problem

$$\begin{cases} -\Delta w = f + t(|u|+\varphi_1), & x\in\Omega,\\ -\Delta w' = g, & x\in\Omega,\\ w = w' = 0, & x\in\partial\Omega.\end{cases}\tag{27}$$

We will prove that there exists a nontrivial fixed point of operator S_0. Since $f\in L^k(\Omega)$ for $k<\frac{n}{\kappa}$ and $g\in L^l(\Omega)$ for $l<\frac{n}{\lambda}$, we have $S_t(u,v)\in W^{2,r}(\Omega)\times\ W^{2,r}(\Omega)$ for $r\in(\frac{n}{2},\min\{\frac{n}{\kappa},\frac{n}{\lambda}\})$.



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Therefore, $S_t \colon X \to X$ is compact. Observe that the right-hand sides in (27) are nonnegative for every $(u,v) \in X$, hence w, w' are nonnegative. Thus S_t has no fixed point beyond the nonnegative cone $K = \{(u',v') \in X : u',v' \geq 0\}$ for any $t \geq 0$.

Let $\|(u,v)\|_X = \varepsilon$ for $\varepsilon > 0$ small, $\theta \in [0,1]$. Assume $(u,v) = \theta S_0(u,v)$. Using L^p-estimates (see [8, Chapter 9]), we have

$$\|u\|_\infty \leq C\|u\|_{2,r} \leq C\|f\|_r \leq C\|a|x|^{-\kappa}\|_r\|v\|_\infty^q \leq C\|v\|_\infty^q,$$

where $\|\cdot\|_{2,r}$ denotes the norm in $W^{2,r}(\Omega)$. Similarly, we obtain $\|v\|_\infty \leq C\|u\|_\infty^p$. Combining the last two estimates, we have

$$\|u\|_\infty \leq C\|u\|_\infty^{pq} \leq C\varepsilon^{pq-1}\|u\|_\infty.$$

This is a contradiction for ε sufficiently small due to the assumption $pq > 1$. Hence $(u,v) \neq \theta S_0(u,v)$ and the homotopy invariance of the topological degree implies

$$\deg(I - S_0, 0, B_\varepsilon) = \deg(I, 0, B_\varepsilon) = 1, \tag{28}$$

where I denotes the identity and $B_\varepsilon := \{(u,v) \in X : \|(u,v)\|_X < \varepsilon\}$.

Theorem 1.1 immediately implies $S_T(u,v) \neq (u,v)$ for T large and $(u,v) \in \overline{B_R} \cap K$ and $S_t(u,v) \neq (u,v)$ for $t \in [0,T]$ and $(u,v) \in (\overline{B_R} \smallsetminus B_R) \cap K$ (where $R > 0$ is large enough), hence we have

$$\deg(I - S_0, 0, B_R) = \deg(I - S_T, 0, B_R) = 0. \tag{29}$$

Equalities (28) and (29) imply $\deg(I - S_0, 0, B_R \smallsetminus \overline{B_\varepsilon}) = -1$, hence there exist $u, v \in (B_R \smallsetminus \overline{B_\varepsilon}) \cap K$ such that $S_0(u,v) = (u,v)$. Finally, the maximum principle implies the positivity of u, v. $\square$

Proof of Theorem 1.3. Basic ideas used in the proof are from [13]. Lemma 2.4 assures the existence of sets Σ_ϕ, Σ_ψ such that $\phi := \chi_{\Sigma_\phi}|x|^{-(\alpha+2)}$, $\psi := \chi_{\Sigma_\psi}|x|^{-(\beta+2)}$ belong to $L^1_\delta(\Omega)$, where

α, β are defined by (6). Let (u, v) be the (positive) very weak solution of

$$\begin{cases} -\Delta u = \phi, & x \in \Omega, \\ -\Delta v = \psi, & x \in \Omega, \\ \quad u = v = 0, & x \in \partial\Omega. \end{cases}$$

Lemma 2.4 then implies

$$u \geq C|x|^{-\alpha}\chi_{\Sigma_\phi}, \quad v \geq C|x|^{-\beta}\chi_{\Sigma_\psi}, \tag{30}$$

hence $u, v \notin L^\infty(\Omega)$. Observe that (30) and (10) imply $a', b' \in L^\infty(\Omega)$, where $a' := \frac{|x|^\kappa \phi}{v^q}$, $b' := \frac{|x|^\lambda \psi}{u^p}$ are nonnegative functions and (u, v) is a very weak solution of (1) with $a = a'$, $b = b'$. $\square$

1. Bidaut-Véron M. F., *Local behaviour of the solutions of a class of nonlinear elliptic systems*, Adv. Differential Equations **5** (2000), 147–192.

2. Bidaut-Véron M. F.and Giacomini H., *A new dynamical approach of Emden-Fowler equations and systems*, Adv. Differential Equations **15** (2010), 1033–1082.

3. Bidaut-Véron M. F. and Yarur C., *Semilinear elliptic equations and systems with measure data: existence and a priori estimates*, Adv. Differential Equations **7** (2002), 257–296.

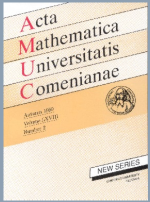
4. Calanchi M. and Ruf B., *Radial and non radial solutions for Hardy-Hénon type elliptic systems*, Calc. Var. **38** (2010), 111–133.

5. De Figueiredo D. G., Peral I. and Rossi J. D., *The critical hyperbola for a Hamiltonian elliptic system with weights*, Ann. Mat. Pura Appl. **187** (2008), 531–545.

6. Del Pino M., Musso M. and Pacard F., *Boundary singularities for weak solutions of semilinear elliptic problems*, J. Funct. Anal. **253** (2007), 241–272.

7. Fazly M., *Liouville type theorems for stable solutions of certain elliptic systems*, Adv. Nonlinear Stud. **12** (2012), 1–17.

8. Gilbarg D. and Trudinger N. S., *Elliptic partial differential equations of second order*, Springer, Berlin, 1998.



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9. Liu F. and Yang J., *Nontrivial solutions of Hardy-Hénon type elliptic systems*, Acta Math. Sci. Ser. B Engl. Ed. **27** (2007), 673–688.

10. Phan Q. H., *Liouville-type theorems and bounds of solutions for Hardy-Hénon elliptic systems*, Adv. Differential Equations **17** (2012), 605–634.

11. Quittner P. and Souplet Ph., *A priori estimates and existence for elliptic systems via bootstrap in weighted Lebesgue spaces*, Arch. Rational Mech. Anal. **174** (2004), 49–81.

12. ______, *Superlinear Parabolic Problems*, Birkhäuser 2007.

13. Souplet Ph., *Optimal regularity conditions for elliptic problems via $L^p_\delta(\Omega)$-spaces*, Duke Math. J. **127** (2005), 175–192.

J. Pačuta, Department of Applied Mathematics and Statistics, Comenius University, Mlynská dolina, 84248 Bratislava, Slovakia, *e-mail*: `julopacuta@gmail.com`