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ON NON–SEPARATING EMBEDDINGS OF GRAPHS IN CLOSED SURFACES

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Abstract. A. A. Zykov [Fundamentals of Graph Theory, Nauka, Moscow, 1987] asks to determine, for a given closed surface S, all graphs G (including disconnected ones) that admit an embedding $i\colon G \hookrightarrow S$ in a closed surface S leaving $S - i(G)$ connected. We anwser this question completely. For connected graphs the results can be formulated as follows: G has an embedding $i\colon G \hookrightarrow S$ with $S - i(G)$ connected if and only if S is non-orientable and $\tilde{\gamma}(S) \geq \beta(G) = |E(G)| - |V(G)| + 1$, or S is orientable and $\gamma(S) \geq \beta(G) - \gamma_M(G)$, where $\gamma_M(G)$ is the maximum genus of G.

An embedding $i : G \hookrightarrow S$ of a graph G in a closed surface S is said to be **non-separating** if the subset $S - i(G)$ of S is connected. In his books [**10,** pp. 445–447] and [**11,** pp. 229–230] Zykov posed the problem of determining, for a given closed surface S, all graphs that admit a non separating embedding in S. He also observed that if S is non-orientable then such an embedding exists for every graph G whose Betti (= cyclomatic) number $\beta(G)$ does not exceed the non-orientable genus $\tilde{\gamma}(S)$ of S. For orientable surfaces the problem has remained open although some further work in this direction was previously done by Khomenko and Yavorskii [**3**].

In this paper we show how this problem can be completely solved in both orientable and non-orientable case. Our solution requires only a few facts about the maximum genus of a graph and the standard surface topology. For terms not defined here we refer the reader to [**7**].

Let G be a connected graph with p vertices and q edges. The (orientable) **maximum genus** $\gamma_M(G)$ of G is the largest among the genera $\gamma(S)$ of orientable surfaces S in which G has a cellular embedding. If G is cellularly embedded with r faces in an orientable surface S of genus $\gamma(S)$ then the Euler formula [**7**] claims that

$$p - q + r = 2 - 2\gamma(S).$$

It follows from this formula that $2\gamma_M(G)$ is bounded from above by $\beta(G) = q - p + 1$. Thus it is natural to consider the difference

$$\xi(G) = \beta(G) - 2\gamma_M(G)$$

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which is called the **Betti deficiency** of G. Note that $\xi(G)+1$ is in fact the minimum number of faces over all orientable cellular embeddings of G, and that $\xi(G)$ has the same parity as $\beta(G)$.

It is widely known that the Betti deficiency can be effectively characterized in purely combinatorial terms [**2, 4, 8**] and can be computed in polynomial time [**1, 2**]. In particular, let us recall that $\xi(G)$ is equal to the minimum number of components with odd number of edges taken over all cotrees of G.

For our purposes it is convenient to extend the definition of the Betti deficiency to disconnected graphs. If G has k components $G_1, G_2, \ldots, G_k$, then we set

$$\xi(G) = \sum \xi(G_i)$$

Now we can state our results.

Theorem 1. *A graph G has a non-separating embedding in an orientable surface S if and only if $\gamma(S) \geq (\beta(G)+\xi(G))/2$.*

Theorem 2. *A graph G has a non-separating embedding in a non-orientable surface S if and only if $\tilde{\gamma}(S) \geq \beta(G)$.*

Proof of Theorem 1. Let S be an orientable surface of genus g and let G be a graph with k components $G_1, G_2, \ldots, G_k$ such that $g \geq (\beta(G)+\xi(G))/2$. We show that G has a non-separating embedding in S.

A non-separating embedding of G can be constructed as follows. Take any orientable surface R and for every component G_i of G take a cellular embedding $j_i \colon G_i \to S_i$ of G_i in some orientable surface S_i. Let F_i be a closed collaring of $j_i(G_i)$ in S_i, i.e., the closure of a "small" open neighbourhood of $j_i(G_i)$ of which $j_i(G_i)$ is a deformation retract. If the embedding j_i has r_i faces then F_i is a bordered surface with r_i boundary components containing $j_i(G_i)$ in its interior. For each F_i and for each boundary component C of F_i remove an open disc D_C from R and identify homeomorphically C with the boundary of D_C in R. The identifications should be made in such a way that the resulting surface T will be orientable. Note that we thus obtain a non-separating embedding j of G in T; we shall refer to j as the **join** of $j_1, j_2, \ldots, j_k$ by R.

Elementary computations show that if $r = \sum r_i$ is the total number of faces in the above cellular embeddings j_i, $i = 1, 2, \ldots, k$, then

$$\gamma(T) = \gamma(R) + \sum \gamma(S_i) + r - k\,. \tag{1}$$

In particular, choosing S_i to have genus $\gamma(S_i) = \gamma_M(G_i) = (\beta(G_i) - \xi(G_i))/2$, R to have genus $\gamma(R) = g - (\beta(G) + \xi(G))/2$ (which by our assumption is non-negative) and using the fact that $r_i = \xi(G_i)+1$ we obtain that $\gamma(T) = g$. Thus T is homeomorphic to S and the required non-separating embedding exists.

Conversely, assume that G is a graph having a non-separating embedding $j\colon G \hookrightarrow S$ in an orientable surface S. We show that $(\beta(G) + \xi(G))/2 \le \gamma(S)$. Take a closed collaring F of $j(G)$ in S. If G has k components $G_1, G_2, \ldots, G_k$ then F is the disjoint union of k bordered surfaces, each containing a component of $j(G)$ in its interior. Let F_i be the component of F containing $j(G_i)$. Then by capping each boundary component of F_i with a 2-cell we obtain a closed surface S_i and a cellular embedding $j_i\colon G_i \hookrightarrow S_i$, $i = 1, 2, \ldots, k$. (This is the well-known "capping operation" of Youngs [**9**].)

Since $S - j(G)$ is connected, so is $S - Int(F) = H$. Thus H is a bordered surface. Obviously, each boundary component of H is a boundary component of some F_i and vice versa. It follows that the number of boundary components of H is equal to the total number of faces in the embeddings $j_i\colon G_i \hookrightarrow S_i$, which we denote by r. By capping each boundary component of H with a 2-cell we obtain a closed surface R, and it is now clear that j is the join of $j_1, j_2, \ldots, j_k$ by R. Hence, employing (1) and the Euler formula for each j_i we finally have

$$\begin{aligned}\gamma(S) &\ge \gamma(R) + \sum \gamma(S_i) + r - k \ge 0 + (\beta(G) - r + k)/2 + r - k \\ &= (\beta(G) + r - k)/2 \ge (\beta(G) + \xi(G))/2\,.\end{aligned}$$

This completes the proof. □

Theorem 2 can be proved basically in the same way, the main difference being that every connected graph has a cellular embedding in a non-orientable surface with a single face [**5, 6**], i.e., the non-orientable analogue of Betti deficiency is constantly 0.

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