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TRACKING INVARIANT MANIFOLDS WITHOUT DIFFERENTIAL FORMS

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Abstract. We present a different proof of a result of Jones et al. [**3**] concerning the inclination of invariant manifolds of singularly perturbed differential equations at exit points from neighborhoods of the "slow manifolds" of such systems.

A frequently studied problem of geometric singular perturbation theory consists in establishing the presence of trajectories of certain types (homoclinic, heteroclinic, satisfying given boundary conditions, etc.) approximating singular ones for the unperturbed problem. A useful tool for this problem has been established in Jones et al. [**2**] and called "Exchange Lemma" by the authors. It resembles the well known λ-lemma (Palis et al. [**4**]) with critical elements of a dynamical system replaced by "slow manifolds" of a singularly perturbed differential equation. The degeneration of transversality in the unperturbed equation in important applications lead the authors of Jones et al. [**3**] establish a more precise version of the Exchange Lemma.

The proof of the Exchange Lemma of Jones et al. [**2**], [**3**] involves differential equations for the evolution of differential forms of tangent vectors along trajectories. The purpose of this paper is to present an alternative proof which avoids differential forms. We believe that, except of being more elementary, it provides additional insight into the geometry of the problem.

We refer the reader to Jones et al. [**2**], [**3**] for the motivation and the application of the Exchange Lemma. In order to facilitate the comparison of our result to Jones et al. [**2**], [**3**] we use freely their notation whenever possible.

As in Jones et al. [**2**], [**3**] we consider a singularly perturbed system

$$\begin{aligned} \varepsilon\dot{x} &= f(x,y,\varepsilon) \\ \dot{y} &= g(x,y,\varepsilon) \end{aligned} \tag{1}$$

with $x \in \mathbb{R}^m,\ y \in \mathbb{R}^n$, $0 < \varepsilon \ll 1$ and f, g being C^2. As usual, by a change of the time scale we can transform the system (1) into the regularly perturbed system

$$\begin{aligned} x' &= f(x,y,\varepsilon) \\ y' &= \varepsilon g(x,y,\varepsilon), \qquad 0 \le \varepsilon \ll 1 \end{aligned} \tag{2}$$

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We assume that S_0 is a normally hyperbolic connected manifold of stationary points of the system (2) for $\varepsilon = 0$, i.e. $f(x, y, 0) = 0$ for $(x, y, 0) \in S_0$ and $D_x f(x, y, 0)$ does not have eigenvalues on the imaginary axis. As argued in Jones et al. [**3**], for $0 < \varepsilon \ll 1$ there is a family of normally hyperbolic manifolds S_ε approaching S_0 for $\varepsilon \to 0$ in a C^1 way. Using the "Fenichel coordinates" (Fenichel [**1**]), in a sufficiently small neighbourhood of S_ε the system can be transformed to the form

$$\begin{aligned} a' &= \Lambda(a, b, y, \varepsilon)a, && \dim a = k \\ b' &= \Gamma(a, b, y, \varepsilon)b, && \dim b = l \\ y' &= \varepsilon[m(y, \varepsilon) + h(a, b, y, \varepsilon)ab], && \dim y = n \end{aligned} \tag{3}$$

for $z := (a, b, y) \in \Omega_\Delta := \{|a| \leqslant \Delta, |b| \leqslant \Delta\} \cap \Omega$, where Δ and ε are sufficiently small, Ω is a fixed compact region,

$$\begin{aligned} &\operatorname{Re}\lambda > \lambda_0 > 0, && \text{for } \lambda \in \text{spectrum of } \Lambda(a, b, y, \varepsilon) \\ &\operatorname{Re}\gamma < \gamma_0 < 0, && \text{for } \gamma \in \text{spectrum of } \Gamma(a, b, y, \varepsilon) \end{aligned}$$

and $h(z(t), \varepsilon)(\cdot, \cdot)$ is a bilinear form.

In these coordinates S_ε is represented by the plane $a = 0$, $b = 0$, k, l are the dimensions of the invariant subspaces of $D_x f(x, y, 0)$ at $(x, y, 0) \in S_0$ corresponding to the part of the spectrum right resp. left to the imaginary axis (note that due to normal hyperbolicity they have to be the same over S_0 and we have $k + l = m$).

Note that in Jones et al. [**2**], the factor b does not appear in the second term of the third equation of (3). The possibility to reduce this term to become bilinear in a, b allowed the authors of Jones et al. [**3**] to improve the estimates of Jones et al. [**2**].

As in Jones et al. [**2**], [**3**] we write $z = (a, b, y)$. We understand the norms $|a|, |b|, |y|$ to be Euclidean and define

$$|z| = |a| + |b| + |y|.$$

We assume that $m(y, \varepsilon)$ is parallelizable over $S_0 \cap \Omega$ to $U \equiv (1, 0, \cdots, 0)$. We can now formulate the

Exchange lemma (Jones et al. [**3**]). *Let $\{M_\varepsilon\}$, $0 < \varepsilon \ll 1$ be a family of $(k+1)$-dimensional invariant manifolds of (3) intersecting the subspace $a = 0$ transversally at $p_\varepsilon = (0, \hat{b}_\varepsilon, \hat{y}_\varepsilon)$ of Ω_Δ where $p_\varepsilon \to p_0$ for $\varepsilon \to 0$ Assume that for $\varepsilon \to 0$ the transversality has the following asymptotics:*

There is a neighborhood V of p_0 such that, for each $p \in M_\epsilon \cap V$, $T_p M_\epsilon$ contains a subspace E_ϵ of codimension 1 transversal to (a', b', y') such that for $(\delta a, \delta b, \delta y) \in E_\epsilon$ one has

$$|\delta b| + |\delta y| = O(\varepsilon^{-r})|\delta a| \tag{4}$$

uniformly in $p \in M_\varepsilon \cap V$ for some $r > 0$.

Fix $l > 0$ and let $p = (\hat{a}, \hat{b}, \hat{y}) \in M_\varepsilon \cap V$ be a point whose trajectory $z(t)$ stays in the set $\Omega \cap \{|a| \leq \Delta\}$ for time $T \geq l/\varepsilon$. Then, M_ε is uniformly $O(\varepsilon^{-\rho/\varepsilon})$ C^1-close to the manifold $b = 0$, $y_i = \hat{y}_i$ for $i > 1$ at $q = z(T)$ for some $\rho > 0$.

Note that our formulation of the lemma is somewhat different to Jones et al. [**3**]. We include the asymptotics of the transversality for $\varepsilon \to 0$ (which in Jones et al. [**2**], [**3**] appears in the comments only) explicitly into the formulation of the lemma. Further, we correct an obvious misprint — the y_i, $i > 1$, components of the points of the manifold M_ε at q are close to their initial values at p, not to 0.

Geometrically, assumption (4) means that the angle between T_pM_ϵ and $\{a = 0\}$ is larger that $C\epsilon^r$ for some $C > 0$; note that

$$\text{angle}\,(T_pM_\epsilon, \{a = 0\}) = \text{angle}\,(\Sigma_\epsilon \cap T_pM_\epsilon, \Sigma_\epsilon \cap \{a = 0\}),$$

where Σ_ϵ is the codimension 1 plane orthogonal to the 1-dimensional subspace $T_pM_\epsilon \cap \{a = 0\}$. Most efficiently, one can choose $E_\epsilon = T_pM_\epsilon \cap E_\epsilon$. More simply, one can take E_ϵ as an intersection of T_pM_ϵ with some fixed codimesnion 1 subspace transversal to the flow, e.g. the tangent plane to $M_\epsilon \cap \{|b| = |\hat{b}_\epsilon|\}$ if $\hat{b}_0 \neq 0$.

The following simple lemma will be used several times in the proof of the theorem.

Lemma. *Let $\alpha < 0 < \beta, \xi_0, R$. Assume that $\xi(t)$ is nonnegative differentiable and satisfies*

$$\xi'(t) \leq \left(\alpha + \zeta e^{\beta(t-T)}\xi(t)\right)\xi(t) + \zeta e^{\alpha t}$$

for $0 \leq t \leq T$, $\varepsilon > 0$ and

$$\xi(0) = \xi_0.$$

Then, for sufficiently large T and sufficiently small $\zeta > 0$ we have

$$\xi(t) \leq e^{\frac{\alpha}{2}t}\left[\xi_0 - \frac{2\zeta}{\alpha}\right].$$

Proof. Fix ζ, $\eta > 0$ and choose T so large that

$$\zeta e^{\beta(t-T)} e^{\frac{\alpha}{2}t}\left[\xi_0 - \frac{2\zeta}{\alpha}\right] < -\frac{\alpha}{2} \qquad \text{for } 0 \leq t \leq T. \tag{5}$$

While $t \geq 0$ is such that

$$\alpha + \zeta e^{\beta(t-T)}\xi(t) < \frac{1}{2}\alpha, \tag{6}$$

we have

$$\xi' \leq \frac{\alpha}{2}\xi + \zeta e^{\alpha t}.$$

Integrating this inequality, for such t we obtain

$$\xi(t) \leq e^{\frac{\alpha}{2}t}\left[\xi_0 - \frac{2\zeta}{\alpha}\right] \tag{7}$$

Because of (5) and (7), by contradiction it follows that (6) and, hence, also (7) remains valid for all $0 \leq t \leq T$ which proves the lemma. □

Proof of the Theorem. For the simpler C^0 part of the Exchange lemma we refer the reader to Jones et al. [**3**, Lemma 3.1]. In particular, we note that from Jones et al. [**3**, Lemma 3.1] it follows

$$|a(t)| = O(\Delta e^{\bar{\lambda}(t-T)}) \text{ and } |b(t)| = O(\Delta e^{-\bar{\gamma}t}). \tag{8}$$

To establish the C^1 extension we have to prove that the tangent plane of T_qM_ε tends to the subspace $b = 0$, $y_i = 0, i > 1$ for $\varepsilon \to 0$ with rate $e^{-\rho/\varepsilon}$. In other words,

$$|\delta b| + \sum_{i>1} |\delta y_i| = O(e^{-\rho/\varepsilon})(|\delta a| + |\delta y_1|)$$

for all $\delta z = (\delta a, \delta b, \delta y) \in T_qM_\varepsilon$. The space T_qM_ε is spanned by vectors $\delta z(T)$ such that $\delta z(t) = (\delta a(t), \delta b(t), \delta y(t))$ are solutions of the linearized equation

$$\begin{aligned} \delta a' &= \Lambda(z(t),\varepsilon)\delta a + D_z\Lambda(z(t),\varepsilon)\delta z a(t) \\ \delta b' &= \Gamma(z(t),\varepsilon)\delta b + D_z\Gamma(z(t),\varepsilon)\delta z b(t) \\ \delta y' &= \varepsilon[h(z(t),\varepsilon)\delta a b + h(z(t),\varepsilon)a\delta b + D_z h(z(t),\epsilon)\delta z a b] \end{aligned} \tag{9}$$

satisfying $\delta z(0) \in T_pM_\varepsilon$.

The idea of the proof is simple. The vectors $\delta z(T)$ with $\delta z(0) \in E_\epsilon$ form a k-dimensional subspace N_ε of T_qM_ε. Because of the estimate (4) and the exponential stretching of $\delta a(t)$, the δa-components of the vectors of N_ε dominate the remaining components by a factor proportional to $e^{\rho/\varepsilon}$. Therefore, N_ε has a complement vector in T_qM_ε with $\delta a = 0$. Integrating (9) backwards we see that for this vector $\delta a(t)$ remains $O(e^{-\rho/\varepsilon})$-small compared to the remaining components of $\delta z(t)$ for all $0 \leq t \leq T$. Integrating (9) once more forward we find that, if $\delta z \in T_qM_\varepsilon$ and $\delta a = 0$ then δy_1 dominates the remaining components of δz by a factor proportional to $e^{\rho/\varepsilon}$. A combination of this estimate with the estimate on the vectors of N_ε concludes the proof.

We now give the details of the proof. As indicated by its outline, unlike in Jones et al. [**2**], [**3**], we will estimate uniformly the ratio of the norms of components of individual tangent vectors from several linear subspaces of solutions of (9). In order to facilitate these estimates we introduce a y-dependent norm of the a and b components as follows:

We define

$$\|a\|_y = \int_{-\infty}^{0} e^{-\lambda_0 t}|e^{\Lambda(0,0,y,0)t}a|\,dt \tag{10}$$
$$\|b\|_y = \int_{0}^{\infty} e^{-\gamma_0 t}|e^{\Gamma(0,0,y,0)t}b|\,dt$$

Because of the uniform convergence of the integrals the norm

$$\|z\| = \|a\|_y + \|b\|_y + |y|$$

depends smoothly on y and is uniformly equivalent to $|z|$.

For a solution $\delta z(t) = (\delta a(t), \delta b(t), \delta y(t))$ of (9) along the solution $z(t) = (a(t), b(t), y(t))$ of (3) in Ω_Δ we have uniformly

$$\|\delta b(t+\tau)\|_{y(t+\tau)} - \|\delta b(t)\|_{y(t)} = \|\delta b(t+\tau)\|_{y(t)} - \|\delta b(t)\|_{y(t)} + O(\varepsilon\tau)\|\delta b(t)\|_{y(t)}. \tag{11}$$

Further, we have (the arguments of $\Gamma(0,0,y,0)$ dropped)

$$\begin{aligned} &\|\delta b(t+\tau)\|_{y(t)} - \|\delta b(t)\|_{y(t)} \\ &\quad \le \|e^{\tau\Gamma}\delta b(t)\|_{y(t)} - \|\delta b(t)\|_{y(t)} + \|\tau(\Gamma(z(t),\varepsilon) - \Gamma)\|_{y(t)}\|\delta b(t)\| \\ &\qquad + \tau\|\Gamma(z(t),\varepsilon)\|(\|\delta a(t)\|_{y(t)} + \|\delta b(t)\|_{y(t)} + |\delta y(t)|)(\|b(t)\|) + o(\tau). \end{aligned} \tag{12}$$

From (10)–(12) it follows

$$\frac{d}{dt}\|\delta b(t)\|_{y(t)} \le \bar\gamma\|\delta b(t)\|_{y(t)} + O(|b(t)|)(\|\delta a(t)\|_{y(t)} + |\delta y(t)|_{y(t)})$$

where

$$\bar\gamma = \gamma_0 + \sup_{\Omega_\Delta}[|\Gamma(a,b,y,\varepsilon) - \Gamma(0,0,y,0)| + O(\Delta)\|\Gamma(z,\varepsilon)\|] + 0(\varepsilon) < 0 \tag{13}$$

provided Δ and ε are sufficiently small. Similarly one proves

$$\frac{d}{dt}\|\delta a(t)\|_{y(t)} \ge \bar\lambda\|\delta a(t)\|_{y(t)} - O(|a(t)|)(\|\delta b(t)\|_{y(t)} + |\delta y(t)|_{y(t)}) \tag{14}$$

$$\frac{d}{dt}\|a(t)\|_{y(t)} \ge \bar\lambda\|a(t)\|_{y(t)}, \quad \frac{d}{dt}\|b(t)\|_{y(t)} \le \bar\gamma\|b(t)\|_{y(t)}. \tag{15}$$

for $\bar\gamma < 0$ possibly larger than in (13) and some $\bar\lambda > 0$.[1]

Since we see no danger of confusion we drop the subscript of the norm $\|\ \|$ in the sequel.

[1] Note that by employing (15) the proof of the C^0-exchange lemma (Jones et al. [**2, Proposition 3.1**], [**3, Lemma 3.1**]) can be slightly simplified as well.

We continue the proof by two estimates on the vectors of T_pM_ε. We assume that Δ has been chosen so small that (12), (15) holds with $\bar\gamma < 0 < \bar\lambda$ for sufficiently small $\varepsilon > 0$.

First, we prove

$$\frac{\|\delta b\|}{|\delta a| + |\delta y|} = O(\varepsilon^{-2r-2}) \tag{16}$$

for each $0 \neq \delta z = (\delta a, \delta b, \delta y) \in T_pM_\varepsilon$.

Each $\delta z \in E_\epsilon$ can be written in the form

$$\delta z = \alpha\widetilde{\delta z} + \beta z' \tag{17}$$

with $\widetilde{\delta z} = (\widetilde{\delta a}, \widetilde{\delta b}, \widetilde{\delta y}) \in E_\epsilon$ such that $\|\widetilde{\delta a}\| = 1$ and $z' = (a', b', y')$ from (3). By linearity, it suffices to prove the result for the case $0 \le \alpha \le 1$, $\beta = 1 - \alpha$.

Since $\|\widetilde{\delta a}\| = 1$, by assumption we have

$$|\widetilde{\delta y}| \le K\varepsilon^{-r}, \quad \|\widetilde{\delta b}\| \le K\varepsilon^{-r}. \tag{18}$$

In addition, we have

$$k\varepsilon \le |y'| \le K\varepsilon, \quad |b'| \le K \tag{19}$$

and, by (8) and (9),

$$\|a'\| = O(\|\hat a\|) \le O(e^{-\bar\lambda/\bar\varepsilon}) \tag{20}$$

for some $K > 1 > k$.

Hence, we have

$$\frac{\|\delta b\|}{\|\delta a\| + |\delta y|} \le \frac{K(1+\varepsilon^{-r})}{D}$$

with $D = \alpha + |\alpha\widetilde{\delta y} + (1-\alpha)y'| - \|a'\|$.

If $\alpha \ge \varepsilon^{r+2}$, (16) follows immediately from (20).

If $\alpha \le \varepsilon^{r+2}$, from (18)–(20) it follows

$$D \ge \left(1 - \varepsilon^{r+2}\right) k\varepsilon - \varepsilon^{r+2}K\varepsilon^{-r} - O(e^{\frac{-\bar\lambda}{\varepsilon}}) \ge \frac{1}{2}k\varepsilon - O(\varepsilon^2).$$

Hence (16) holds in this case as well.

As the second initial estimate we prove that for each $\delta z = (\delta a, \delta b, \delta y) \in T_pM_\varepsilon$ such that

$$\|\delta a\| = O(e^{-\frac{\lambda_1}{\varepsilon}})(\|\delta b\| + |\delta y|) \tag{21}$$

for some $\lambda_1 \in (0, \bar{\lambda})$ we have

$$\sum_{i>1} |\delta y_i| = O(e^{-\frac{\lambda_2}{\varepsilon}})|\delta y_1| \tag{22}$$

for some $0 < \lambda_2 < \lambda_1$.

To carry out the proof we express δz as

$$\delta z = \widetilde{\delta z} + \rho z'$$

with $\widetilde{\delta z} \in E_\epsilon,\ z'$ from (3) and $\rho \in \mathbb{R}$. By (4) and (21) we have

$$\begin{aligned}
\|\widetilde{\delta b}\| + |\widetilde{\delta y}| &\leq O(\varepsilon^{-r})\|\widetilde{\delta a}\| \leq O(\varepsilon^{-r})(\|\delta a\| + |\rho||a'|) \\
&\leq O(\varepsilon^{-r})(\|\delta a\| + |\rho| O(e^{-\bar{\lambda}/\varepsilon})) \\
&\leq O(\varepsilon^{-r})O(e^{-\lambda_1/\varepsilon})(\|\delta b\| + |\delta y| + |\rho|) \\
&\leq O(e^{-\lambda_3/\varepsilon})(\|\widetilde{\delta b}\| + |\widetilde{\delta y}| + |\rho|),
\end{aligned}$$

hence

$$|\widetilde{\delta y}| \leq \|\widetilde{\delta b}\| + |\widetilde{\delta y}| \leq \left(1 - O(e^{-\lambda_3/\varepsilon})\right)^{-1} O(e^{-\lambda_3/\varepsilon})|\rho| = O(e^{-\lambda_3/\varepsilon})|\rho|$$

for some $0 < \lambda_3 < \lambda_1$. Thus we have

$$\delta y = \rho y' + \widetilde{\delta y} = \varepsilon \rho \left[U + O(e^{-\lambda_3/\varepsilon})\right],$$

which implies (22).

Using the lemma we now turn (14), (16) and (22) into estimates for the tangent vectors along the trajectory of p and eventually for the vectors of $T_q M_\varepsilon$.

For a solution $\delta z(t) = (\delta a(t), \delta b(t), \delta y(t))$ of (9) with $0 \neq \delta z(0) \in E_\epsilon$ we denote

$$\mu(t) := \frac{\|\delta b(t)\| + |\delta y(t)|}{\|\delta a(t)\|}.$$

We have

$$\begin{aligned}
\mu' &= \frac{1}{\|\delta a\|}(\|\delta b\|' + |\delta y|') - \mu \frac{\|\delta a\|'}{\|\delta a\|} \\
&\leq \frac{1}{\|\delta a\|}\left[(\bar{\gamma} + O(\Delta))\|\delta b\| + O(\|b\|)(\|\delta b\| + |\delta y|) + O(\|b\|) \cdot \|\delta a\|\right] \\
&\quad + \frac{1}{\|\delta a\|}\varepsilon \left[O(\|b\|)(\|\delta a\| + \|\delta b\| + |\delta y|) + O(\|a\|)\|\delta b\|\right] \\
&\quad + \frac{\mu}{\|\delta a\|}\left[(-\bar{\lambda} + O(\Delta))\|\delta a\| + O(\|a\|)(|\delta y| + \|\delta b\|)\right] \\
&\leq (\alpha + O(\|a\|)\mu)\mu + O(\|b\|)
\end{aligned} \tag{23}$$

where $\alpha := \bar{\gamma} - \bar{\lambda} + O(\Delta) < 0$ for Δ sufficiently small.

From (4), (8) and the Lemma we conclude

$$\frac{\|\delta b(t)\| + |\delta y(t)|}{\|\delta a(t)\|} = O(\varepsilon^{-r} e^{\frac{\alpha}{2}t}), \qquad \text{for } 0 \le t \le T, \tag{24}$$

provided $(\delta a(0), \delta b(0), \delta y(0) \in E_\epsilon$ and ε is sufficiently small (so $T \ge l/\varepsilon$ is sufficiently large). In particular, we have

$$\frac{\|\delta b\| + |\delta y|}{\|\delta a\|} \le O(e^{-\lambda_4/\varepsilon}) \tag{25}$$

for some $\lambda_4 > 0$ and every $(\delta a, \delta b, \delta y) \in N_\varepsilon$, where $N_\epsilon = \{\delta z(T) : \delta z(t)$ is a solution of (8) with $z(0) \in E_\epsilon$.

In a similar way, we estimate

$$\nu(t) := \frac{\|\delta b(t)\|}{\|\delta a(t)\| + |\delta y(t)|}$$

for $\delta z(0) = (\delta a(0), \delta b(0), \delta y(0)) \in T_p M_\varepsilon$. As for μ, for ν we obtain the differential inequality (23). Applying the Lemma, from this inequality and (16) we obtain

$$\frac{\|\delta b(t)\|}{\|\delta a(t)\| + |\delta y(t)|} = O(\varepsilon^{-2r-2} e^{\frac{\alpha}{2}t}) \tag{26}$$

for $0 \le t \le T$ and

$$\|\delta b\| = O(e^{-\lambda_5/\varepsilon})(\|\delta a\| + |\delta y|) \tag{27}$$

for some $\lambda_5 > 0$ and all $(\delta a, \delta b, \delta y) \in T_q M_\varepsilon$.

Since $T_q M_\varepsilon$ has dimension $k+1$ and, because of (25), has a k-dimensional subspace projecting to the subspace $a = 0$ isomorphically, there exists a nonzero vector $(0, \beta, \eta) \in T_q M_\varepsilon$.

Using the Lemma backwards in a similar way as it was used forwards to obtain (24) and (26) one concludes that if $\delta z(t) = (\delta a(t), \delta b(t), \delta y(t))$ is a solution with $\delta z(T) = (0, \beta, \eta) \in T_q M_\varepsilon$ then

$$\|\delta a(t)\| = O(e^{\beta(t-T)})(\|\delta b(t)\| + |\delta y(t)|) \tag{28}$$

for some $\beta > 0$ and

$$\frac{\|\delta a(0)\|}{\|\delta b(0)\| + |\delta y(0)|} = O(e^{-\lambda_1/\varepsilon}) \tag{29}$$

for some $\lambda_1 > 0$.

By (22), for $\delta z(0) \in T_p M_\varepsilon$ satisfying (29) we have

$$\sum_{i>1} |\delta y_i(0)| = O(e^{-\lambda_2/\varepsilon}) \|\delta y_1(0)\| \tag{30}$$

Further, for $\delta z(0)$ satisfying (29), from (27) it follows

$$\begin{aligned} \|\delta a(t)\| &= O(e^{\beta(t-T)})(\|\delta b(t)\| + |\delta y(t)|) \\ &= O(e^{\beta(t-T)}) \left[O(\varepsilon^{-2r-2} e^{\frac{\alpha}{2}t})(\|\delta a(t)\| + |\delta y(t)|) + |\delta y(t)| \right] \end{aligned}$$

hence

$$(1 - r(t))\|\delta a(t)\| = O(e^{\beta(t-T)}) \left[O(\varepsilon^{-2r-2} e^{\frac{\alpha}{2}t}) + 1 \right] |\delta y(t)|$$

where $r(t) = O(e^{\beta(t-T)} \varepsilon^{-2r-2} e^{\frac{\alpha}{2}t}) \le \frac{1}{2}$ for ε sufficiently small (hence $T \ge l/\varepsilon$ large). Thus,

$$\|\delta a(t)\| = O(e^{\beta_1(t-T)}) |\delta y(t)| \tag{31}$$

for some $\beta_1 > 0$ and, by (24),

$$\|\delta b(t)\| = \left[O(\varepsilon^{-2r-2} e^{\frac{\alpha}{2}t}) \right] |\delta y(t)|. \tag{32}$$

From (31), (32) we obtain

$$\begin{aligned} |\delta y|' &= O(\varepsilon)[O(\|b\|)\|\delta a\| + O(\|a\|)\|\delta b\| + O(\|a\|\|b\|)|\delta y|] \\ &= O(\varepsilon) \Big[O(e^{\bar{\gamma} t}) O(e^{\beta_1(t-T)}) + O(e^{\bar{\lambda}(t-T)})(O(\varepsilon^{-2r-2} e^{\frac{\alpha}{2}t}) \\ &\quad + O(e^{\bar{\gamma} t}) O(e^{\bar{\lambda}(t-T)})) \Big] |\delta y| \\ &= O(e^{-\lambda_6/\varepsilon}) |\delta y| \end{aligned} \tag{33}$$

Since the integral of the square bracket of (33) is bounded on $0 \le t \le T$ independently of $\varepsilon > 0$, integrating we obtain

$$|\delta y(t)| = O(|\delta y(0)|) \tag{34}$$

Substituting (34) into (33) and integrating once more we conclude

$$|\delta y(T) - \delta y(0)| = O(\varepsilon^{-\lambda_6/\varepsilon}) |\delta y(0)|.$$

Therefore, we have

$$\delta y(T) = (1 + O(e^{-\lambda_6/\varepsilon})) \delta y(0)$$

and by (30)

$$\sum_{i>1} |\eta_i| = O(e^{-\lambda_7/\varepsilon}) |\eta_1| \tag{35}$$

for some $\lambda_7 > 0$. Summarizing, we conclude that any vector $(\delta a, \delta b, \delta y) \in T_q M_\varepsilon$ satisfies

$$\|\delta b\| \leq O(e^{-\lambda_5/\varepsilon})(\|\delta a\| + |\delta y|) \tag{36}$$

by (27) and can be written as

$$(\delta a, \delta b, \delta y) = (\widetilde{\delta a}, \widetilde{\delta b}, \widetilde{\delta y}) + q(0, \beta, \eta) \tag{37}$$

with $q \in \mathbb{R}$, where $(\widetilde{\delta a}, \widetilde{\delta b}, \widetilde{\delta y}) \in N_\varepsilon$ satisfies

$$\|\widetilde{\delta b}\| + |\widetilde{\delta y}| = O(e^{-\lambda_4/\varepsilon})\|\widetilde{\delta a}\|, \tag{38}$$

by (25) and η satisfies

$$\sum_{i>1} |\eta_i| \leq O(e^{-\lambda_7/\varepsilon})|\eta_1|. \tag{39}$$

Denote $\rho = \min\{\lambda_4, \lambda_5, \lambda_7\}$. From (37) and (38) it follows

$$\begin{aligned} |q\eta_1| &\leq |\delta y_1| + |\widetilde{\delta y_1}| \leq |\delta y_1| + O(e^{-\rho/\varepsilon})\|\widetilde{\delta a}\| \\ &= |\delta y_1| + O(e^{-\rho/\varepsilon})\|\delta a\|. \end{aligned} \tag{40}$$

Using (40), from (36)–(39) we obtain

$$\begin{aligned} \|\delta b\| + \sum_{i>1} |\delta y_i| &\leq O(e^{-\rho/\varepsilon})\|\delta a\| + \sum_{i>1} |\widetilde{\delta y_i}| + |q| \sum_{i>1} |\eta_i| \\ &\leq O(e^{-\rho/\varepsilon})[\|\delta a\| + |q\eta_1|] \\ &\leq O(e^{-\rho/\varepsilon})[\|\delta a\| + |\delta y_1|]. \end{aligned}$$

This completes the proof. □

Remark. After this paper was finished the author got acquainted with the PhD. thesis of Tin [**5**] in which the Exchange Lemma is extended to manifolds M_ε of dimension higher than $k + 1$. Our proof seems to extend to the latter case without problems.

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