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A NEW CLASS OF ANALYTIC FUNCTIONS INVOLVING CERTAIN FRACTIONAL DERIVATIVE OPERATORS

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ABSTRACT. The present paper systematically investigates a new class of functions involving certain fractional derivative operators. Characterization and distortion theorems, and other interesting properties of this class of functions are studied. Further, the modified Hadamard product of several functions belonging to this class are also investigated.

1. Introduction, Definitions and Preliminaries

The theory of fractional calculus has recently found interesting applications in the theory of analytic functions. The classical definition of Riemann-Liouville in fractional calculus operators [**5**] and their various other generalizations ([**14**]; see also [**13**]) have fruitfully been applied in obtaining, for example, the characterization properties, coefficient estimates, distortion inequalities, and convolution structures for various subclasses of analytic functions ([**7**], [**8**], [**9**], [**10**], [**11**], [**12**], [**15**] and [**16**]) and the works in the research monographs [**3**], [**6**], [**17**] and [**18**]. The purpose of the present paper is to systematically study a new class of analytic functions involving a certain fractional derivative operator (defined below by (1.2)).

In Section 1 we give the necessary details and definitions of the class of analytic functions and fractional derivative operators. Section 2 describes the characterization property for the functions belonging to the class $S_{\lambda,\mu,\eta}(\alpha,\beta,m)$ defined below, and Section 3 gives the distortion theorems. Its further properties (including those related to Hadamard product of several functions) are discussed in Sections 4 and 5, respectively. The significant relationships and relevance with other results are also invariably mentioned.

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Denote by A the class of functions $f(z)$ defined by

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \qquad (a_n \geq 0;\ n \in N), \tag{1.1}$$

which are analytic in the unit disk $U = \{z : |z| < 1\}$.

We introduce the class $S_{\lambda,\mu,\eta}(\alpha, \beta, m)$ of analytic functions $f(z)$ belonging to A and satisfying the condition:

$$\left| \frac{\Delta_{z,m}^{\lambda,\mu,\eta} f(z) - 1}{\Delta_{z,m}^{\lambda,\mu,\eta} f(z) + (1 - 2\alpha)} \right| < \beta \qquad (z \in U), \tag{1.2}$$

for

$$0 \leq \lambda < 1,\ \mu < 1,\ 0 \leq \alpha < 1,\ 0 < \beta \leq 1,\ m \in N \text{ and } \eta > \max\{\lambda, \mu\} - 1, \tag{1.3}$$

where the function $\Delta_{z,m}^{\lambda,\mu,\eta} f(z)$ is defined by

$$\Delta_{z,m}^{\lambda,\mu,\eta} f(z) = L(\lambda, \mu, \eta, m) z^{\frac{\mu}{m} - 1} D_{0,z,\frac{1}{m}}^{\lambda,\mu,\eta} f(z), \tag{1.4}$$

such that $0 \leq \lambda < 1$, $\mu < 1$, $\eta > \max\{\lambda, \mu\} - 1$ and $m \in N$; and

$$L(\lambda, \mu, \eta, m) = \frac{\Gamma(1 - \mu + m)\Gamma(1 + \eta - \lambda + m)}{\Gamma(1 + m)\Gamma(1 + \eta - \mu + m)}\,, \tag{1.5}$$

where the operator $D_{0,z,\frac{1}{m}}^{\lambda,\mu,\eta}$ is a modified fractional derivative operator of Saigo [**14**] ([**10**]), and is defined as follows:

Definition. For $0 \leq \alpha < 1$; $\beta, \eta \in R$ and $m \in N$,

$$\begin{aligned} D_{0,z,m}^{\alpha,\beta,\eta} f(z) = \frac{d}{dz} \Bigg(& \frac{z^{-m(\beta - \alpha)}}{\Gamma(1 - \alpha)} \int_0^z (z^m - t^m)^{-\alpha} f(t) \\ & \times F\Big(\beta - \alpha, 1 - \eta; 1 - \alpha; 1 - \frac{t^m}{z^m}\Big) d(t^m) \Bigg). \end{aligned} \tag{1.6}$$

The function $f(z)$ is analytic in a simply-connected region of the z-plane containing the origin, with the order

$$f(z) = O(|z|^{\varepsilon}),\ z \to 0, \tag{1.7}$$

where

$$\varepsilon > \max\{0, m(\beta - \eta)\} - m. \tag{1.8}$$

The multiplicity of $(z^m - t^m)^{-\alpha}$ in (1.6) is removed by requiring $\log(z^m - t^m)$ to be real when $(z^m - t^m) > 0$, and is assumed to be well defined in the unit disk.

The operator defined by (1.6) include the well-known Riemann-Liouville and Erdélyi-Kober operators of fractional calculus. Indeed, we have

$$D_{0,z;1}^{\alpha,\alpha;\eta} f(z) = D_z^{\alpha} f(z), \tag{1.9}$$

where D_z^{α} is the familiar Riemann-Liouville fractional derivative operator [**5**].

Also,

$$D_{0,z,1}^{\alpha,1,\eta} z f(z) = E_{0,z}^{-\alpha,-\eta} f(z) + (\alpha - \eta) E_{0,z}^{1-\alpha,\eta} f(z), \tag{1.10}$$

in terms of the Erdélyi-Kober operator [**14**] (see also [**13**]).

2. Characterization Property

Before stating and proving our main assertions, we need the following result to be used in the sequel:

Lemma 1 ([**10**]). *If* $0 \le \alpha < 1$, $m \in N$; $\beta, \eta \in R$, *and* $k > \max\{0, m(\beta - \eta)\} - m$, *then*

$$D_{0,z,m}^{\alpha,\beta,\eta} z^k = \frac{\Gamma\left(1 + \frac{k}{m}\right)\Gamma\left(1 + \eta - \beta + \frac{k}{m}\right)}{\Gamma\left(1 - \beta + \frac{k}{m}\right)\Gamma\left(1 + \eta - \alpha + \frac{k}{m}\right)} z^{k - m\beta}. \tag{2.1}$$

We investigate the characterization property for the function $f(z) \in A$ to belong to $S_{\lambda,\mu,\eta}(\alpha, \beta, m)$, thereby, obtaining the coefficient bounds. We prove the following:

Theorem 1. *Let* $f(z)$ *be defined by* (1.1). *Then,* $f(z) \in S_{\lambda,\mu,\eta}(\alpha, \beta, m)$ *if and only if*

$$\sum_{n=2}^{\infty} \Phi_n(\lambda, \mu, \eta, m)(1 + \beta) a_n \le 2\beta(1 - \alpha), \tag{2.2}$$

where

$$\Phi_n(\lambda, \mu, \eta, m) = L(\lambda, \mu, \eta, m) M(\lambda, \mu, \eta, m, n), \tag{2.3}$$

with $L(\lambda,\mu,\eta,m)$ *defined by* (1.5), *and*

$$M(\lambda,\mu,\eta,m,n)=\frac{\Gamma(1+\eta-\mu+nm)\Gamma(1+nm)}{\Gamma(1+\eta-\lambda+nm)\Gamma(1-\mu+nm)}\,, \tag{2.4}$$

under the conditions given by (1.3). *The result* (2.2) *is sharp.*

Proof. Suppose that (2.2) holds true, and let $|z|=1$. Then, on using (1.4), (1.5) and (2.1), we have

$$\begin{aligned}&\left|\Delta_{z,m}^{\lambda,\mu,\eta}f(z)-1\right|-\beta\left|\Delta_{z,m}^{\lambda,\mu,\eta}f(z)+(1-2\alpha)\right|\\&=\left|-\sum_{n=2}^{\infty}\Phi_n(\lambda,\mu,\eta,m)a_nz^{n-1}\right|-\beta\left|2(1-\alpha)-\sum_{n=2}^{\infty}\Phi_n(\lambda,\mu,\eta,m)a_nz^{n-1}\right|\\&\leq\sum_{n=2}^{\infty}\Phi_n(\lambda,\mu,\eta,m)(1+\beta)a_n-2\beta(1-\alpha)\leq 0,\end{aligned}$$

by hypothesis, where $\Phi_n(\lambda,\mu,\eta,m)$ is given by (2.3).

Therefore it follows that $f(z)\in S_{\lambda,\mu,\eta}(\alpha,\beta,m)$.

Conversely, let $f(z)$ defined by (1.1) be such that $f(z)\in S_{\lambda,\mu,\eta}(\alpha,\beta,m)$. Then, in view of (1.2), we have

$$\left|\frac{\Delta_{z,m}^{\lambda,\mu,\eta}f(z)-1}{\Delta_{z,m}^{\lambda,\mu,\eta}f(z)+(1-2\alpha)}\right|<\beta\qquad(z\in U)$$

$$=\frac{\left|\sum_{n=2}^{\infty}\Phi_n(\lambda,\mu,\eta,m)a_nz^{n-1}\right|}{\left|2(1-\alpha)-\sum_{n=2}^{\infty}\Phi(\lambda,\mu,\eta,m)a_nz^{n-1}\right|}<\beta_\phi\quad(z\in U).$$

Since $|Re(z)|\leq|z|$, for all z, we get

$$Re\left[\frac{\sum_{n=2}^{\infty}\Phi_n(\lambda,\mu,\eta,m)a_nz^{n-1}}{2(1-\alpha)-\sum_{n=2}^{\infty}\Phi_n(\lambda,\mu,\eta,m)a_nz^{n-1}}\right]<\beta\,. \tag{2.5}$$

Now choosing the values of z on the real axis, simplifying and letting $z\to 1$ through the real values, we get

$$\sum_{n=2}^{\infty}\Phi_n(\lambda,\mu,\eta,m)a_n\leq 2\beta(1-\alpha)-\beta\sum_{n=2}^{\infty}\Phi_n(\lambda,\mu,\eta,m)a_n, \tag{2.6}$$

which yields (2.2).

We also note that the assertion (2.2) is sharp and the extremal function is given by

$$f(z) = z - \frac{2\beta(1-\alpha)}{(1+\beta)\Phi_n(\lambda,\mu,\eta,m)} z^n . \tag{2.7}$$

□

Remark 1. If $\mu = \lambda = m = 1$, then in view of (1.4), (1.5) and (1.9), we have

$$\Delta_{z,1}^{1,1,\eta} f(z) = f'(z), \tag{2.8}$$

and also the class

$$S_{1,1,\eta}(\alpha,\beta,1) = P^*(\alpha,\beta), \tag{2.9}$$

where $P^*(\alpha,\beta)$ is the class of functions studied by Gupta and Jain [**2**].

Remark 2. If $\mu = \lambda$, then in view of (1.4), (1.5) and (1.9), we have

$$\Delta_{z,1}^{\lambda,\lambda,\eta} f(z) = \Gamma(2-\lambda) z^{\lambda-1} D_z^{\lambda} f(z), \tag{2.10}$$

and the class

$$S_{\lambda,\lambda,\eta}(\alpha,\beta,1) = P_\lambda^*(\alpha,\beta), \tag{2.11}$$

where $P_\lambda^*(\alpha,\beta)$ is the class studied by Srivastava and Owa [**15**]. By virtue of (2.10) and (2.11), Theorem 1 corresponds to the result [**15**, p. 177, Theorem 1].

The following consequences of Theorem 1 are worth noting:

Corollary 1. *Let the function* $f(z)$ *defined by* (1.1) *belong to the class* $S_{\lambda,\mu,\eta}(\alpha,\beta,m)$. *Then*

$$a_n \leq \frac{2\beta(1-\alpha)}{(1+\beta)\Phi_n(\lambda,\mu,\eta,m)}, \qquad \forall\, n \geq 2, \tag{2.12}$$

where $\Phi_n(\lambda,\mu,\eta,m)$ *is given by* (2.3).

Remark 3. From (2.12), we express

$$a_n \leq \frac{2\beta(1-\alpha)}{(1+\beta)\Phi_n(\lambda,\mu,\eta,m)} = K \cdot \frac{\Gamma(1+\eta-\lambda+mn)\Gamma(1-\mu+mn)}{\Gamma(1+\eta-\mu+mn)\Gamma(1+mn)} ,$$

where

$$K = \frac{2\beta(1-\alpha)\Gamma(1+m)\Gamma(1+\eta-\mu+m)}{(1+\beta)\Gamma(1-\mu+m)\Gamma(1+\eta-\lambda+m)} \leq 1,$$

which is observed to be true for $0 \le \alpha < 1$, $0 < \beta \le 1$, $0 \le \lambda \le \mu < 1$, $\eta \in R_+$ and $m \in N$.

Using the asymptotics for the ratio of gamma functions [**14**, p. 17] for finite large n, we note that

$$\frac{\Gamma(1+\eta-\lambda+mn)\Gamma(1-\mu+mn)}{\Gamma(1+\eta-\mu+mn)\Gamma(1+mn)} \sim (mn)^{-\lambda} \le n \qquad (0 \le \lambda < 1).$$

The assertion (2.12) of Corollary 1 therefore satisfies

$$a_n \le \frac{2\beta(1-\alpha)}{(1+\beta)\Phi_n(\lambda,\mu,\eta,m)} \le n, \qquad \forall\, n \ge 2, \tag{2.13}$$

for $0 \le \alpha < 1$, $0 < \beta \le 1$, $0 \le \lambda \le \mu < 1$, $\eta \in R_+$ and $m \in N$.

Thus, if T denotes the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} C_n z^n \qquad (z \in U), \tag{2.14}$$

that are analytic and univalent in U, then there do exist functions $f(z) \in S_{\lambda,\mu,\eta}(\alpha,\beta,m)$ with $0 \le \alpha < 1$, $0 < \beta \le 1$, $0 \le \lambda \le \mu < 1$, $\eta \in R_+$ and $m \in N$, not necessarily in the class T, for which the celebrated Bieberbach conjecture (now de Brange's theorem)

$$|C_n| \le n \qquad (n \ge 2), \tag{2.15}$$

holds true ([1]).

3. Distortion Theorems

Theorem 2. *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $S_{\lambda,\mu,\eta}(\alpha,\beta,m)$. *Then,*

$$|f(z)| \ge |z| - \frac{2\beta(1-\alpha)}{(1+\beta)\Phi_2(\lambda,\mu,\eta,m)}|z|^2 \tag{3.1}$$

and

$$|f(z)| \le |z| + \frac{2\beta(1-\alpha)}{(1+\beta)\Phi_2(\lambda,\mu,\eta,m)}|z|^2, \tag{3.2}$$

for $z \in U$, *where* $\Phi_2(\lambda,\mu,\eta,m)$ *is given by* (2.3) *holds under the conditions given by* (1.3).

Proof. If $f(z) \in S_{\lambda,\mu,\eta}(\alpha,\beta,m)$, then by virtue of Theorem 1, we have

$$\begin{aligned} \Phi_2(\lambda,\mu,\eta,m)(1+\beta)\sum_{n=2}^{\infty} a_n &\le \sum_{n=2}^{\infty} \Phi_n(\lambda,\mu,\eta,m) a_n (1+\beta) \\ &\le 2\beta(1-\alpha). \end{aligned} \tag{3.3}$$

This yields

$$\sum_{n=2}^{\infty} a_n \leq \frac{2\beta(1-\alpha)}{(1+\beta)\Phi_2(\lambda,\mu,\eta,m)}\,. \tag{3.4}$$

Now

$$|f(z)| \geq |z| - |z|^2 \sum_{n=2}^{\infty} a_n \geq |z| - \frac{2\beta(1-\alpha)}{(1+\beta)\Phi_2(\lambda,\mu,\eta,m)}|z|^2. \tag{3.5}$$

Also,

$$|f(z)| \leq |z| + |z|^2 \sum_{n=2}^{\infty} a_n \leq |z| + \frac{2\beta(1-\alpha)}{(1+\beta)\Phi_2(\lambda,\mu,\eta,m)}|z|^2, \tag{3.6}$$

which proves the assertions (3.1) and (3.2). □

Theorem 3. *Let the function $f(z)$ defined by* (1.1) *be in the class $S_{\lambda,\mu,\eta}(\alpha,\beta,m)$. Then,*

$$\left|D_{0,z,\frac{1}{m}}^{\lambda,\mu,\eta} f(z)\right| \geq \frac{|z|^{1-\frac{\mu}{m}}}{L(\lambda,\mu,\eta,m)}\left(1 - \frac{2\beta(1-\alpha)}{(1+\beta)}|z|\right), \tag{3.7}$$

and

$$\left|D_{0,z,\frac{1}{m}}^{\lambda,\mu,\eta} f(z)\right| \leq \frac{|z|^{1-\frac{\mu}{m}}}{L(\lambda,\mu,\eta,m)}\left(1 + \frac{2\beta(1-\alpha)}{(1+\beta)}|z|\right), \tag{3.8}$$

for $z \in U$ if $\mu \leq m$ and $z \in U - \{0\}$ if $\mu > m$, where $L(\lambda,\mu,\eta,m)$ is given by (1.5), *under the condition given by* (1.3).

Proof. Using (1.1), (1.5) and (2.1), we observe that

$$\begin{aligned}\left|L(\lambda,\mu,\eta,m)\cdot z^{\frac{\mu}{m}} D_{0,z,\frac{1}{m}}^{\lambda,\mu,\eta} f(z)\right| &= \left|z - \sum_{n=2}^{\infty} \Phi_n(\lambda,\mu,\eta,m) a_n z^n\right| \\ &\geq |z| - \sum_{n=2}^{\infty} \Phi_n(\lambda,\mu,\eta,m) a_n |z|^n \\ &\geq |z| - |z|^2 \sum_{n=2}^{\infty} \Phi_n(\lambda,\mu,\eta,m) a_n \\ &\geq |z| - |z|^2 \frac{2\beta(1-\alpha)}{(1+\beta)}\,,\end{aligned}$$

because $f \in S_{\lambda,\mu,\eta}(\alpha,\beta,m)$ by hypothesis. Thus, the assertion (3.7) is proved. The assertion (3.8) can be proved in a similar manner. □

Remark 4. When $\mu = \lambda$ and $n = 1$, then Theorems 2 and 3 give the corresponding distortion properties obtained by Srivastava and Owa [**15**, p. 179, Theorem 2].

The following consequences of Theorems 2 and 3 are worth mentioning here:

Corollary 2. *Under the hypothesis of Theorem 2, $f(z)$ is included in a disk with its centre at the origin and radius r given by*

$$r = 1 + \frac{2\beta(1-\alpha)}{(1+\beta)\Phi_2(\lambda,\mu,\eta,m)}\,. \tag{3.9}$$

Corollary 3. *Under the hypothesis of Theorem 3, $D^{\lambda,\mu,\eta}_{0,z,\frac{1}{m}} f(z)$ is included in a disk with its centre at the origin and radius R given by*

$$R = \frac{1}{L(\lambda,\mu,\eta,m)}\left\{1 + \frac{2\beta(1-\alpha)}{1+\beta}\right\}. \tag{3.10}$$

4. Further Properties of $S_{\lambda,\mu,\eta}(\alpha,\beta,m)$

We next study some interesting properties of the class $S_{\lambda,\mu,\eta}(\alpha,\beta,m)$.

Theorem 4. *Let $0 \le \lambda < 1$, $\mu < 1$, $0 \le \alpha < 1$, $0 < \beta \le 1$, $0 \le \alpha' < 1$, $0 < \beta' \le 1$, $m \in N$ and $\eta > \max\{\lambda,\mu\} - 1$. Then*

$$S_{\lambda,\mu,\eta}(\alpha,\beta,m) = S_{\lambda,\mu,\eta}(\alpha',\beta',m), \tag{4.1}$$

if and only if

$$\frac{\beta(1-\alpha)}{(1+\beta)} = \frac{\beta'(1-\alpha')}{(1+\beta')}\,. \tag{4.2}$$

Proof. First assume that $f(z) \in S_{\lambda,\mu,\eta}(\alpha,\beta,m)$ and let the condition (4.2) hold true. By using assertion (2.2) of Theorem 1, we have then

$$\sum_{n=2}^{\infty} \Phi_n(\lambda,\mu,\eta,m)a_n \le \frac{2\beta(1-\alpha)}{(1+\beta)} = \frac{2\beta'(1-\alpha')}{(1+\beta')}\,, \tag{4.3}$$

which readily shows that $f(z) \in S_{\lambda,\mu,\eta}(\alpha',\beta',m)$ (again by virtue of Theorem 1). Reversing the above steps, we can establish the other part of the equivalence of (4.1).

Conversely, the assertion (4.1) can easily be used to imply the condition (4.2) and this completes the proof of Theorem 4. □

Remark 5. For $0 \le \lambda < 1$, $\mu < 1$, $0 \le \alpha < 1$, $0 < \beta \le 1$, $m \in N$ and $\eta > \max\{\lambda,\mu\} - 1$ it follows from (4.1) that

$$S_{\lambda,\mu,\eta}(\alpha,\beta,m) = S_{\lambda,\mu,\eta}\left(\frac{1-\beta+2\alpha\beta}{1+\beta}, 1, m\right) \tag{4.4}$$

Theorem 5. *Let* $0 \leq \lambda < 1$, $\mu < 1$, $0 \leq \alpha_1 \leq \alpha_2 < 1$, $0 < \beta \leq 1$, $m \in N$ *and* $\eta > \max\{\lambda, \mu\} - 1$. *Then*

$$S_{\lambda,\mu,\eta}(\alpha_1, \beta, m) \supset S_{\lambda,\mu,\eta}(\alpha_2, \beta, m). \tag{4.5}$$

Proof. The result follows easily from Theorem 1. □

Theorem 6. *Let* $0 \leq \lambda < 1$, $\mu < 1$, $0 \leq \alpha < 1$, $0 \leq \beta_1 \leq \beta_2 \leq 1$, $m \in N$ *and* $\eta > \max\{\lambda, \mu\} - 1$. *Then*

$$S_{\lambda,\mu,\eta}(\alpha, \beta_1, m) \subset S_{\lambda,\mu,\eta}(\alpha, \beta_2, m). \tag{4.6}$$

Let$f(z) \in S_{\lambda,\mu,\eta}(\alpha, \beta_1, m)$. Then by virtue of Theorem 1 we have

$$\sum_{n=2}^{\infty} \Phi_n(\lambda, \mu, \eta, m) a_n \leq \frac{2\beta_1(1-\alpha)}{1+\beta_1} = 1 - \frac{1-\beta_1+2\alpha\beta_1}{1+\beta_1}\,. \tag{4.7}$$

Now in view of the inequalities

$$0 \leq \frac{1-\beta_2+2\alpha\beta_2}{1+\beta_2} \leq \frac{1-\beta_1+2\alpha\beta_1}{1+\beta_1} < 1, \qquad (0 \leq \alpha < 1,\ 0 < \beta_1 \leq \beta_2 \leq 1) \tag{4.8}$$

we find that

$$\sum_{n=2}^{\infty} \Phi_n(\lambda, \mu, \eta, m) a_n \leq 1 - \frac{1-\beta_2+2\alpha\beta_2}{1+\beta_2} = \frac{2\beta_2(1-\alpha)}{1+\beta_2}\,, \tag{4.9}$$

implying by virtue of Theorem 1 that $f(z) \in S_{\lambda,\mu,\eta}(\alpha, \beta_2, m)$, and so assertion (4.6) is established.

Corollary 4. *Let* $0 \leq \lambda < 1$, $\mu < 1$, $0 \leq \alpha_1 \leq \alpha_2 < 1$, $0 \leq \beta_1 \leq \beta_2 \leq 1$, $m \in N$ *and* $\eta > \max\{\lambda, \mu\} - 1$. *Then*

$$S_{\lambda,\mu,\eta}(\alpha_2, \beta_1, m) \subset S_{\lambda,\mu,\eta}(\alpha_1, \beta_1, m) \subset S_{\lambda,\mu,\eta}(\alpha_1, \beta_2, m). \tag{4.10}$$

Theorem 7. *Let* $0 \leq \lambda_1 \leq \lambda_2 \leq \mu < 1$, $0 \leq \alpha < 1$, $0 < \beta \leq 1$, $m \in N$ *and* $\eta \in R_+$. *Then*

$$S_{\lambda_1,\mu,\eta}(\alpha, \beta, m) \supset S_{\lambda_2,\mu,\eta}(\alpha, \beta, m). \tag{4.11}$$

Proof. Suppose $f(z)$ defined by (1.1) belongs to $S_{\lambda_2,\mu,\eta}(\alpha, \beta, m)$. Applying Theorem 1, we obtain

$$\sum_{n=2}^{\infty} \Phi_n(\lambda_1 \mu, \eta, m)(1+\beta) a_n \leq \sum_{n=2}^{\infty} \Phi_n(\lambda_2, \mu, \eta, m)(1+\beta) a_n \leq 2\beta(1-\alpha), \tag{4.12}$$

since

$$1 \le \Phi_n(\lambda_1, \mu, \eta, m) \le \Phi_n(\lambda_2, \mu, \eta, m) \le n, \tag{4.13}$$

for $0 \le \lambda_1 \le \lambda_2 \le \mu < 1$, $m \in N$, $n \ge 2$, and $\eta \in R_+$.

The validity of the inequalities in (4.13) is observed from the following:

In view of the arguments in Remark 3, we note that

$$1 \le \frac{\Gamma(1-\mu+m)\Gamma(1+\eta-\lambda_1+m)}{\Gamma(1+m)\Gamma(1+\eta-\mu+m)} \le n \qquad (n \ge 2)$$

for $0 < \lambda_1 \le \mu < 1$, $m \in N$ and $\eta \in R_+$.

Also,

$$1 \le \frac{\Gamma(1+\eta-\mu+mn)\Gamma(1+mn)}{\Gamma(1+\eta-\lambda_1+mn)\Gamma(1-\mu+mn)} \sim (mn)^{\lambda_1} \le n \qquad (n \ge 2)$$

for $0 < \lambda_1 \le \mu < 1$, $m \in N$ and $\eta \in R_+$.

Similar bound hold true for the above gamma quotients (wherein λ_1 is replaced by λ_2) under the conditions that $0 < \lambda_2 \le \mu < 1$, $m \in N$ and $\eta \in R_+$.

The dominant expressions of $\Phi_n(\lambda_1, \mu, \eta, m)$ and $\Phi_n(\lambda_2, \mu, \eta, m)$ thus satisfy

$$1 \le \frac{\Gamma(1+\eta-\lambda_1+m)}{\Gamma(1+\eta-\lambda_1+mn)} \le \frac{\Gamma(1+\eta-\lambda_2+m)}{\Gamma(1+\eta-\lambda_2+mn)} \le n \qquad (n \ge 2),$$

provided that $0 \le \lambda_1 \le \lambda_2 \le \mu < 1$, $m \in N$ and $\eta \in R_+$.

Hence, from (4.12) it follows that $f(z) \in S_{\lambda,\mu,\eta}(\alpha, \beta, m)$ (in view of Theorem 1), which proves (4.11) of Theorem 7. □

We now recall the following known results:

Lemma 2 ([**2**]). *A function $f(z)$ defined by* (1.1) *is in the class $P^*(\alpha, \beta)$ if and only if*

$$\sum_{n=2}^{\infty} n(1+\beta)a_n \le 2\beta(1-\alpha). \tag{4.14}$$

The result is sharp, the extremal function being

$$f(z) = z - \frac{2\beta(1-\alpha)}{n(1+\beta)} z^n \qquad (n \in N). \tag{4.15}$$

Lemma 3 ([**15**, p. 177, Theorem 1]). *A function $f(z)$ defined by* (1.1) *is in the class $P^*_\lambda(\alpha, \beta)$ if and only if*

$$\sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} (1+\beta)a_n \le 2\beta(1-\alpha). \tag{4.16}$$

Theorem 8. *Let $0 \le \lambda \le \mu < 1$, $0 \le \alpha < 1$, $0 \le \beta \le 1$, $m \in N$ and $\eta \in R_+$. Then*

$$P^*(\alpha,\beta) \subset P^*_\lambda(\alpha,\beta) \subset S_{\lambda,\mu,\eta}(\alpha,\beta,m), \tag{4.17}$$

where $P^*(\alpha,\beta)$ and $P^*_\lambda(\alpha,\beta)$ are the classes defined by (2.9) and (2.11), respectively.

Proof. Let $f(z)$ defined by (1.1) belong to the class $P^*(\alpha,\beta)$. Then, by using Lemma 2 and Lemma 3, we have

$$\begin{aligned}\sum_{n=2}^{\infty}\Phi_n(\lambda,\mu,\eta,m)(1+\beta)a_n &\le \sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}(1+\beta)a_n \\ &\le \sum_{n=2}^{\infty} n(1+\beta)a_n \le 2\beta(1+\alpha),\end{aligned} \tag{4.18}$$

since

$$1 \le \Phi_n(\lambda,\mu,\eta,m) \le \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} \le n, \tag{4.19}$$

for $0 \le \lambda \le \mu < 1$, $m \in N$, $\eta > R_+$, $n \ge 2$, and $0 \le \alpha < 1$, $0 < \beta \le 1$. □

Now (4.18) in conjunction with Theorem 1 yields the desired result (4.17).

5. Results Involving Hadamard Product

In this section we study interesting properties and theorems for the class of functions $S_{\lambda,\mu,\eta}(\alpha,\beta,m)$ involving the modified Hadamard product of several functions. Let $f(z)$ be defined by (1.1) and let

$$g(z) = z - \sum_{n=2}^{\infty} b_n z^n \qquad (b_n \ge 0). \tag{5.1}$$

Then the modified Hadamard product of $f(z)$ and $g(z)$ is defined by

$$f * g(z) = z - \sum_{n=2}^{\infty} a_n b_n z^n. \tag{5.2}$$

The following result reveals an interesting property of the modified Hadamard product of several functions.

Theorem 9. *Let the functions $f_1(z)$, $f_2(z)$, ... , $f_r(z)$ defined by*

$$f_i(z) = z - \sum_{n=2}^{\infty} C_{n,i}\, z^n \qquad (C_{n,i} \geq 0), \tag{5.3}$$

be in the class $S_{\lambda,\mu,\eta}(\alpha_i, \beta_i, m)$; $i = 1, 2, \ldots, r$, respectively. Also, let

$$\Phi_2(\lambda, \mu, \eta, m)\Big(1 + \min_{1\leq i\leq r} \beta_i\Big) \geq 2. \tag{5.4}$$

Then,

$$f_1 * f_2 * \cdots * f_r(z) \in S_{\lambda,\mu,\eta}\Big(\prod_{i=1}^{r} \alpha_i, \prod_{i=1}^{r} \beta_i, m\Big). \tag{5.5}$$

Proof. By hypothesis, $f_i(z) \in S_{\lambda,\mu,\eta}(\alpha_i, \beta_i, m)$, $\forall\, i = 1, 2, \ldots, r$; therefore, by Theorem 1, we have

$$\sum_{n=2}^{\infty} \Phi_n(\lambda, \mu, \eta, m)(1 + \beta_i)C_{n,i} \leq 2\beta_i(1 - \alpha_i), \qquad \forall\, i = 1, 2, \ldots, r; \tag{5.6}$$

and

$$\sum_{n=2}^{\infty} C_{n,i} \leq \frac{2\beta_i(1 - \alpha_i)}{(1 + \beta_i)\Phi_2(\lambda, \mu, \eta, m)}, \qquad \forall\, i = 1, 2, \ldots, r. \tag{5.7}$$

For β_i satisfying $0 < \beta_i \leq 1$ $(i = 1, \ldots, r)$, we observe that

$$\sum_{n=2}^{\infty}\Phi_n(\lambda, \mu, \eta, m)\Big[1 + \prod_{i=1}^{r} \beta_i\Big] \prod_{i=1}^{r} C_{n,i} \leq \sum_{n=2}^{\infty} \Phi_n(\lambda, \mu, \eta, m)(1 + \beta_r) \prod_{i=1}^{r} C_{n,i}$$
$$= \sum_{n=2}^{\infty}\Big\{\Phi_n(\lambda, \mu, \eta, m)(1 + \beta_r)C_{n,r}\Big\} \prod_{i=1}^{r-1} C_{n,i}.$$

Using (5.6) for any fixed $i = r$, and (5.7) for the rest, it follows that

$$\begin{aligned}\sum_{n=2}^{\infty}\Phi_n(\lambda, \mu, \eta, m)\Big[1 + \prod_{i=1}^{r} \beta_i\Big] \prod_{i=1}^{r} C_{n,i} &\\ \leq \frac{[2\beta_r(1 - \alpha_r)]\Big[2^{r-1} \prod_{i=1}^{r-1} \{\beta_i(1 - \alpha_i)\}\Big]}{\prod_{i=1}^{r-1} (1 + \beta_i)\, \{\Phi_2(\lambda, \mu, \eta, m)\}^{r-1}} &\\ \leq 2\prod_{i=1}^{r} \beta_i\Big[1 - \prod_{i=1}^{r} \alpha_i\Big] \left[\frac{2}{\Phi_2(\lambda, \mu, \eta, m)\Big\{1 + \min_{1\leq i\leq r} \beta_i\Big\}}\right]^{r-1} &\\ \leq 2\prod_{i=1}^{r} \beta_i\Big[1 - \prod_{i=1}^{r} \alpha_i\Big],\end{aligned} \tag{5.8}$$

because in view of (5.4):

$$0 < \frac{2}{\Phi_2(\lambda,\mu,\eta,m)\left[1+\min\limits_{1\le i\le r}\beta_i\right]} \le 1. \tag{5.9}$$

Hence with the aid of Theorem 1, the assertion (5.5) is proved. □

For $\alpha_i = \alpha$ and $\beta_i = \beta$, $i = 1, 2, \ldots, r$; Theorem 1 yields the following result:

Corollary 5. *Let each of the functions $f_1(z)$, $f_2(z)$, ... , $f_r(z)$ defined by* (5.3) *be in the same class $S_{\lambda,\mu,\eta}(\alpha,\beta,m)$. Also, let*

$$\Phi_2(\lambda,\mu,\eta,m)(1+\beta) \ge 2. \tag{5.10}$$

Then

$$f_1 * f_2 * \cdots * f_r(z) \in S_{\lambda,\mu,\eta}(\alpha^r,\beta^r,m). \tag{5.11}$$

Theorem 11. *Let the functions $f_i(z)$ $(i = 1, 2)$, defined by* (5.3) *be in the class $S_{\lambda,\mu,\eta}(\alpha,\beta,m)$. Then*

$$f_1 * f_2(z) \in S_{\lambda,\mu,\eta}(\sigma,\beta,m), \tag{5.12}$$

where

$$\sigma = \sigma(\alpha,\beta,\lambda,\mu,\eta,m) = 1 - \frac{2\beta(1-\alpha)^2}{(1+\beta)\Phi_2(\lambda,\mu,\eta,m)}\,. \tag{5.13}$$

The result is sharp.

Proof. In view of Theorem 1, we need to prove the following:

$$\sum_{n=2}^{\infty} \frac{\Phi_n(\lambda,\mu,\eta,m)(1+\beta)C_{n,1}C_{n,2}}{2\beta(1-\sigma)} \le 1, \tag{5.14}$$

where σ is function given by (5.13).

By Cauchy-Schwarz inequality it follows from (2.2) of Theorem 1 that

$$\sum_{n=2}^{\infty} \frac{(1+\beta)\Phi_n(\lambda,\mu,\eta,m)}{2\beta(1-\alpha)} \cdot \sqrt{C_{n,1}C_{n,2}} \le 1\,. \tag{5.15}$$

Let us find largest σ such that

$$\begin{aligned}\sum_{n=2}^{\infty} \frac{(1+\beta)\Phi_n(\lambda,\mu,\eta,m)}{2\beta(1-\sigma)}C_{n,1}C_{n,2} \\ \le \sum_{n=2}^{\infty} \frac{(1+\beta)\Phi_n(\lambda,\mu,\eta,m)}{2\beta(1-\alpha)}\sqrt{C_{n,1}C_{n,2}}\,,\end{aligned} \tag{5.16}$$

which implies

$$\sqrt{C_{n,1}C_{n,2}} \le \frac{1-\sigma}{1-\alpha} \quad \text{with } n \ge 2. \tag{5.17}$$

In view of (5.15) it is sufficient to find largest Ψ such that

$$\frac{2\beta(1-\alpha)}{(1+\beta)\Phi_n(\lambda,\mu,\eta,m)} \le \frac{1-\sigma}{1-\alpha}\,, \tag{5.18}$$

which yields

$$\sigma \le 1 - \frac{2\beta(1-\alpha)^2}{(1+\beta)\Phi_n(\lambda,\mu,\eta,m)}\,. \tag{5.19}$$

That is

$$\sigma \le 1 - \frac{2\beta(1-\alpha)^2}{(1+\beta)}\theta_1(n), \tag{5.20}$$

where

$$\theta_1(n) = \frac{1}{\Phi_n(\lambda,\mu,\eta,m)}\,. \tag{5.21}$$

Noting that $\theta_1(n)$ is a decreasing function of n $(n \ge 2)$ for fixed $\lambda \cdot \mu, \eta, m$ satisfying $0 \le \lambda \le \mu < 1$, $m \in N$ and $\eta \in R_+$ since we have for large n:

$$\frac{\theta_1(n+1)}{\theta_1(n)} \sim \frac{(n+1)^{-\lambda}}{n^{-\lambda}} = \left(1 - \frac{1}{1+n}\right)^{\lambda} \le 1$$

for $n \ge 2$, $0 \le \lambda < 1$; and under the aforementioned constraints.

Hence

$$\sigma \le \sigma(\alpha,\beta,\lambda,\eta,m) = 1 - \frac{2\beta(1-\alpha)^2}{(1+\beta)}\theta_1(2). \tag{5.22}$$

In view of (5.14), (5.18), (5.20) and (5.22), the assertion (5.12) is hence proved.□

Lastly, by considering the functions

$$f_i(z) = z - \frac{2\beta(1-\alpha)}{(1+\beta)\Phi_2(\lambda,\mu,\eta,m)}z^2, \qquad (i = 1,2), \tag{5.23}$$

it can be shown that the result is sharp.

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