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DOMAINS WITH CONVEX HYPERBOLIC RADIUS

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Abstract. The hyperbolic radius of a domain on the Riemann sphere is equal to the reciprocal of the density of the hyperbolic metric. In the present paper, it is proved that the hyperbolic radius is a convex function if and only if the complement of the domain is a convex set.

1. Introduction

A domain D on the Riemann sphere $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is said to be hyperbolic if $\overline{\mathbb{C}} \setminus D$ contains at least three points. For $w \in D$, the hyperbolic radius $R(D, w)$ is defined by $R(D, w) = |f'(0)|$, where f is a covering map of the unit disk $\mathbb{U} = \{z : |z| < 1\}$ onto D with $f(0) = w$. Hyperbolic radius is closely related to the maximal solution of Liouville's equation and metrics of constant negative curvature [**1**].

Minda and Wright [**10**] established that the hyperbolic radius $R(D, w)$ of a convex hyperbolic domain $D \subset \mathbb{C}$ is a concave function of w, thus strengthening the theorem of Caffarelli and Friedman [**2**]. Later Kim and Minda [**6**] proved that the concavity of $R(D, w)$ is equivalent to the convexity of D. Here and in what follows we do not assume that the domain of a convex or concave function is a convex set.

The aim of the present paper is to describe domains with convex hyperbolic radius in geometric terms. The method from [**10**] does not seem to work in this case. By employing a different technique, we shall show that $R(D, w)$ is convex in $D \setminus \{\infty\}$ if and only if $\mathbb{C} \setminus D$ is a convex set.

2. Preliminary Results

Let $\mathbf{M}$ denote the set of all univalent meromorphic functions in the unit disk $\mathbb{U}$ with $f(0) = 0$, $f'(0) > 0$. The class $\mathbf{A}$ is defined to be a collection of all members of $\mathbf{M}$ that are analytic in $\mathbb{U}$. Define $\mathbf{M}^c = \{f \in \mathbf{M} : \mathbb{C} \setminus f(\mathbb{U}) \text{ is convex}\}$. Let $\mathbf{P}$ denote the set of all analytic functions in $\mathbb{U}$ with positive real part and $f(0) = 1$.

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For $f \in \mathbf{M}$ and $p \in \overline{\mathbb{U}} \setminus \{0\}$, define

$$[f,p](z) = \frac{2\bar{p}z}{1-\bar{p}z} - \frac{2p}{z-p} - \left(1 + \frac{zf''(z)}{f'(z)}\right).$$

For $f \in \mathbf{M} \setminus \mathbf{A}$, let $\hat{f} = [f, f^{-1}(\infty)]$, where f^{-1} is the inverse of f.

Lemma 2.1. *Function $[f,p]$ is analytic in $\mathbb{U}$ if and only if either $f \in \mathbf{M} \setminus \mathbf{A}$ and $p = f^{-1}(\infty)$ or $f \in \mathbf{A}$ and $|p| = 1$.*

Proof. The 'only if' part of the statement is trivial. In case of $f \in \mathbf{A}$ and $|p| = 1$, function $[f,p]$ is analytic in $\mathbb{U}$ by its definition. Let $f \in \mathbf{M} \setminus \mathbf{A}$, $p = f^{-1}(\infty)$, and $c = \lim_{z \to p} f(z)(z-p)$. Then asymptotic expansions

$$f'(z) = -\frac{c}{(z-p)^2} + O(1), \quad f''(z) = \frac{2c}{(z-p)^3} + O(1) \quad (z \to p)$$

hold. Therefore,

$$[f,p](z) = -\frac{2p}{z-p} - \frac{2cp(z-p)^{-3}}{-c(z-p)^{-2}} + O(1) = O(1) \quad (z \to p),$$

which implies the analyticity of $[f,p]$. This proves the lemma. □

Lemma 2.2. *(a) If $f \in \mathbf{M}^c \setminus \mathbf{A}$, then $\hat{f} \in \mathbf{P}$.*
(b) If $f \in \mathbf{M}^c \cap \mathbf{A}$, then $[f,p] \in \mathbf{P}$ for some $p \in \partial\mathbb{U}$.

Proof. (a) Let $p = f^{-1}(\infty)$. Then $p \in \mathbb{U} \setminus \{0\}$. For $0 < p < 1$ statement (a) was proved by Pfaltzgraff and Pinchuk [**11**], see also [**8**]. For arbitrary $p \in \mathbb{U} \setminus \{0\}$, let $g(z) = \frac{|p|}{p} f(pz/|p|)$. It is easy to see that $g \in \mathbf{M}^c \setminus \mathbf{A}$, $g(|p|) = \infty$, and $\hat{f}(z) = \hat{g}(pz/|p|)$. Thus $\hat{f} \in \mathbf{P}$.

(b) For $n > \operatorname{dist}(0, \mathbb{C} \setminus f(\mathbb{U}))$ let $D_n = f(\mathbb{U}) \cup \{z : |z| > n\}$. Then $\mathbb{C} \setminus D_n$ is convex. There is a unique function $f_n \in \mathbf{M}^c \setminus \mathbf{A}$ that maps $\mathbb{U}$ onto D_n. Since $D_{n+1} \subset D_n$, the function $f_n^{-1} \circ f_{n+1}$ maps $\mathbb{U}$ into itself. By Schwarz Lemma, $|f_n^{-1}(f_{n+1}(z))| \le |z|$ for all $z \in \mathbb{U}$. Letting $z = f_{n+1}^{-1}(\infty)$ yields $|f_n^{-1}(\infty)| \le |f_{n+1}^{-1}(\infty)|$. Taking a subsequence, we can assume that $\{f_n^{-1}(\infty)\}$ converge to some point p of $\overline{\mathbb{U}} \setminus \{0\}$. By Carathéodory kernel theorem [**5**, p.56] $f_n \to f$ and $\hat{f}_n \to [f,p]$ locally uniformly in $\mathbb{U} \setminus \{p\}$. Since $\hat{f}_n \in \mathbf{P}$, it follows that $[f,p] \in \mathbf{P}$. Lemma 2.1 implies $|p| = 1$.

The proof is complete. □

Remark 2.3. Functions f with $\hat{f} \in \mathbf{P}$ have been also considered by Miller [**9**] and Royster [**12**].

3. Main Result

Define the cone

$$C(\zeta, \theta, \beta) = \{\zeta + \rho e^{i\varphi} : \rho > 0, |\varphi - \theta| < \beta/2\}$$

with opening angle β at the point $\zeta \in \mathbb{C}$.

Theorem 3.1. *(a) Let $D \subset \overline{\mathbb{C}}$ be a hyperbolic domain. If $\mathbb{C} \setminus D$ is convex, then for any $w \in D \setminus \{\infty\}$ and $\varphi \in \mathbb{R}$*

$$\frac{d^2}{dt^2} R(D, w + te^{i\varphi})\big|_{t=0} \geq 0. \tag{1}$$

Equality is attained in (1) *if and only if one of the following conditions holds:*

(i) *D is a half-plane;*
(ii) *$D = C(\zeta, \theta, \beta)$, where $\beta > \pi$ and $e^{-i\varphi}(w - \zeta) \in \mathbb{R}$.*

(b) Let D be a hyperbolic domain such that (1) *holds for all $w \in D \setminus \{\infty\}$ and $\varphi \in \mathbb{R}$. Then $\mathbb{C} \setminus D$ is convex.*

Proof. (a) Without loss of generality we may assume that $w = \varphi = 0$ and $R(D, 0) = 1$. Then (1) can be rewritten as

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{-2}(R(D, \varepsilon) + R(D, -\varepsilon) - 2) \geq 0. \tag{2}$$

There exists a unique $f \in \mathbf{M}^c$ that maps $\mathbb{U}$ onto D. Denote its Taylor coefficients at zero by c_k $(k = 1, 2, \dots)$. Since $|c_1| = R(D, 0) = 1$, it follows that $c_1 = 1$. For $0 < \varepsilon < \operatorname{dist}(0, \partial D)$, let $z_1 = f^{-1}(\varepsilon)$ and $z_2 = f^{-1}(-\varepsilon)$. Combining expansions $z_1 + c_2 z_1^2 + o(\varepsilon^2) = \varepsilon$ and $z_2 + c_2 z_2^2 + o(\varepsilon^2) = -\varepsilon$ $(\varepsilon \downarrow 0)$ yields

$$z_1 + z_2 = -c_2(z_1^2 + z_2^2) + o(\varepsilon^2) \quad (\varepsilon \downarrow 0). \tag{3}$$

Since $|1 + w| = 1 + \operatorname{Re} w + (\operatorname{Im} w)^2/2 + o(|w|^2)$ $(w \to 0)$, we have

$$\begin{aligned} |f'(z_i)| = 1 &+ \operatorname{Re}(2c_2 z_i + 3c_3 z_i^2) \\ &+ 2(\operatorname{Im}(c_2 z_i))^2 + o(\varepsilon^2) \quad (\varepsilon \downarrow 0,\ i = 1, 2). \end{aligned} \tag{4}$$

Combining (4), (3), and relations $z_1 = \varepsilon + o(\varepsilon)$, $z_2 = -\varepsilon + o(\varepsilon)$ $(\varepsilon \downarrow 0)$ yields

$$|f'(z_1)| + |f'(z_2)| = 2 + 2\varepsilon^2 \operatorname{Re}(3c_3 - 2c_2^2) + 4\varepsilon^2(\operatorname{Im} c_2)^2 + o(\varepsilon^2) \quad (\varepsilon \downarrow 0),$$

$$\begin{aligned} R(D, \varepsilon) + R(D, -\varepsilon) &= |f'(z_1)|(1 - |z_1|^2) + |f'(z_2)|(1 - |z_2|^2) \\ &= 2 + 2\varepsilon^2(\operatorname{Re}(3c_3 - 2c_2^2) + 2(\operatorname{Im} c_2)^2 - 1) + o(\varepsilon^2) \quad (\varepsilon \downarrow 0). \end{aligned}$$

Because

$$\begin{aligned} &\lim_{\varepsilon \downarrow 0} \varepsilon^{-2}(R(D, \varepsilon) + R(D, -\varepsilon) - 2) \\ &\quad = 2(\operatorname{Re}(3c_3 - 2c_2^2) + 2(\operatorname{Im} c_2)^2 - 1) = 2(3\operatorname{Re}(c_3 - c_2^2) + |c_2|^2 - 1), \end{aligned} \tag{5}$$

inequality (2) is equivalent to

$$3\operatorname{Re}(c_3 - c_2^2) + |c_2|^2 \geq 1. \tag{6}$$

By Lemma 2.2 there is $p \in \overline{\mathbb{U}} \setminus \{0\}$ such that $[f, p] \in \mathbf{P}$. The Taylor series for $[f, p]$ at 0 has the form

$$[f, p](z) = 1 + 2(\bar{p} + 1/p - c_2)z + 2(\bar{p}^2 + 1/p^2 + 2c_2^2 - 3c_3)z^2 + \cdots.$$

Let $\bar{p}+1/p = re^{i\varphi}$, where $\varphi\in\mathbb{R}$ and $r=|\bar{p}+1/p| = |p|+1/|p|\ge 2$. It follows from Carathéodory's lemma [**4**, p.41] that $|re^{i\varphi}-c_2|\le 1$. Let $c_2 = re^{i\varphi}+\rho e^{i\psi}$, where $\psi\in\mathbb{R}$, $0\le\rho\le 1$. The identity

$$\bar{p}^2+\frac{1}{p^2} = \left(\bar{p}+\frac{1}{p}\right)^2 - 2\frac{\bar{p}}{p} = r^2e^{2i\varphi}-2e^{2i\varphi} = (r^2-2)e^{2i\varphi}$$

implies

$$\begin{aligned}[f,p](z) &= 1-2\rho e^{i\psi}z + 2((r^2-2)e^{2i\varphi}+2c_2^2-3c_3)z^2+\cdots \\ &= 1+a_1z+a_2z^2+\cdots .\end{aligned} \tag{7}$$

It is easy to see that for $\alpha\in\mathbb{R}$

$$\tau_\alpha(\zeta) = \frac{(1+e^{i\alpha})\zeta+1-e^{i\alpha}}{(1-e^{i\alpha})\zeta+1+e^{i\alpha}}$$

is a conformal automorphism of the right half-plane which fixes 1. Hence the function

$$\tau_\alpha([f,p](z)) = 1+e^{i\alpha}a_1z+e^{i\alpha}(a_2-(1-e^{i\alpha})a_1^2/2)z^2+\cdots \tag{8}$$

belongs to **P**. It follows from Carathéodory's lemma that

$$\operatorname{Re}(a_2-(1-e^{i\alpha})a_1^2/2))\le 2.$$

Passing to the supremum over all $\alpha\in\mathbb{R}$ yields

$$\operatorname{Re}(a_2-a_1^2/2)+|a_1|^2/2\le 2$$

which is equivalent to

$$(r^2-2)\cos 2\varphi+\operatorname{Re}(2c_2^2-3c_3)-\rho^2\cos 2\psi+\rho^2\le 1.$$

Therefore,

$$\begin{aligned}&3\operatorname{Re}(c_3-c_2^2)+|c_2|^2\ge \\ &\ge |c_2|^2-\operatorname{Re}c_2^2+(r^2-2)\cos 2\varphi+\rho^2(1-\cos 2\psi)-1 \\ &= 2(\operatorname{Im}c_2)^2+(r^2-2)(1-2\sin^2\varphi)+2\rho^2\sin^2\psi-1 \\ &= 2(r\sin\varphi+\rho\sin\psi)^2-2(r^2-2)\sin^2\varphi+2\rho^2\sin^2\psi+r^2-3 \\ &= 4(\sin\varphi+\rho\sin\psi)^2+4\rho(r-2)\sin\varphi\sin\psi+r^2-3 \\ &\ge -4(r-2)+r^2-3=(r-2)^2+1\ge 1.\end{aligned} \tag{9}$$

This proves (6), and (1) follows.

Suppose that equality is attained in (1). Then (9) also becomes an equality. This implies $r=2$ and $|p|=1$. Since $|p|=1$, it follows from Lemma 2.1 that $\infty\notin D$. By equality statement in Carathéodory's lemma [**4**, p.41], there are $\alpha\in\mathbb{R}$ and $\mu\in[0,1]$ such that

$$\tau_\alpha([f,p](z)) = \mu\frac{1+e^{i\alpha/2}z}{1-e^{i\alpha/2}z}+(1-\mu)\frac{1-e^{i\alpha/2}z}{1+e^{i\alpha/2}z}.$$

Hence,

$$[f,p](z) = \frac{1+2(2\mu-1)\cos(\alpha/2)z+z^2}{1+2(2\mu-1)i\sin(\alpha/2)z-z^2} = \frac{1+2vz+z^2}{1+2iuz-z^2}, \tag{10}$$

where $u = (2\mu-1)\sin(\alpha/2)$, $v = (2\mu-1)\cos(\alpha/2)$. Combining (7) and (10) yields $\rho e^{i\psi} = ui - v$ and $u = \rho\sin\psi$. Since (9) is supposed to be an equality, we have $\sin\varphi = -\rho\sin\psi = -u$ which implies $p = e^{-i\varphi} \in \{p_1, p_2\}$, where $p_n = (-1)^n\sqrt{1-u^2} + iu$, $n = 1, 2$. Recalling the definition of $[f,p]$ and using $|p| = 1$ we obtain

$$(\log f'(z))' = \frac{f''(z)}{f'(z)} = \frac{4}{p-z} + 2\,\frac{z+v-iu}{z^2-2iuz-1}. \tag{11}$$

It is easy to see that $z^2 - 2iuz - 1 = (z-p_1)(z-p_2)$.

Case 1. $|u| < 1$. Let $\gamma = v/\sqrt{1-u^2}$. Integrating (11) yields

$$\begin{aligned}
\log f'(z) &= -4\int_0^z \frac{d\zeta}{\zeta-p} + \int_0^z \frac{2\zeta-2iu}{\zeta^2-2iu\zeta-1}\,d\zeta + 2v\int_0^z \frac{d\zeta}{(\zeta-p_1)(\zeta-p_2)} \\
&= -4\log(1-z/p) + \log(1+2iuz-z^2) - \gamma\log\frac{1-z/p_1}{1-z/p_2}, \\
f'(z) &= \left(\frac{1-z/p_1}{1-z/p_2}\right)^{-\gamma}\frac{(1-z/p_1)(1-z/p_2)}{(1-z/p)^4}.
\end{aligned}$$

Recall that p is equal to either p_1 or p_2. If $p = p_1$, then

$$\begin{aligned}
f'(z) &= \left(\frac{1-z/p_2}{1-z/p_1}\right)^{1+\gamma}(1-z/p_1)^{-2}, \\
f(z) &= \frac{1}{(4+2\gamma)\sqrt{1-u^2}}\left\{1 - \left(\frac{1-z/p_2}{1-z/p_1}\right)^{2+\gamma}\right\}.
\end{aligned}$$

This implies $D = C\left(\left((4+2\gamma)\sqrt{1-u^2}\right)^{-1}, \theta, (2+\gamma)\pi\right)$ for some $\theta \in \mathbb{R}$.

If $p = p_2$, then

$$\begin{aligned}
f'(z) &= \left(\frac{1-z/p_1}{1-z/p_2}\right)^{1-\gamma}(1-z/p_2)^{-2}, \\
f(z) &= \frac{1}{(4-2\gamma)\sqrt{1-u^2}}\left\{\left(\frac{1-z/p_1}{1-z/p_2}\right)^{2-\gamma} - 1\right\}.
\end{aligned}$$

Thus $D = C\left(\left((2\gamma-4)\sqrt{1-u^2}\right)^{-1}, \theta, (2-\gamma)\pi\right)$ for some $\theta \in \mathbb{R}$. Taking into account that $\gamma \in [-1, 1]$, we conclude that domain D is a cone with opening angle not less than π at some point on the real axis. Therefore, D satisfies one of the conditions (i), (ii).

Case 2. $|u| = 1$. This implies $p_1 = p_2 = p = iu$ and $v = 0$. Integrating (11) yields

$$\log f'(z) = -2\log(1 - z/p),$$
$$f'(z) = \frac{1}{(1 - z/p)^2},$$
$$f(z) = \frac{z}{1 - z/p}.$$

In this case domain D satisfies (i).

It remains to verify that each of the conditions (i), (ii) implies equality in (1). This follows directly from the identity

$$R\left(C(\zeta, \theta, \beta), \zeta + \rho e^{i(\theta+\delta)}\right) = \frac{2\beta\rho}{\pi} \cos\frac{\pi\delta}{\beta},$$

which holds for all $\rho > 0$ and $|\delta| < \beta/2$. Claim (a) is proved.

(b) Let D be such a domain that (1) holds for all $a \in D \setminus \{\infty\}$ and $\varphi \in \mathbb{R}$. If $\mathbb{C} \setminus D$ is not convex, then there exist such points $a, b \in \mathbb{C} \setminus D$ that $ta + (1-t)b \in D$ for $0 < t < 1$. The function $R(D, ta + (1 - t)b)$ is convex on the interval $(0, 1)$ and vanishes in its ends. Therefore, $R(D, ta + (1 - t)b) \leq 0$ for $0 < t < 1$. This contradicts the definition of hyperbolic radius.

The proof is complete. □

4. Concluding Remarks

The fact that $R(D, w)$ is concave for convex D [**10**] leads to a non-covering theorem for convex univalent functions [**7**]. From Theorem 3.1, a covering theorem for convex meromorphic functions can be derived as follows. Consider function $f \in \mathbf{M}^c$ that has Taylor expansion $f(z) = z + c_2 z^2 + \ldots$ at the origin. One can easily show [**7**, p. 146] that

$$R(D, w) = 1 + 2\operatorname{Re}(c_2 w) + o(|w|) \quad (w \to 0).$$

By Theorem 3.1, $R(D, w) \geq 1 + 2\operatorname{Re}(c_2 w)$ for all $w \in f(\mathbb{U}) \setminus \{\infty\}$. Because $R(D, w)$ vanishes on $\partial f(\mathbb{U})$, we have the following result.

Corollary 4.1. *If a function f in $\mathbf{M}^c$ has Taylor expansion $f(z) = z + c_2 z^2 + \ldots$ at 0, then*

$$\left\{w \in \mathbb{C} :\ \operatorname{Re}(c_2 w) > -\frac{1}{2}\right\} \subset f(\mathbb{U}).$$

Example of the function $f(z) = \dfrac{z}{1 - z}$ shows that the constant $-\dfrac{1}{2}$ in Corollary 4.1 is the maximal possible.

Remark 4.2. Coefficient estimate (6) is the reverse of known Trimble's inequality [**13**] which is valid in the different class of univalent functions.

Remark 4.3. Class $\mathbf{M}^c$ is related to class MC from the recent paper of Yamashita [**14**], where some other sharp coefficient estimates were proposed.

In view of [**1**] it is natural to ask whether Theorem 3.1 will remain true if one replaces $R(D,w)$ with the inner radius $r(D,w)$ of D (see [**5**] or [**3**] for definition). Since for simply connected domains these two radii coincide [**1**], statement (a) holds in this case as well. However, statement (b) fails. The domain $D = \overline{\mathbb{C}} \setminus (\overline{\mathbb{U}} \cup \{2\})$ gives a counterexample. Indeed,

$$r(D,w) = r(\overline{\mathbb{C}} \setminus \overline{\mathbb{U}}, w) = |w|^2 - 1$$

for all $w \in D \setminus \{\infty\}$. Hence, $r(D,w)$ is convex in $D \setminus \{\infty\}$, while $\mathbb{C} \setminus D$ is not a convex set.

Problem 4.4. *Hyperbolic radius can also be defined for certain domains in* $\mathbb{R}^n$, $n > 2$ [**1**]. *Does Theorem* 3.1 *hold for such domains?*

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