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MAXIMAL OPERATORS, LEBESGUE POINTS AND QUASICONTINUITY IN STRONGLY NONLINEAR POTENTIAL THEORY

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Abstract. Many maximal functions defined on some Orlicz spaces $\mathbf{L}_A$ are bounded operators on $\mathbf{L}_A$ if and only if they satisfy a capacitary weak inequality. We show also that $(m, A)-$quasievery x is a Lebesgue point for f in $\mathbf{L}_A$ sense and we give an $(m, A)-$ quasicontinuous representative for f when $\mathbf{L}_A$ is reflexive.

1. Introduction

The first part of this paper describes the connection between some maximal operators defined in Orlicz spaces, and capacities in this spaces. Theorem 1 states that maximal operators of strong type (A, A), satisfy a capacitary weak type inequality. The converse is the main of Theorem 2. More precisely, for N-functions satisfying the Δ_2 condition, maximal operators verifying a capacitary weak type inequality are of weak type (A, A). If in addition the conjugate N-function A^* satisfies also the Δ_2 condition, then these operators are of strong type (A, A). Theorem 3 deals with a limiting case which connects the capacity of compact set and its Lebesgue measure.

All results in this part generalize those given in [1] for the case of Lebesgue classes.

The second part is devoted to establish some results about Lebesgue points and quasicontinuity for Orlicz spaces.

By a theorem of Lebesgue, almost every point is a Lebesgue point. And if $f \in L^p$ for some p, $1 \le p < \infty$, then almost every x is a Lebesgue point in the sense that

$$\lim_{r\to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y) - f(x)|^p \, dy = 0.$$

This result is generalized in [4] to Orlicz spaces $\mathbf{L}_A$ for A satisfying the Δ_2 condition. We give a new proof of this result and we improve it in the first part of Theorem 4.

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On the other hand, Lars Hedberg proved the following result (see [2, Chapter 6, Th 6.2.1] or [14, Chapter 3, Th 3.10.2]): *Let* $1 < p < \infty$ *and* $m > 0$ *be such that* $mp \leq N$. *If* $f = \mathcal{G}_m * g$, $g \in L^p$, *then for every* $\epsilon > 0$ *there is an open set* U *with Bessel capacity less than* ϵ, *and such that*

$$\lim_{r \to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y) - f(x)|^p \, dy = 0$$

uniformly on U^c.

We generalize this result in the second part of Theorem 4 to reflexive Orlicz spaces. The proof depends on a density argument (which needs that A verifies the Δ_2 condition) and on a weak type estimate involving the maximal Hardy-Littlewood function (which needs that A^* verifies the Δ_2 condition).

2. Preliminaries

2.1. Orlicz spaces

Let $A : R \to R^+$ be an *N-function*, i.e. A is continuous, convex, with $A(t) > 0$ for $t > 0, \lim\limits_{t \to 0} \frac{A(t)}{t} = 0, \lim\limits_{t \to \infty} \frac{A(t)}{t} = +\infty$ and A is even.

Equivalently, A admits the representation: $A(t) = \int\limits_0^{|t|} a(x)dx$, where $a : R^+ \to R^+$ is non-decreasing, right continuous, with $a(0) = 0$, $a(t) > 0$ for $t > 0$ and $\lim\limits_{t \to +\infty} a(t) = +\infty$.

The N-function A^* *conjugate* to A is defined by $A^*(t) = \int\limits_0^{|t|} a^*(x)dx$, where a^* is given by $a^*(s) = \sup\{t : a(t) \leq s\}$.

Let A be an N-function and let Ω be an open set in R^N. We note $\mathcal{L}_A(\Omega)$ the set, called an *Orlicz class*, of measurable functions f, on Ω, such that

$$\rho(f, A, \Omega) = \int_\Omega A(f(x))dx < \infty.$$

Let A and A^* be two conjugate N-functions and let f be a measurable function defined almost everywhere in Ω. The *Orlicz norm of* f, $||f||_{A,\Omega}$ or $||f||_A$ if there is no confusion, is defined by

$$||f||_A = \sup \left\{ \int_\Omega |f(x)g(x)| \, dx : g \in \mathcal{L}_{A^*}(\Omega) \text{ and } \rho(g, A^*, \Omega) \leq 1 \right\}.$$

The set $\mathbf{L}_A(\Omega)$ of measurable functions f, such that $||f||_A < \infty$ is called an *Orlicz space*. When $\Omega = R^N$, we set $\mathbf{L}_A$ in place of $\mathbf{L}_A(R^N)$.

The *Luxemburg norm* $|||f|||_{A,\Omega}$ or $|||f|||_A$ if there is no confusion, is defined in $\mathbf{L}_A(\Omega)$ by

$$|||f|||_A = \inf \left\{ r > 0 : \int_\Omega A\left(\frac{f(x)}{r}\right) dx \leq 1 \right\}.$$

Let A be an N-function. We say that A *verifies the* Δ_2 *condition* if there exists a constant $C > 0$ such that $A(2t) \leq CA(t)$ for all $t \geq 0$.

We denote by $C(A)$ the smallest constant C such that $A(2t) \leq CA(t)$ for all $t \geq 0$.

We recall the following results. Let A be an N-function and a its derivative. Then

1. A verifies the Δ_2 condition if and only if one of the following holds:

i) $\forall r > 1, \exists k = k(r) : (\forall t \geq 0, A(rt) \leq kA(t))$,

ii) $\exists \alpha > 1 : (\forall t \geq 0,\, ta(t) \leq \alpha A(t)\,)$,

iii) $\exists \beta > 1 : (\forall t \geq 0,\, ta^*(t) \geq \beta A^*(t)\,)$,

iv) $\exists d > 0 : \left(\forall t \geq 0, \left(\frac{A^*(t)}{t}\right)' \geq d\frac{a^*(t)}{t}\right)$.

Moreover, α in *ii*) and β in *iii*) can be chosen such that $\alpha^{-1} + \beta^{-1} = 1$. We note $\alpha(A)$ the smallest α such that *ii*) holds. By a simple computation we have $C(A) \leq 2^\alpha$. See [5].

2. If A verifies the Δ_2 condition, then

i) $\forall t \geq 1, A(t) \leq A(1)t^\alpha$ and $\forall t \leq 1, A(t) \geq A(1)t^\alpha$,

ii) $\forall t \geq 1, A^*(t) \geq A^*(1)t^\beta$ and $\forall t \leq 1, A^*(t) \leq A^*(1)t^\beta$.

See for instance [7, 9, 11]. For more details on the theory of Orlicz spaces, see [3, 9, 11].

2.2. Capacity and Bessel kernels

We define a *capacity* as a positive set function C given on a σ-additive class of sets Γ, which contains compact sets and has the properties:

(i) $C(\emptyset) = 0$.

(ii) If X and Y are in Γ and $X \subset Y$, then $C(X) \leq C(Y)$.

(iii) If X_i, $i = 1, 2, ...$ are in Γ, then $C(\bigcup\limits_{i\geq 1} X_i) \leq \sum\limits_{i\geq 1} C(X_i)$.

Let k be a positive and integrable function in R^N and let A be an N-function. For $X \subset R^N$, we define

$$C_{k,A}(X) = \inf\{A(|||f|||_A) : f \in \mathbf{L}_A^+ \text{ and } \ k * f \geq 1 \text{ on } \ X\}$$

$$C'_{k,A}(X) = \inf\{|||f|||_A : f \in \mathbf{L}_A^+ \text{ and } \ k * f \geq 1 \text{ on } \ X\}$$

where $k * f$ is the usual convolution. The sign + deals with positive elements in the considered space. From [6] $C'_{k,A}$ is a capacity.

If a statement holds except on a set X where $C_{k,A}(X) = 0$, then we say that the statement holds $C_{k,A} - quasieverywhere$ (abbreviated $C_{k,A} - q.e$ or $(k, A) - q.e$ if there is no confusion).

For $m > 0$, the *Bessel kernel*, $\mathcal{G}_m$, is most easily defined through its Fourier transform $\mathfrak{F}(\mathcal{G}_m)$ as:

$$[\mathfrak{F}(\mathcal{G}_m)]\,(x) = (2\pi)^{-\frac{N}{2}} \left(1 + |x|^2\right)^{-\frac{m}{2}}$$

where $[\mathfrak{F}(f)](x) = (2\pi)^{-\frac{N}{2}} \int f(y)e^{-ixy}dy$ for $f \in \mathbf{L}^1$. $\mathcal{G}_m$ is positive, in $\mathbf{L}^1$ and verifies the equality: $\mathcal{G}_{r+s} = \mathcal{G}_r * \mathcal{G}_s$.

In the sequel, we put $B_{m,A} = C_{\mathcal{G}_m,A}$ and $B'_{m,A} = C'_{\mathcal{G}_m,A}$. We write $(m, A) - q.e.$ in place of $B_{m,A} - q.e.$ We denote $\mathcal{I}_m(x) = |x|^{m-N}$ the Riesz kernel. We have (see for instance [2])

$$\mathcal{G}_m(x) \sim \mathcal{I}_m(x), \text{when } |x| \to 0, \text{ with } 0 < m < N, \tag{2.1}$$

On the other hand, for every $c < 1$,

$$\mathcal{G}_m(x) = O(e^{-c|x|}), \text{when } |x| \to \infty, \text{ with } 0 < m. \tag{2.2}$$

Another inequality which serves in this paper is

$$\mathcal{G}_m(x) \le C\mathcal{G}_m(x+y), \ |x| \ge 2, \ |y| \le 1\,. \tag{2.3}$$

3. Maximal operators and capacity.

For $i, j \in N$, let $\theta_{i,j}$ be a complex valued function defined on R^N and such that $\theta_{i,j} \in \mathbf{L}_B$ for all N-functions B.

Let the sequence $(\theta_j)_j$ be such that

1. $\theta_{i,j} * f \to \theta_j * f$ in $\mathbf{L}_B$ for all $f \in \mathbf{L}_B$
2. $\theta_j * f_n \to \theta_j * f$ in $\mathbf{L}_B$ if $f_n \to f$ in $\mathbf{L}_B$.

Define the *maximal operator* $\mathcal{M}$

$$\mathcal{M}(f) = \sup_j |\theta_j * f\,| \tag{3.1}$$

and assume that $\mathcal{M}(f)$ is Lebesgue measurable on R^N.

An operator $H : \mathbf{L}_A \to \mathbf{L}_A$ is of weak type (*A*,*A*) if

$$\forall f \in \mathbf{L}_A, \forall t > 0, \ \mathbf{m}(\{x : |H(f)(x)| > t\}) \le \frac{1}{A\left(\frac{Ct}{|||f|||_A}\right)}$$

where C is a constant dependent only on A, and $\mathbf{m}$ is the Lebesgue measure on R^N.

H is of strong type (*A*,*A*) if

$$\forall f \in \mathbf{L}_A, \ |||H(f)|||_A \le C|||f|||_A$$

where C is a constant dependent only on A. For more details, see [13].

Theorem 1. *Let A be an N-function and $\mathcal{M}$ the maximal operator defined by* (3.1). *Suppose $\mathcal{M}$ is of strong type (A,A). Then*

$$\forall f \in \mathbf{L}_A, \forall t > 0, C_{k,A}(\{x : \mathcal{M}(k * f)(x) > t\}) \le A\left(C_A \frac{|||f|||_A}{t}\right).$$

C_A is the constant in the strong type.

Proof. It is easy to see that if $\theta_j \in \mathbf{L}_B$ for all B, then

$$\theta_j * (k * f) = k * (\theta_j * f).$$

In general case, if $\theta_{i,j} * f \to \theta_j * f$ in $\mathbf{L}_A$, then by [6, Théorème 4], there is a subsequence $(\theta'_{i,j})_i$ such that

$$\theta'_{i,j} * (k * f) = k * (\theta'_{i,j} * f) \to k * (\theta_j * f) \quad C_{k,A} - q.e.$$

Since $k * f \in \mathbf{L}_A$, we get

$$k * (\theta'_{i,j} * f) = \theta'_{i,j} * (k * f) \to \theta_j * (k * f) \text{ in } \mathbf{L}_A \,.$$

Hence

$$\theta_j * (k * f) = k * (\theta_j * f) \quad C_{k,A} - q.e.$$

There exists X_j such that $C_{k,A}(X_j) = 0$ and for all $x \notin X_j$,

$$\theta_j * (k * f)(x) = k * (\theta_j * f)(x) \,.$$

We get for $x \notin X_j$,

$$|\theta_j * (k * f)(x)| = |k * (\theta_j * f)(x)| \leq k * |\theta_j * f| \, (x).$$

Put $X = \bigcup_j X_j$. Then $C_{k,A}(X) = 0$ and

$$\mathcal{M}(k * f)(x) \leq k * \mathcal{M}(f)(x) \quad C_{k,A} - q.e.$$

It follows that for all $t > 0$,

$$C_{k,A} \left(\{x : \mathcal{M}(k * f)(x) > t\} \right) \leq C_{k,A} \left\{ x : k * \mathcal{M}(f)(x) > t \right\}.$$

From [6, Théorème 3], we deduce for all $t > 0$,

$$C_{k,A} \left(\{x : \mathcal{M}(k * f)(x) > t\} \right) \leq A \left(C_A \frac{|||f|||_A}{t} \right).$$

This completes the proof. □

Remark 1. *If we suppose in addition that A verifies the Δ_2 condition, then there exists a constant C' dependent only on A, such that for all $t > 0$,*

$$C_{k,A} \left(\{x : \mathcal{M}(k * f)(x) > t\} \right) \leq C' A \left(\frac{|||f|||_A}{t} \right).$$

Lemma 1. *Let $f \in \mathbf{L}_A$. Then there exists $\lambda > 0$ such that*

$$\int A \left(\frac{\mathcal{G}_m * f - f}{\lambda} \right) dx \to 0 \textit{ as } m \to 0.$$

Proof. We have $\mathcal{G}_m * f \to f$ a.e. as $m \to 0$. On the other hand, there is a constant $\gamma > 0$ such that $\dfrac{f}{\gamma} \in \mathcal{L}_A$. Let $\lambda = 2\gamma$. Then

$$A \left(\frac{\mathcal{G}_m * f - f}{\lambda} \right) \leq 2^{-1} A \left(\frac{2\mathcal{G}_m * f}{\lambda} \right) + 2^{-1} A \left(\frac{2f}{\lambda} \right).$$

Jensen's inequality gives

$$A\left(\frac{2\mathcal{G}_m * f}{\lambda}\right) \leq A\left(\frac{2f}{\lambda}\right) * \mathcal{G}_m.$$

The desired result follows by Vitali's Theorem. □

Theorem 2. *Let A be an N-function satisfying the Δ_2 condition, and let $\mathcal{M}$ be the maximal operator defined by* (3.1)*. Choose $k = \mathcal{G}_m$ with $m > 0$. Let C be a constant dependent only on A and such that for all $t > 0$ and all $f \in \mathbf{L}_A$,*

$$C_{k,A}\left(\{x : \mathcal{M}(\mathcal{G}_m * f)(x) > t\}\right) \leq CA\left(\frac{|||f|||_A}{t}\right).$$

Then $\mathcal{M}$ is of weak type (A, A).

Proof. Let X be a set and $f \in \mathbf{L}_A^+$ such that $\mathcal{G}_m * f \geq 1$ on X. Then

$$\mathbf{m}(X) \leq \int_X (\mathcal{G}_m * f)dx \leq |||\mathcal{G}_m * f|||_A \ \|\chi_X\|_{A^*}$$

where χ_X is the characteristic function of X.

The identity $\|\chi_X\|_{A^*} = \mathbf{m}(X)A^{-1}\left(\frac{1}{\mathbf{m}(X)}\right)$ gives

$$\frac{1}{A^{-1}\left(\frac{1}{\mathbf{m}(X)}\right)} \leq C'_{\mathcal{G}_m,A}(X).$$

This implies

$$\mathbf{m}\left(\{x : \mathcal{M}(\mathcal{G}_m * f)(x) > t\}\right) \leq \frac{1}{A\left(\frac{Ct}{|||f|||_A}\right)}.$$

Note that if $s = \inf(m, b)$, then

$$\mathcal{G}_m * f - \mathcal{G}_b * f = \mathcal{G}_s * (\mathcal{G}_{m-s} * f - \mathcal{G}_{b-s} * f).$$

This implies

$$\mathbf{m}\left(\{x : \mathcal{M}(\mathcal{G}_m * f - \mathcal{G}_b * f)(x) > t\}\right) \leq \frac{1}{A\left(\frac{Ct}{|||\mathcal{G}_{m-s} * f - \mathcal{G}_{b-s} * f|||_A}\right)}.$$

By the previous Lemma, $\mathcal{G}_m * f \to f$ in $\mathbf{L}_A$ as $m \to 0$, since A verifies the Δ_2 condition. By the sublinearity of $\mathcal{M}$, $\left(\mathcal{M}(\mathcal{G}_m * f)\right)_m$ is Cauchy in measure as $m \to 0$. Thus $\left(\mathcal{M}(\mathcal{G}_m * f)\right)_m$ converges in measure to a function h, as $m \to 0$. This implies

$$\mathbf{m}\left(\{x : |h(x)| > t\}\right) \leq \frac{1}{A\left(\frac{Ct}{2|||f|||_A}\right)}.$$

There exists a subsequence $\left(\mathcal{M}(\mathcal{G}_{m'} * f)\right)_{m'}$ of the sequence $\left(\mathcal{M}(\mathcal{G}_m * f)\right)_m$ such that $\mathcal{M}(\mathcal{G}_{m'} * f) \to h$ *a.e.* And there exists a subsequence $\left(\mathcal{M}(\mathcal{G}_{m"} * f)\right)_{m"}$ of the sequence $\left(\mathcal{M}(\mathcal{G}_{m'} * f)\right)_{m'}$ such that

$$\theta_j * (\mathcal{G}_{m"} * f) \to \theta_j * f \ a.e.$$

Hence there exists X_j such that $\mathbf{m}(X_j) = 0$ and $\theta_j * f(x) \leq h(x)$ for $x \notin X_j$. Thus $\mathcal{M}(f)(x) \leq h(x)$ *a.e.* This gives

$$\mathbf{m}\left(\{x : |\mathcal{M}(f)(x)| > t\}\right) \leq \frac{1}{A\left(\frac{Ct}{2|||f|||_A}\right)}.$$

Then $\mathcal{M}$ is of weak type (A, A). □

Corollary 1. *If in addition to hypothesis of Theorem 2 we suppose that A^* verifies the Δ_2 condition, then $\mathcal{M}$ is of strong type (A, A).*

Proof. From Theorem 2, $\mathcal{M}$ is of weak type (A, A) for all A satisfying the Δ_2 condition. $\mathcal{M}$ is then of weak type (p, p) for all $1 < p < \infty$. The Marcinkiewicz interpolation Theorem shows that $\mathcal{M}$ is of strong type (p, p) for all $1 < p < \infty$. By [7] and [13] $\mathcal{M}$ is of strong type (A, A). □

Theorem 3. *Let $(k_i)_i$ be a sequence of positive integrable functions on R^N such that*

1. $\int k_i(x)dx \to 1,$ *as* $i \to \infty$
2. $\int_{\{|x| \geq \delta\}} k_i(x)dx \to 0,$ *as* $i \to \infty$.

Then for any compact K in R^N, $\lim_{i\to\infty} C_{k_i,A}(K) = A\left[\frac{1}{A^{-1}\left(\frac{1}{\mathbf{m}(K)}\right)}\right]$.

Proof. Let $f \in \mathbf{L}_A^+$ such that $k_i * f \geq 1$ on K. Then

$$\mathbf{m}(K) \leq \int_K (k_i * f)dx \leq |||k_i * f|||_A \, \|\chi_K\|_{A^*}$$

where χ_K is the characteristic function of K.

But $\|\chi_K\|_{A^*} = \mathbf{m}(K)A^{-1}\left(\frac{1}{\mathbf{m}(K)}\right)$, and by [10] (see also [7] for a simple proof)

$$|||k_i * f|||_A \leq \|k_i\|_1 \, |||f|||_A \, .$$

Hence

$$\frac{1}{A^{-1}\left(\frac{1}{\mathbf{m}(K)}\right)} \leq \|k_i\|_1 \, |||f|||_A \, .$$

This implies

$$\frac{1}{A^{-1}\left(\frac{1}{\mathbf{m}(K)}\right)} \leq \|k_i\|_1 \, C'_{k_i,A}(K) \, .$$

Thus

$$\frac{1}{A^{-1}\left(\frac{1}{\mathbf{m}(K)}\right)} \leq \liminf_{i\to\infty} C'_{k_i,A}(K).$$

On the other hand, let O be a bounded open set such that $K \subset O$ and let ϵ be such that $0 < \epsilon < 1$. Then there is i_0 such that for $i \geq i_0$, we have $k_i * \chi_O \geq 1 - \epsilon$ on K.

Since $\chi_O \in \mathbf{L}_A$, we deduce that $C'_{k_i,A}(K) \leq \dfrac{|||\chi_O|||_A}{1-\epsilon}$. From the identity $|||\chi_O|||_A = \dfrac{1}{A^{-1}\left(\frac{1}{\mathbf{m}(O)}\right)}$, we have

$$\limsup_{i\to\infty} C'_{k_i,A}(K) \leq (1-\epsilon)^{-1}\frac{1}{A^{-1}\left(\frac{1}{\mathbf{m}(O)}\right)}.$$

This implies $\limsup\limits_{i\to\infty} C'_{k_i,A}(K) \leq \dfrac{1}{A^{-1}\left(\frac{1}{\mathbf{m}(K)}\right)}$. Thus

$$\lim_{i\to\infty} C'_{k_i,A}(K) = \frac{1}{A^{-1}\left(\frac{1}{\mathbf{m}(K)}\right)}.$$

The proof is complete. □

4. Lebesgue point and quasicontinuity

Recall that if $f \in L^1_{loc}$, a point $x \in R^N$ is called a Lebesgue point for f if

$$\lim_{r\to 0}\frac{1}{|B(x,r)|}\int_{B(x,r)} |f(y)-f(x)|\, dy = 0.$$

Here $|B(x,r)|$ is the Lebesgue measure of $B(x,r)$ on R^N.

By a theorem of Lebesgue, almost every point is a Lebesgue point. On the other hand, if $f \in L^p$ for some p, $1 \leq p < \infty$, then almost every x is a Lebesgue point in the sense that

$$\lim_{r\to 0}\frac{1}{|B(x,r)|}\int_{B(x,r)} |f(y)-f(x)|^p\, dy = 0.$$

See [12, Section I.5.7].

This result is generalized in [4] to Orlicz spaces $\mathbf{L}_A$ for A satisfying the Δ_2 condition. More precisely

Lemma 2. [4] *Let A be an N-function verifying the Δ_2 condition and $\alpha = \alpha(A)$. Then*

$$\lim_{r\to 0} r^{\frac{-N}{\alpha}} |||f_x|||_{A,B(x,r)} = 0 \ \text{ a.e. on } R^N.$$

Here f_x is defined by $f_x(y) = f(y) - f(x)$.

We shall give a new proof of this result.

Lemma 3. *Let A be an N-function verifying the Δ_2 condition and $\alpha = \alpha(A)$. Then, for all $t \geq 0$ and all $0 < s \leq 1$,*

$$A(s^{\frac{-1}{\alpha}}t) \leq C(A)s^{-1}A(t).$$

Proof. If $s = 1$, the result is obvious.

Let $s < 1$, and q be the smallest positive integer such that $s^{\frac{-1}{\alpha}} \leq 2^q$. Then

$$q \geq \frac{Log(s^{\frac{-1}{\alpha}})}{Log2} \quad \text{and} \quad q - 1 \leq \frac{Log(s^{\frac{-1}{\alpha}})}{Log2} = K(s, \alpha).$$

Since $2^\alpha \geq C(A)$, we get

$$C(A)^q \leq C(A).C(A)^{K(s,\alpha)} \leq C(A).e^{(\alpha Log2).K(s,\alpha)} = C(A)s^{-1},$$

and

$$A(s^{\frac{-1}{\alpha}}t) \leq A(2^q t) \leq C(A)^q A(t) \leq C(A)s^{-1}A(t).$$

The proof is finished. □

Now we give a new proof of Lemma 2.

New Proof of Lemma 2. Since the function $A \circ f_x$ is locally integrable, by [13, Section I.5.7] we have

$$\lim_{r\to 0} r^{-N} \int_{B(x,r)} (A \circ f_x)(y)dy = 0 \text{ a.e. on } R^N.$$

Lemma 3 implies

$$\int_{B(x,r)} A(r^{\frac{-N}{\alpha}} f_x)(y)dy \leq C(A)r^{-N} \int_{B(x,r)} (A \circ f_x)(y)dy.$$

Hence

$$\lim_{r\to 0} \int_{B(x,r)} A(r^{\frac{-N}{\alpha}} f_x)(y)dy = 0 \text{ a.e. on } R^N.$$

The result follows since A verifies the Δ_2 condition.

Lemma 4. *Let A be an N-function satisfying the Δ_2 condition. Then there is a constant C such that $\forall u \geq 1$, $u^{\frac{1}{\alpha}} \leq CA^{-1}(u)$.*

Proof. Let $u \geq 1$. Then

$$A(u^{\frac{1}{\alpha}}) \leq A(1)u.$$

This implies

$$u^{\frac{1}{\alpha}} \leq A^{-1}[A(1)u] \leq A^{-1}(\beta u),$$

where $\beta = \sup(1, A(1))$.

From the inequality $\beta A(t) \leq A(\beta t)$, valid for all t, we get

$$A^{-1}[\beta A(t)] \leq \beta t.$$

Hence

$$A^{-1}(\beta u) \leq \beta A^{-1}(u).$$

So

$$u^{\frac{1}{\alpha}} \leq \beta A^{-1}(u).$$

The proof is finished. □

Recall that the Hardy-Littlewood maximal function of a locally integrable function f is

$$M(f)(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, dy.$$

Lemma 5. *Let A be an N-function such that A^* satisfies the Δ_2 condition. Let m be a positive number and $f = \mathcal{G}_m * g$, $g \in \mathbf{L}_A^+$. Let $E_s = \{x : M(f)(x) > s\}$. Then there exists a constant C independent of f such that*

$$B'_{m,A}(E_s) \leq \frac{C}{s} |||g|||_A.$$

Proof. Let χ be the normalized characteristic function of the unit ball, and for $r > 0$, define χ_r by $\chi_r(x) = r^N \chi(\frac{x}{r})$. Then

$$\chi_r * f(x) = \chi_r * \mathcal{G}_m * g(x) \leq \mathcal{G}_m * Mg(x).$$

Thus

$$M(f)(x) = \sup_{r>0} \chi_r * f(x) \leq \mathcal{G}_m * Mg(x).$$

This implies

$$\{x : M(f)(x) > s\} \subset \{x : \mathcal{G}_m * Mg(x) > s\}.$$

We get by the definition of $B'_{m,A}$, $B'_{m,A}(E_s) \leq \frac{1}{s} |||Mg|||_A$.

Since A^* satisfies the Δ_2 condition, there is a constant C such that $|||Mg|||_A \leq C|||g|||_A$. (See for instance [8]). The Lemma follows. $\square$

Remark 2. *We can also derive quickly the Lemma from Theorem 1. In fact, we are in the conditions of this theorem because M is of strong type since A^* satisfies the Δ_2 condition.*

Lemma 6. *Let A be an N-function such that A and A^* satisfy the Δ_2 condition. Let m be a positive number such that $0 < \alpha m \leq N$, and $f = \mathcal{G}_m * g$, $g \in \mathbf{L}_A^+$. Let $E_s = \left\{ x : \sup_{r>0} |B(x,r)|^{\frac{-1}{\alpha}} |||f|||_{A,B(x,r)} > s \right\}$. Then there exists a constant C independent of f such that*

$$B'_{m,A}(E_s) \leq \frac{C}{s} |||g|||_A$$

for all $s \geq |||g|||_A$.

Proof. Let $s \geq |||g|||_A$ and $x_0 \in E_s$. Then there exists r such that

$$|B(x_0,r)|^{\frac{-1}{\alpha}} |||f|||_{A,B(x_0,r)} > s.$$

Now the inequality

$$|||f|||_A \leq ||\mathcal{G}_m||_1 |||g|||_A$$

implies $|||\frac{f}{s}|||_A \leq 1$, since $||\mathcal{G}_m||_1 = 1$. Hence

$$|B(x_0,r)| < 1.$$

We set $g = g_1 + g_2$, where $g_1(x) = 0$ for $|x - x_0| > 2r$, and $g_1(x) = g(x)$ for $|x - x_0| \leq 2r$. Then

$$s < |B(x_0,r)|^{\frac{-1}{\alpha}} \left[|||g_1 * \mathcal{G}_m|||_{A,B(x_0,r)} + |||g_2 * \mathcal{G}_m|||_{A,B(x_0,r)}\right].$$

So that either

$$s < 2\,|B(x_0,r)|^{\frac{-1}{\alpha}}\, |||g_1 * \mathcal{G}_m|||_{A,B(x_0,r)} \tag{4.1}$$

or

$$s < 2\,|B(x_0,r)|^{\frac{-1}{\alpha}}\, |||g_2 * \mathcal{G}_m|||_{A,B(x_0,r)}. \tag{4.2}$$

On the other hand, by [2, Lemma 3.1.1], for any $x \in B(x_0,r)$,

$$\frac{g_1 * \mathcal{G}_m(x)}{s} \leq \frac{1}{s}\int_{B(x,3r)} \mathcal{G}_m(x-y)g_1(y)dy \leq KM\left(\frac{g_1(x)}{s}\right) r^m.$$

If the inequality (4.1) holds, we get

$$r^{\frac{N}{\alpha}} < \frac{K}{s}|||g_1 * \mathcal{G}_m|||_{A,B(x_0,r)} \leq \frac{K''}{s} r^m |||Mg_1|||_{A,B(x_0,r)} \leq \frac{K'''}{s} r^m |||g_1|||_{A,B(x_0,2r)}.$$

So

$$r^{\frac{N}{\alpha}-m} < \frac{K'''}{s}|||g|||_{A,B(x_0,2r)}. \tag{4.3}$$

Remark that when $N = m\alpha$, then (4.3) cannot occur if $s \geq K'''|||g|||_A$ since always

$$|||g|||_{A,B(x_0,r)} \leq |||g|||_A.$$

If the inequality (4.2) holds, then we claim that

$$Cg * \mathcal{G}_m(x) > s. \tag{4.4}$$

In fact, if $x_1, x_2 \in B(x,r)$ and y outside of $B(x_0,2r)$, then

$$\frac{|x_2 - y|}{3} \leq |x_1 - y| \leq 3\,|x_2 - y|\,,$$

and

$$|x_2 - y| - 2r \leq |x_1 - y| \leq |x_2 - y| + 2r.$$

By the estimates (2.1) and (2.3) for Bessel kernels, we have

$$\mathcal{G}_m(x_1 - y) \leq C\mathcal{G}_m(x_2 - y).$$

So for any $x_1 \in B(x,r)$

$$g_2 * \mathcal{G}_m(x_1) \leq C \inf_{x\in B(x_0,r)} g_2 * \mathcal{G}_m(x) \leq C \inf_{x\in B(x_0,r)} g * \mathcal{G}_m(x).$$

Hence

$$s < 2C\,|B(x_0,r)|^{\frac{-1}{\alpha}} \inf_{x\in B(x_0,r)} g * \mathcal{G}_m(x)|||1|||_{A,B(x_0,r)}.$$

But

$$|||1|||_{A,B(x_0,r)} = \frac{1}{A^{-1}\left(|B(x_0,r)|^{-1}\right)}.$$

So

$$s < 2C\frac{|B(x_0,r)|^{\frac{-1}{\alpha}}}{A^{-1}\left(|B(x_0,r)|^{-1}\right)}\inf_{x\in B(x_0,r)} g*\mathcal{G}_m(x).$$

By Lemma 4 we have

$$s < K_1 \inf_{x\in B(x_0,r)} g*\mathcal{G}_m(x).$$

This implies the claim. Let U be the set of all $x \in E_s$ and satisfying (4.3). Then by (4.4),

$$Cg*\mathcal{G}_m(x) > s \text{ on } E_s\setminus U.$$

So

$$B'_{m,A}(E_s\setminus U) \le \frac{C}{s}|||g|||_A.$$

By the simple covering Vitali lemma, see [2, Theorem 1.4.1], there are disjoint balls $\{B(x_i,2r_i)\}_1^\infty$ such that

$$r_i^{\frac{N}{\alpha}-m} < \frac{K}{s}|||g|||_{A,B(x_i,2r_i)},$$

and

$$U \subset \bigcup_1^\infty B(x_i,10r_i).$$

We may take $10r_i < 1$, for all i. We have, by the subadditivity of $B'_{m,A}$ (see [6])

$$B'_{m,A}(U) \le \sum_1^\infty B'_{m,A}\left(B(x_i,10r_i)\right).$$

By [5, Lemma 2] we get

$$B'_{m,A}\left(B(x_i,10r_i)\right) \le Cr_i^{-m}2^{-q_i}.$$

Here q_i is the greatest positive integer such that $q_i \le \dfrac{Log(r_i^{-N})}{Log(C(A))}$.

A simple computation shows that $2^{-q_i} \le 2r_i^{\frac{N}{\alpha}}$. This implies

$$B'_{m,A}(U) \le C\sum_1^\infty r_i^{\frac{N}{\alpha}-m} \le \sum_1^\infty \frac{K'}{s}|||g|||_{A,B(x_i,2r_i)}.$$

From the definition of the Orlicz norm we get easily

$$\sum_1^\infty ||g||_{A,B(x_i,2r_i)} \le ||g||_A.$$

The equivalence

$$|||g|||_{A,\Omega} \le ||g||_{A,\Omega} \le 2|||g|||_{A,\Omega},$$

valid for all Ω, implies

$$B'_{m,A}(U) \le \frac{K}{s}|||g|||_A.$$

Since $B'_{m,A}(E_s) \le B'_{m,A}(E_s\setminus U) + B'_{m,A}(U)$, the lemma follows. $\square$

Recall the definition of quasicontinuity.

Definition 1. Let $\mathcal{C}$ be a capacity on R^N and let f be a function defined $\mathcal{C}-$quasieverywhere on R^N or on some open subset of R^N. Then f is said to be $\mathcal{C}-$quasicontinuous if for every $\epsilon > 0$, there is an open set O such that $\mathcal{C}(O) < \epsilon$ and $f\,|_{O^c} \in C(O^c)$.

In other words, the restriction of f to the complement of O is continuous in the induced topology.

We write $(m, A)-$quasicontinuous in place of $B'_{m,A}-$quasicontinuous.

Let A be an N-function and $m > 0$. We define the space of Bessel potentials $\mathbf{L}_{m,A}$ by

$$\mathbf{L}_{m,A} = \{\psi = G_m * f : f \in \mathbf{L}_A\},$$

and a norm on $\mathbf{L}_{m,A}$ by $|||\psi|||_{m,A} = |||f|||_A$ if $\psi = G_m * f$.

Theorem 4. *Let A be an N-function such that A and A^* satisfy the Δ_2 condition and let $\alpha = \alpha(A)$. Let m be a positive number and $f = \mathcal{G}_m * g \in \mathbf{L}_{m,A}$, $0 < m\alpha < N$. Then $(m, A)-$quasievery x is a Lebesgue point for f in $\mathbf{L}_A-$sense, i.e.*

$$\lim_{r\to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)dy = \widetilde{f}(x) \text{ exists,}$$

and

$$\lim_{r\to 0} r^{\frac{-N}{\alpha}} |||f_x|||_{A,B(x,r)} = 0\,,$$

where f_x is defined as $f_x(y) = f(y) - \widetilde{f}(x)$.

Moreover, the convergence is uniform outside an open set of arbitrarily small $(m, A)-$capacity, $\widetilde{f}$ is an $(m, A)-$quasicontinuous representative for f, and

$$\widetilde{f}(x) = \mathcal{G}_m * g \ (m, A) - q.e.$$

Proof. Let $f = \mathcal{G}_m * g \in \mathbf{L}_{m,A}$ and define χ_r as in the proof of Lemma 5. We denote by $\mathbf{S}$ the Schwartz class of rapidly decreasing infinitely differentiable functions on R^N. For $\epsilon > 0$, there exists $g_0 \in \mathbf{S}$ such that $|||g - g_0|||_A < \epsilon$, since A verifies the Δ_2 condition. Then $f_0 = \mathcal{G}_m * g_0 \in \mathbf{S}$ and $\lim_{r\to 0} \chi_r * f_0 = f_0$.

Let $\delta > 0$ and define

$$\Omega_\delta f(x) = \sup_{0<r<\delta} (\chi_r * f)(x) - \inf_{0<r<\delta} (\chi_r * f)(x).$$

We have

$$\Omega_\delta f(x) \le \Omega_\delta (f - f_0)(x) + \Omega_\delta f_0(x).$$

By uniform continuity we can choose δ such that $\Omega_\delta f_0(x) < \epsilon$, for all x.

On the other hand

$$|\chi_r * (f - f_0)(x)| \le M(f - f_0)(x),$$

so

$$\Omega_\delta f(x) \le 2M(f - f_0)(x) + \epsilon.$$

Let $\epsilon < \frac{s}{2}$. Then

$$\{x : \Omega_\delta f(x) > s\} \subset \left\{x : 2M(f - f_0)(x) > \frac{s}{2}\right\}.$$

Lemma 5 implies

$$B'_{m,A}\left(\{x : \Omega_\delta f(x) > s\}\right) \le \frac{C}{s} |||g - g_0|||_A \le \frac{C\epsilon}{s}. \tag{4.5}$$

Choose $s = 2^{-n}$, and $\epsilon = 4^{-n}$ for $n = 1, 2, ...,$ and denote the corresponding δ by δ_n. Set

$$D_n = \left\{x : \Omega_{\delta_n} f(x) > 2^{-n}\right\}.$$

Then

$$B'_{m,A}(D_n) \le C2^{-n}.$$

If we set $F_p = \bigcup\limits_{n=p}^{\infty} D_n$, we get

$$B'_{m,A}(F_p) \le C \sum_{n=p}^{\infty} 2^{-n},$$

which tends to 0 as p tends to ∞. Whence

$$B'_{m,A}\left(\bigcap_{p=1}^{\infty} F_p\right) = 0.$$

If $x \notin F_p$, then $\Omega_\delta f(x) \le 2^{-n}$ for $\delta \le \delta_n$ and all $n \ge p$. This implies that $\lim\limits_{r\to 0} \chi_r * f(x) = \widetilde{f}(x)$ exists if $x \notin \bigcap_{p=1}^{\infty} F_p$ and uniformly outside F_p for any p. This proves the first part of the theorem.

To prove the second part, we define

$$\Omega_{A,\delta}\left(f - \widetilde{f}(x)\right)(x) = \sup_{0<r\le\delta} |B(x,r)|^{\frac{-1}{\alpha}} |||f_x|||_{A,B(x,r)},$$

where f_x is defined as $f_x(y) = f(y) - \widetilde{f}(x)$. We choose $\epsilon > 0$, g_0, and $f_0 = \mathcal{G}_m * g_0$ as before. Then $\widetilde{f_0} = f_0$ and as before we can choose δ so small that $\Omega_{A,\delta}\left(f_0 - \widetilde{f_0}(x)\right)(x) < \epsilon$ for all x. We have

$$\begin{aligned}
\Omega_{A,\delta}\left(f - \widetilde{f}(x)\right)(x) \quad &\le \quad \Omega_{A,\delta}\left(f - f_0 - (\widetilde{f}(x) - f_0(x))\right)(x) \\
&\qquad + \Omega_{A,\delta}\left(f_0 - \widetilde{f_0}(x)\right)(x) \\
&\le \quad \sup_{0<r\le\delta} |B(x,r)|^{\frac{-1}{\alpha}} |||f - f_0|||_{A,B(x,r)} \\
&\qquad + \sup_{0<r\le\delta} |B(x,r)|^{\frac{-1}{\alpha}} \left|\widetilde{f}(x) - f_0(x)\right| |||1|||_{A,B(x,r)} + \epsilon\,.
\end{aligned}$$

We know that

$$|||1|||_{A,B(x,r)} = \frac{1}{A^{-1}\left(\frac{1}{|B(x,r)|}\right)}.$$

From Lemma 4, there is a constant Q such that

$$\frac{|B(x,r)|^{\frac{-1}{\alpha}}}{A^{-1}\left(\frac{1}{|B(x,r)|}\right)} \leq Q\,.$$

Whence

$$\Omega_{A,\delta}\left(f - \widetilde{f}(x)\right)(x) \leq \sup_{r>0} |B(x,r)|^{\frac{-1}{\alpha}}\, |||f - f_0|||_{A,B(x,r)} + Q\left|\widetilde{f}(x) - f_0(x)\right| + \epsilon.$$

If $\epsilon < \frac{s}{3}$, then

$$\left\{x:\ \Omega_{A,\delta}\left(f - \widetilde{f}(x)\right)(x) > s\right\}$$

$$\subset \left\{x : \sup_{r>0} |B(x,r)|^{\frac{-1}{\alpha}}\, |||f - f_0|||_{A,B(x,r)} > \frac{s}{3}\right\} \cup \left\{x : \left|\widetilde{f}(x) - f_0(x)\right| > \frac{s}{3Q}\right\}.$$

We know that

$$\left|\widetilde{f}(x) - f_0(x)\right| \leq \mathcal{G}_m * |g - g_0|\,(x)\ (m,A) - q.e.$$

So by the definition of capacity we get

$$B'_{m,A}\left(\left\{x : \left|\widetilde{f}(x) - f_0(x)\right| > \frac{s}{3Q}\right\}\right) \leq \frac{3Q}{s}|||g - g_0|||_A.$$

Lemma 6 applied to $\mathcal{G}_m * |g - g_0|$ gives

$$B'_{m,A}\left(\left\{x : \sup_{r>0} |B(x,r)|^{\frac{-1}{\alpha}}\, |||f - f_0|||_{A,B(x,r)} > \frac{s}{3}\right\}\right) \leq \frac{3C}{s}|||g - g_0|||_A.$$

Hence

$$B'_{m,A}\left(\left\{x : \Omega_{A,\delta}\left(f - \widetilde{f}(x)\right)(x) > s\right\}\right) \leq \frac{C'\epsilon}{s}\,. \tag{4.6}$$

The estimate (4.6) gives the conclusion as the estimate (4.5) for the first part. □

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