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ON AN INEQUALITY FOR ENTIRE FUNCTIONS

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Abstract. It is shown that the entire function $F(z) = \sum_{n=0}^{\infty} \mathrm{e}^{-v(n)} z^n$ satisfies an inequality: $|F(z)| \geq MF(|z|)$ for some $M > 0$ and for a set of z in the complex plane.

1. Introduction

The entire function $\mathrm{e}^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ trivially satisfies the inequality: $|\,\mathrm{e}^z\,| \leq \mathrm{e}^{|z|}$ for all z in the complex plane. It is of some interest to know the set of z for which $|\mathrm{e}^z| \geq M\mathrm{e}^{|z|}$ for some $M > 0$. Indeed, if e^x is real-valued then for any M, $0 \leq M \leq 1$, $|\,\mathrm{e}^x\,| \geq M\,\mathrm{e}^{|x|}$ for all $x \geq 0$.

Here we are concerned with a more general function

$$F(z) \sum_{n=0}^{\infty} \mathrm{e}^{-v(n)} \cdot z^n \tag{1}$$

where $v(x)$ is a suitable real-valued function so that $F(z)$ becomes an entire function and satisfies the inequality:

$$|F(z)| \geq MF\,(|z|) \tag{$**$}$$

for some $M > 0$ and for a set of z in the complex plane.

Inequalities of the type $(**)$ are useful in evaluating, among other results, the glb or minorant of an entire function in the same way as the inequality of the type $|F(z)| \leq MF\,(|z|)$ is used in obtaining the lub or majorant. For details on entire functions and their properties, the reader may consult the references [**1**]–[**7**].

Note that if $v(x) = \left(x + \frac{1}{2}\right) \log x - x$, then by using the Stirling's formula: $n! \approx \sqrt{2\pi n} \cdot n^n\,\mathrm{e}^{-n}$, we see that

$$1 + \sum_{n=1}^{\infty} \mathrm{e}^{-(n+\frac{1}{2})\log n + n} \cdot z^n \leq 1 + c \sum_{n=1}^{\infty} \frac{z^n}{n!}$$

for some $c > 0$.

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2. The Theorem

We use $z = r\,\mathrm{e}^{i\theta} = x + iy$ $\left(i = \sqrt{-1}\right)$ throughout and prove:

Theorem. *Let $v(x)$ be a twice differentiable real-valued function defined for all $x \geq 0$ such that*

(i) *$v(x)$ is increasing and $\longrightarrow \infty$ as $x \longrightarrow \infty$;*
(ii) *$v'(x)$ is increasing and $\longrightarrow \infty$ as $x \longrightarrow \infty$;*
(iii) *$v''(x)$ is decreasing and $\longrightarrow 0$ as $x \longrightarrow \infty$;*
(iv) *$v''(x) \geq 0$ for all $x \geq 0$;*
(v) *There exist numbers $\alpha > 0$, $\beta > 0$ and $x_0 > 0$ such that for all $x \geq x_0$, $\alpha \leq xv''(x) \leq \beta$.*

Then $F(z)$, given by (1), is an entire function and there is $r_0 > 0$ such that for a fixed $r \geq r_0$ and for all $z = x + iy$ with its y-coordinate satisfying the inequality:

$$|y| \leq \left(\sqrt{2}r\right) \Big/ \left(\frac{\alpha}{v''(x)} + 2\right),$$

we have

$$|F(z)| \geq MF\left(|z|\right)$$

for some $M > 0$, depending on r, but $0 \leq M \leq 1$ for all z.

Proof. If $a_n = \mathrm{e}^{-v(n)} \cdot z^n$, then

$$\left|\frac{a_{n+1}}{a_n}\right| = |\,\mathrm{e}^{-v'(\eta)}\,| \cdot |z| \to 0 \qquad (n < \eta < n+1)$$

as $n \to \infty$ for all z by using the Mean Value Theorem and the hypothesis (ii). So $F(z)$ is indeed an entire function.
Put

$$F(|z|) = F(r) = \sum_{n=0}^{\infty} \mathrm{e}^{-v(n)} \cdot r^n \tag{2}$$

The maximum term [**5**] or [**7**] of the series (2) is given as follows: clearly

$$\frac{d}{dx}\left(\mathrm{e}^{-v(x)} \cdot r^x\right) = \mathrm{e}^{-v(x)} \cdot r^x \left(\log r - v'(x)\right) = 0$$

if and only if

$$\log r - v'(x) = 0 \tag{3}$$

has a solution for a fixed r. Since $v'(x)$ is continuous (because $v''(x)$ exists), (3) has a solution. Let x be the largest solution of (3) for a fixed r and let $\xi = [x]$, the integral part of x, denote the index. Then

$$T(r) = \mathrm{e}^{-v(\xi)} \cdot r^{\xi}$$

is the maximum term of the series (2) for a fixed r. Clearly $|x - \xi| \leq 1$ and so

$$\xi - 1 < \xi \leq x \leq \xi + 1.$$

We choose

$$n_0 = \left[x - \frac{\alpha}{v''(x)}\right] \tag{4}$$

$$n_1 = \left[x + \frac{\alpha}{v''(x)}\right]. \tag{5}$$

From (4) and (5), we have

$$n_0 \leq x - \frac{\alpha}{v''(x)}, \qquad x - n_0 \leq \frac{\alpha}{v''(x)} + 1 \tag{6}$$

and

$$\left.\begin{array}{l} n_1 \leq x + \dfrac{\alpha}{v''(x)} \Rightarrow n_1 + 1 \leq x + \dfrac{\alpha}{v''(x)} + 1 \\ \text{and} \quad x - n_1 - 1 \leq -\dfrac{\alpha}{v''(x)}. \end{array}\right\} \tag{7}$$

We choose r large enough so that

$$n_0 + 1 < \xi - 1 < \xi = [x] \leq x \leq \xi + 1 < n_1. \tag{8}$$

Write

$$F(z) = T(r)\sum_{n=0}^{\infty} \mathrm{e}^{-v(n)+v(\xi)} \cdot r^{n-\xi} \cdot \mathrm{e}^{\mathrm{i}\,n\theta}. \tag{9}$$

Put

$$H(z) = \sum_{n=0}^{\infty} \mathrm{e}^{-v(n)+v(\xi)} \cdot r^{n-\xi} \cdot \mathrm{e}^{\mathrm{i}\,n\theta} = S_1 + S_2 + S_3, \tag{10}$$

where

$$\left.\begin{array}{lcl} S_1 & = & \sum_{n=0}^{n_0} \mathrm{e}^{-v(n)+v(\xi)} \cdot r^{n-\xi} \cdot \mathrm{e}^{\mathrm{i}\,n\theta} \\ S_2 & = & \sum_{n=n_0+1}^{n_1} \mathrm{e}^{-v(n)+v(\xi)} \cdot r^{n-\xi} \cdot \mathrm{e}^{\mathrm{i}\,n\theta} \\ S_3 & = & \sum_{n=n_1+1}^{\infty} \mathrm{e}^{-v(n)+v(\xi)} \cdot r^{n-\xi} \cdot \mathrm{e}^{\mathrm{i}\,n\theta}. \end{array}\right\} \tag{11}$$

First we show that

$$\lim_{r\to\infty} |S_1| = 0. \tag{12}$$

From (11), we have

$$|S_1| \leq \left(\mathrm{e}^{-v(n_0)+v(\xi)} \cdot r^{n_0-\xi}\right) \left(\sum_{n=0}^{n_0} \mathrm{e}^{-v(n)+v(n_0)} \cdot r^{n-n_0}\right).$$

Set

$$A_1 = \left(\mathrm{e}^{-v(n_0)+v(\xi)} \cdot r^{n_0-\xi}\right), \qquad B_1 = \sum_{n=0}^{n_0} \mathrm{e}^{-v(n)+v(n_0)} \cdot r^{n-n_0}.$$

Then

$$|S_1| \le A_1 \times B_1. \tag{13}$$

Using Taylor's theorem and the equation (3), we estimate upper bounds of A_1, B_1.

$$\begin{aligned}
A_1 &= \left(\mathrm{e}^{v(\xi)} \cdot r^{-\xi}\right)\left(\mathrm{e}^{-v(n_0)} \cdot r^{n_0}\right)\\
&= \left(\mathrm{e}^{v(\xi)} \cdot r^{-\xi}\right)\left(\mathrm{e}^{-\{v(x)+(n_0-x)v'(x)+\frac{1}{2}(n_0-x)^2 v''(n_0+\theta(x-n_0)\}} \cdot \mathrm{e}^{n_0 v' x)}\right)\\
&\qquad\qquad (0<\theta<1 \text{ and } (\log r = v'(x))\\
&= \left(\mathrm{e}^{v(\xi)-v(x)+xv'(x)-\xi v'(x)}\right) \times \left(\mathrm{e}^{-\frac{1}{2}(n_0-x)^2 v''(n_0+\theta(x-n_0))}\right)\\
&= C_1 \times D_1 \qquad (14)
\end{aligned}$$

where

$$\begin{aligned}
C_1 &= \mathrm{e}^{v(\xi)-v(x)} \cdot \mathrm{e}^{(x-\xi)v'(x)}\\
&= \mathrm{e}^{(\xi-x)v'(x)+\frac{1}{2}(\xi-x)^2 v'(\xi+\theta(x-\xi))} \cdot \mathrm{e}^{(x-\xi)v'(x)}\\
&= \mathrm{e}^{\frac{1}{2}(\xi-x)^2 v''(\xi+\theta(x-\xi))}\\
&\ge \mathrm{e}^0 = 1
\end{aligned}$$

(because $\frac{1}{2}(\xi-x)^2 v''(\xi+\theta(x-\xi)) \ge 0$).
On the other hand, since $|x-\xi| \le 1$ and $\xi+\theta(x-\xi) \ge \min\{x,\xi\} \ge x-1$, we have

$$C_1 \le \mathrm{e}^{\frac{1}{2}(1)^2 v''(x-1)} \to \mathrm{e}^0 = 1 \quad \text{as} \quad x \to \infty,$$

because of (iii).
Thus we have shown that $\lim_{x\to\infty} C_1 = 1$.
But $x \to \infty \Rightarrow r \to \infty$ and so $\lim_{r\to\infty} C_1 = 1$.
Also

$$D_1 = \mathrm{e}^{-\frac{1}{2}(n_0-x)^2 v''(n_0+\theta(x-n_0))} \le \mathrm{e}^{-\frac{1}{2}(n_0-x)^2 v''(x)}$$

because $n_0 + \theta(x-n_0) < x$.

In view of (6)

$$D_1 \le \mathrm{e}^{-\frac{1}{2}\left(-\frac{\alpha}{v''(x)}\right)^2 \cdot v''(x)} = \mathrm{e}^{-\frac{1}{2}\frac{\alpha^2}{v'(x)}} \to 0$$

as $x \to \infty$ by (iii). Since $x \to \infty \Rightarrow r \to \infty$, we have shown that

$$\lim_{r\to\infty} D_1 = 0 \quad \Rightarrow \quad \lim_{r\to\infty} A_1 = 0. \qquad \text{(from (14))}$$

To find an upper bound for B_1, we have

$$\begin{aligned} B_1 &= \sum_{n=0}^{n_0} \mathrm{e}^{-v(n)+v(n_0)} \cdot r^{n-n_0} \\ &= \sum_{n=0}^{n_0} \mathrm{e}^{-\left\{(n-n_0)v'(n_0)+\frac{1}{2}(n-n_0)^2 v''(n_0+\theta(n-n_0))\right\}} \cdot \mathrm{e}^{(n-n_0)v'(x)} \\ &= \sum_{n=0}^{n_0} \mathrm{e}^{(n-n_0)\left(v'(x)-v'(n_0)\right)} \cdot \mathrm{e}^{-\frac{1}{2}(n-n_0)^2 v''(n_0+\theta(n-n_0))} \\ &\le \sum_{n=0}^{n_0} \mathrm{e}^{(n-n_0)\left(v'(x)-v'(n_0)\right)} \qquad \left(\text{since } -\frac{1}{2}(n-n_0)^2\, v''\left(n_0+\theta\left(n-n_0\right)\right) \le 0\right) \end{aligned}$$

Put $n - n_0 = -m$. Then

$$B_1 \quad \le \sum_{m=0}^{n_0} \mathrm{e}^{-m\left(v'(x)-v'(n_0)\right)} \quad \le \frac{1}{1-\mathrm{e}^{-(v'(x)-v'(n_0))}}$$

because $n_0 < x \Rightarrow v'(n_0) \le v'(x)$ and so the resulting geometric series is convergent. To obtain an upper bound for

$$v'(n_0) - v'(x) = (n_0 - x)\, v''\left(n_0 + \theta\left(x - n_0\right)\right) \tag{15}$$

where $0 < \theta < 1$, we observe that

$$n_0 < x \text{ and } n_0 + \theta\left(x - n_0\right) < x \;\Rightarrow\; v''(x) \le v''\left(n_0 + \theta\left(x - n_0\right)\right).$$

From (6), we have

$$n_0 - x \le -\frac{\alpha}{v''(x)}$$

and so from (15) we have

$$v'(n_0) - v'(x) \le \left(-\frac{\alpha}{v''(x)}\right) v''(x) = -\alpha.$$

And so

$$B_1 \le \frac{1}{1-\mathrm{e}^{-\alpha}} < \infty.$$

This shows that

$$\lim_{r\to\infty} |S_1| = \lim_{r\to\infty} A_1 \cdot \lim_{r\to\infty} B_1 = 0.$$

Next we prove that

$$\lim_{r\to\infty} |S_3| = 0 \tag{16}$$

From (11) we have:

$$\begin{aligned}|S_3| &\le \sum_{n=n_1+1}^{\infty} \mathrm{e}^{-v(n)+v(\xi)} \cdot r^{n-\xi} \\ &= \left(\mathrm{e}^{-v(n_1+1)+v(\xi)} \cdot r^{n_1+1-\xi}\right)\left(\sum_{n=n_1+1}^{\infty} \mathrm{e}^{-v(n)+v(n_1+1)} \cdot r^{n-n_1-1}\right) \\ &= A_3 \times B_3,\end{aligned}$$

where

$$A_3 = \mathrm{e}^{-v(n_1+1)+v(\xi)} \cdot r^{n_1+1-\xi} \qquad B_3 = \sum_{n=n_1+1}^{\infty} \mathrm{e}^{-v(n)+v(n_1+1)} \cdot r^{n-n_1-1}.$$

To see that $B_3 < \infty$, we consider

$$\begin{aligned}B_3 &= \sum_{n=n_1+1}^{\infty} \mathrm{e}^{-\left\{(n-n_1-1)v'(n_1+1)+\frac{1}{2}(n-n_1-1)^2 v''(n_1+1+\theta(n-n_1-1))\right\}} \cdot \mathrm{e}^{(n-n_1-1)v'(x)} \\ &\qquad\qquad (0<\theta<1 \text{ and } \log r = v'(x)) \\ &= \sum_{n=n_1+1}^{\infty} \mathrm{e}^{(n-n_1-1)(v'(x)-v'(n_1+1))} \cdot \mathrm{e}^{-\frac{1}{2}(n-n_1-1)^2 v''(n_1+1+\theta(n-n_1-1))} \\ &\le \sum_{n=n_1+1}^{\infty} \mathrm{e}^{(n-n_1-1)(v'(x)-v'(n_1+1))} \\ &\qquad \left(\text{since} \quad -\frac{1}{2}(n-n_1-1)^2 v''(n_1+1+\theta(n-n_1-1)) \le 0\right)\end{aligned}$$

Put $m = n - n_1 - 1$.
Then

$$\begin{aligned}B_3 &\le \sum_{m=0}^{\infty} \mathrm{e}^{m(v'(x)-v'(n_1+1))} \\ &\le \frac{1}{1-\mathrm{e}^{v'(x)-v'(n_1+1)}} \qquad (17)\end{aligned}$$

because the geometric series is convergent, since $x < n_1+1 \Rightarrow v'(x) < v'(n_1+1)$.
Further,

$$\begin{aligned}v'(x) - v'(n_1+1) &= (x-n_1-1)\, v''(x+\theta(n_1+1-x)) \\ &\qquad (0<\theta<1) \\ &\le (x-n_1-1)\, v''(n_1+1) \qquad (18)\end{aligned}$$

because $x < n_1+1$ and $x+\theta(n_1+1-x) < n_1+1$ imply

$$v''(n_1+1) \le v''(x+\theta(n_1+1-x)).$$

Now if $x \ge x_0$, then $n_1+1 \ge x_0$ and from hypothesis (v),

$$v''(n_1+1) \ge \frac{\alpha}{n_1+1}.$$

But from (7),

$$n_1 + 1 \leq x + \frac{\alpha}{v''(x)} + 1$$

and so

$$v''\left(n_1+1\right) \geq \frac{\alpha}{x + \dfrac{\alpha}{v''(x)} + 1}. \tag{19}$$

But then from (18), we have

$$\begin{aligned} v'(x) - v'(n_1+1) &\leq (x - n_1 - 1)\frac{\alpha}{x + \dfrac{\alpha}{v''(x)} + 1} \\ &\leq \left(\frac{-\alpha}{v''(x)}\right)\left(\frac{\alpha}{x + \dfrac{\alpha}{v''(x)} + 1}\right) \\ &= \frac{-\alpha^2}{xv''(x) + \alpha + v''(x)}. \end{aligned}$$

Since $x \geq x_0 > 0$ implies $v''(x) \leq v''(0)$, and so from hypothesis (v) again,

$$xv''(x) + \alpha + v''(x) \leq \alpha + \beta + v''(0). \tag{20}$$

But then

$$v'(x) - v'\left(n_1+1\right) \leq \frac{-\alpha^2}{\alpha + \beta + v''(0)}$$

and so from (17), we have

$$B_3 \leq \frac{1}{1 - e\dfrac{-\alpha 2}{\alpha + \beta + v''(0)}} < \infty.$$

As for A_3, we have:

$$\begin{aligned} A_3 &= \left(\mathrm{e}^{v(\xi)} \cdot r^{-\xi}\right)\left(\mathrm{e}^{-v(n_1+1)} \cdot r^{n_1+1}\right) \\ &= \left(\mathrm{e}^{v(\xi)} \cdot r^{-\xi}\right)\Big(\mathrm{e}^{-\ \left\{v(x)+(n_1+1-x)v'(x)+\frac{1}{2}(n_1+1-x)^2 v''(x+\theta(n_1+1-x))\right\}} \\ &\qquad \cdot \mathrm{e}^{(n_1+1)v'(x)}\Big) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (0 < \theta < 1) \\ &= \left(\mathrm{e}^{-v(\xi)-v(x)+(-\xi+x)v'(x)}\right)\left(\mathrm{e}^{-\frac{1}{2}(n_1+1-x)^2 v"(x+\theta(n_1+1-x))}\right) \\ &= C_3 \times D_3, \qquad \text{say.} \end{aligned}$$

Since $C_3 = C_1$, $\qquad \lim_{r\to\infty} C_3 = 1.$

Also

$$x + \theta\left(n_1 + 1 - x\right) < n_1 + 1$$

implies

$$v''\left(x + \theta\left(n_1 + 1 - x\right)\right) \geq v''\left(n_1 + 1\right)$$

and so

$$D_3 \le \mathrm{e}^{-\frac{1}{2}\left(n_1+1-x\right)^2 v''\left(n_1+1\right)}. \tag{21}$$

Using (7) and (19), we have

$$\begin{aligned}
D_3 &\le \mathrm{e}^{-\frac{1}{2}\left(\frac{\alpha}{v''(x)}+1\right)^2\left(\frac{\alpha}{x+\frac{\alpha}{v''(x)}+1}\right)} \\
&\le \mathrm{e}^{-\frac{1}{2}\left(\frac{\alpha}{v''(x)}+1\right)\left(\frac{\alpha}{v''(x)}+1\right)\left(\frac{\alpha}{x+\frac{\alpha}{v''(x)}+1}\right)} \\
&\le \mathrm{e}^{-\frac{1}{2}\left(\frac{\alpha}{v''(x)}+1\right)\left(\frac{\alpha^2}{xv''(x)+\alpha+v''(x)}+\frac{\alpha}{x+\frac{\alpha}{v''(x)}+1}\right)} \\
&\le \mathrm{e}^{-\frac{1}{2}\left(\frac{\alpha}{v''(x)}+1\right)\left(\frac{\alpha^2}{xv''(x)+\alpha+v''(x)}\right)} \\
&\qquad\qquad\qquad\qquad \left(\text{since} \quad \frac{\alpha}{x+\frac{\alpha}{v''(x)}+1} \ge 0\right) \\
&\le \mathrm{e}^{-\frac{1}{2}\left(\frac{\alpha}{v''(x)}+1\right)\left(\frac{\alpha^2}{\alpha+\beta+v''(0)}\right)} \qquad\qquad (\text{by } (20)) \\
&\longrightarrow 0 \quad \text{as } \ x \to \infty.
\end{aligned}$$

And so $\lim\limits_{x\to\infty} D_3 = 0 \quad \Rightarrow \quad \lim\limits_{r\to\infty} D_3 = 0 \quad \Rightarrow \quad \lim\limits_{r\to\infty} A_3 = 0.$

This proves that $\lim\limits_{r\to\infty} |S_3| = 0.$

Now consider

$$S_2 = \sum_{n=n_0+1}^{n_1} \mathrm{e}^{-v(n)+v(\xi)} \cdot r^{n-\xi} \cdot \mathrm{e}^{\mathrm{i}\,n\theta}.$$

First we assume that for n, $n_0+1 \le n \le n_1, |(n-\xi)\theta| \le \dfrac{\pi}{2}$ and show that $|S_2| \ge 1$.

Clearly S_2 can be expressed as:

$$S_2 = \mathrm{e}^{i\xi\theta} \sum_{n=n_0+1}^{n_1} \mathrm{e}^{-v(n)+v(\xi)} \cdot r^{n-\xi} \cdot \mathrm{e}^{\mathrm{i}(n-\xi)\theta}$$

and so

$$\begin{aligned}
|S_2| &\ge \left| \mathrm{Re} \sum_{n=n_0+1}^{n_1} \mathrm{e}^{-v(n)+v(\xi)} \cdot r^{n-\xi} \cdot \mathrm{e}^{\mathrm{i}(n-\xi)\theta} \right| \\
&\ge \left| \sum_{n=n_0+1}^{n_1} \mathrm{e}^{-v(n)+v(\xi)} \cdot r^{n-\xi} \cdot \cos(n-\xi)\theta \right|.
\end{aligned}$$

In view of our assumption: $|(n-\xi)\,\theta| \leq \dfrac{\pi}{2}$ for $n_0+1 \leq n \leq n_1$, we have

$$\cos(n-\xi)\,\theta \geq 0$$

for $n_0+1 \leq n \leq n_1$ and so

$$\begin{aligned} |S_2| &\geq \cos(\xi-\xi)\,\theta \cdot r^{(\xi-\xi)}\,\mathrm{e}^{-v(\xi)+v(\xi)} \\ &\quad + \sum_{\substack{n=n_0+1 \\ n\neq\xi}}^{n_1} \mathrm{e}^{-v(n)+v(\xi)} \cdot r^{n-\xi} \cdot \cos(n-\xi)\theta \\ &\geq 1+0 \end{aligned}$$

and so $|S_2| \geq 1$, if

$$|(n-\xi)\,\theta| \leq \frac{\pi}{2} \text{ for } n_0+1 \leq n \leq n_1.$$

Now we show that the assumption:

$|(n-\xi)\,\theta| \leq \dfrac{\pi}{2}$, $n_0+1 \leq n \leq n_1$ is implied by those $z = x + \mathrm{i}\,y$ whose y-coordinate satisfies the condition

$$|y| \leq \frac{\sqrt{2}r}{\dfrac{\alpha}{v''(x)}+2}.$$

Let $\theta \geq 0$. Then $|(n-\xi)\,\theta| \leq \dfrac{\pi}{2}$, $n_0+1 \leq n \leq n_1$ yields

$$(n_1+1-\xi)\,\theta \leq \frac{\pi}{2} \qquad \text{and} \qquad (\xi-n_0)\,\theta \leq \frac{\pi}{2}.$$

But

$$\begin{aligned} |n_1+1-\xi| &\leq |n_1+1-x| + |x-\xi| \\ &\leq |n_1+1-x| + 1 = (n_1+1-x)+1 \end{aligned}$$

(since $x \leq n_1+1$) and from (7)

$$n_1+1-x \leq \frac{\alpha}{v''(x)}+1$$

implies $|n_1+1-\xi| \leq \left(\dfrac{\alpha}{v''(x)}+1\right)+1 = \dfrac{\alpha}{v''x)}+2.$ Also

$$\begin{aligned} |\xi-n_0| &\leq |\xi-x| + |x-n_0| \leq 1 + |x-n_0| \\ &\leq 1+x-n_0 && \text{(since } n_0 < x\text{)} \\ &\leq 1+\left(\frac{\alpha}{v''(x)}+1\right) && \text{(from (6))} \\ &= \frac{\alpha}{v''(x)}+2. \end{aligned}$$

Thus the assumption:

$$|(n-\xi)\,\theta| \leq \frac{\pi}{2} \text{ for } n_0+1 \leq n \leq n_1$$

is implied by the condition that

$$\left(\frac{\alpha}{v''(x)}+2\right)\theta \le \frac{\pi}{2} \text{ for } \theta \ge 0$$

or

$$\theta \le \left(\frac{\pi}{2}\right)\Big/\left(\frac{\alpha}{v''(x)}+2\right). \tag{22}$$

Set

$$\delta_x = \left(\frac{\pi}{2}\right)\Big/\left(\frac{\alpha}{v''(x)}+2\right) \ge 0. \tag{23}$$

To estimate δ_x we use Euler's formular see [**7**]:

$$\sin(\pi\omega) = (\pi\omega)\prod_{k=1}^{\infty}\left(1-\frac{\omega^2}{k^2}\right).$$

Substituting θ for $\pi\omega$, we get

$$\sin(\theta) = \theta\prod_{k=1}^{\infty}\left(1-\frac{\theta^2}{\pi^2k^2}\right) \tag{24}$$

The product in (24) is ≥ 0 if for all $k \ge 1$, $1-\dfrac{\theta^2}{\pi^2k^2} \ge 0$. In particular, $1-\dfrac{\theta^2}{\pi^2} \ge 0$ if and only if $\theta \le \pi$ for $\theta \ge 0$. Clearly

$$\frac{\alpha}{v''(x)}+2 \ge 2$$

for all $x \ge 0$ and so

$$\frac{\pi}{4} \ge \left(\frac{\pi}{2}\right)\Big/\left(\frac{\alpha}{v''(x)}+2\right) \implies 0 \le \theta \le \delta_x \le \frac{\pi}{4}.$$

Further, if $0 \le \theta_1 \le \theta_2 \le \pi$, then $\theta_1^2 \le \theta_2^2$ implies

$$1-\frac{\theta_1^2}{\pi^2k^2} \ge 1-\frac{\theta_2^2}{\pi^2k^2}$$

for all $k \ge 1$ and so

$$\prod_{k=1}^{\infty}\left(1-\frac{\theta_1^2}{\pi^2k^2}\right) \ge \prod_{k=1}^{\infty}\left(1-\frac{\theta_2^2}{\pi^2k^2}\right).$$

But then $0 \le \theta \le \delta_x \le \dfrac{\pi}{4}$ implies

$$\frac{\sin\theta}{\theta} = \prod_{k=1}^{\infty}\left(1-\frac{\theta^2}{\pi^2k^2}\right) \ge \prod_{k=1}^{\infty}\left(1-\frac{\delta_x^2}{\pi^2k^2}\right) \ge \prod_{k=1}^{\infty}\left(1-\frac{\left(\frac{\pi}{4}\right)^2}{\pi^2k^2}\right). \tag{25}$$

But the last term $= \dfrac{2\sqrt{2}}{\pi}$ if we put $\theta = \dfrac{\pi}{4}$ in (24).

Thus from (25), we obtain

$$\theta \leq \frac{\pi \sin\theta}{2\sqrt{2}}.$$

Remark: As the referee pointed out, the last inequality can also be obtained by elementary means, noting that $\frac{d}{dx}\left(\frac{\sin x}{x}\right) < 0$ in $(0, \frac{\pi}{4}]$ and the value of $\frac{\sin x}{x} = \frac{2\sqrt{2}}{\pi}$ at $x = \frac{\pi}{4}$. Now the inequality in (22) will be satisfied if we set

$$\frac{\pi \sin\theta}{2\sqrt{2}} \leq \left(\frac{\pi}{2}\right) \Big/ \left(\frac{\alpha}{v''(x)} + 2\right) \quad \text{or} \quad \sin\theta \leq \sqrt{2} \Big/ \left(\frac{\alpha}{v''(x)} + 2\right)$$

$$\Longrightarrow \qquad |y| = r\sin\theta \leq \frac{\sqrt{2}r}{\frac{\alpha}{v''(x)} + 2}. \tag{26}$$

Thus we have shown that if $z = x + \mathrm{i}\, y$ is such that

$$|y| \leq \frac{\sqrt{2}r}{\frac{\alpha}{v''(x)} + 2},$$

then

$$|S_2| \geq 1.$$

Since we have already established that

$$\lim_{r\to\infty} |S_1| = 0 = \lim_{r\to\infty} |S_3|$$

and the fact from (9) that

$$|F(z)| \geq T(r)\left(|S_2| - |S_1| - |S_3|\right)$$

the proof of the theorem follows, since $F(r)$ and $T(r)$ differ only slightly (see [**7**]). In other words, there is $r_0 > 0$ such that $r \geq r_0$ and for all $z = x + iy$ with

$$|y| \leq \left(\sqrt{2}r\right) \Big/ \left(\frac{\alpha}{v''(x)} + 2\right)$$

we have $|F(z)| \geq MF(r)$ for some $M > 0$. Since $f(r) \geq |f(z)|$, clearly $0 \leq M \leq 1$ for all z. □

Example. As a particular case of the theorem take:

$$F(z) = \sum_{n=0}^{\infty} \mathrm{e}^{-(n+1)\log(n+1)+n} \cdot z^n = \sum_{n=0}^{\infty} \frac{\mathrm{e}^n}{(n+1)^{n+1}} \cdot z^n \ .$$

Here $v(x) = (x+1)\log(x+1) - x$, for $x \geq 0$, $v'(x) = \log(x+1)$ and $v''(x) = \frac{1}{x+1}$.

Choose $x_0 = 1$, then for all $x \geq 1$, $\frac{1}{2} \leq xv''(x) = \frac{x}{x+1} \leq 1$, i.e. $\alpha = \frac{1}{2}$, $\beta = 1$ for all $x \geq 1$.

Thus $v(x)$ satisfies all the conditions of the theorem. For the index of the maximum term of

$$F(r) = \sum_{n=0}^{\infty} \frac{\mathrm{e}^n}{(n+1)^{n+1}} \cdot r^n$$

we solve: $v'(x) = \log r$, i.e. $\log(x+1) = \log r \Rightarrow x = r-1$ and so $\xi = [x] = [r]-1$.

Here $\delta_x = \dfrac{\frac{\pi}{2}}{\dfrac{\alpha}{v''(x)}+2} = \dfrac{\pi}{x+5} \leq \dfrac{\pi}{6}$ for all $x \geq 1$ and so $0 \leq |\theta| \leq \dfrac{\pi}{6}$. Thus if for $z = x + iy$,

$$|y| \leq \frac{2\sqrt{2}r}{x+5} \leq \frac{\sqrt{2}}{3}r, \quad x \geq 1$$

then there is r_0 such that for $r \geq r_0$ and for all those $z = x + \mathrm{i}\,y$ for which $|y| \leq \dfrac{\sqrt{2}}{3}r$, we have

$$|F(z)| \geq MF(r)$$

for some $M > 0$, depending on r.

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