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FREE SYSTEMS OF ALGEBRAS AND ULTRACLOSED CLASSES

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ABSTRACT. There is considered the concept of the so-called free system of algebras for an ultraclosed class of algebras of a fixed arithmetic type. Certain free systems exist for such a class if and only if the class is defined by finite disjunctions of identities where the operational symbols are interpreted as operational variables for fundamental operations of an algebra.

1. INTRODUCTION AND SUMMARY

Among various concepts in algebra one of the most useful is that of the free algebra. G. Birkhoff [**1**] defines (for a fixed type τ) a free algebra in a class of algebras of type τ. Especially, free algebras are related with equationally defined classes and can be characterized by identities of type τ.

The subject of this paper concerns with a similar connection between the concept of the *free system of algebras* for a class of algebras and classes which are defined by disjunctions of identities where the operational symbols are interpreted as *operational variables* for fundamental operations of an algebra.

Ordinary disjunctions of identities have been investigated as so-called power identities on semigroups [**2**], [**4**], especially and also in a more general form [**7**].

In this paper the use of a disjunction of identities corresponds to the concept of a disjunction of second-order formulas

$$\forall X_1 \dots \forall X_m \forall x_1 \dots \forall x_n (w_1 \approx w_2)$$

(for short $w_1 \approx w_2$) on an algebra where $X_1, \dots, X_m$ are operational variables for fundamental operations of the algebra and $x_1, \dots, x_n$ are the individual variables in the terms w_1 and w_2. A second-order formula $w_1 \approx w_2$ is a hyperidentity by Yu. Movsisyan [**5**].

Another concept of hyperidentity was introduced by W. Taylor where an identity $w_1 \approx w_2$ becomes to a hyperidentity on an algebra if the operational variables are variables for derived operations of the algebra [**6**]. In this paper we do not apply the concept of a hyperidenty in the sense of W. Taylor.

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The paper deals with a class K of algebras of several types τ but of a fixed *arithmetic type* $\mathcal{N}$, i.e., for each algebra it is $\mathcal{N}$ the set of the arities of the operations [**5**]. Especially, each algebra has a reduct of any type.

We introduce a free system for K (with respect to a set X of individual variables and a set F of operational symbols) to be a set $U \subseteq K$ such that each $\mathcal{C} \in U$ is a homomorphic image of the termalgebra $\mathcal{T}(X, F)$ and each homomorphism

$$\mathcal{T}(X, F) \longrightarrow \mathcal{A}$$

from $\mathcal{T}(X, F)$ into any reduct $\mathcal{A}$ (of the appropriate type) of an algebra $\mathcal{B} \in K$ is the composition

$$\mathcal{T}(X, F) \stackrel{\text{onto}}{\longrightarrow} \mathcal{C} \longrightarrow \mathcal{A}$$

of a homomorphism from $\mathcal{T}(X, F)$ onto some $\mathcal{C} \in U$ with a homomorphism from $\mathcal{C}$ into $\mathcal{A}$.

Furthermore, we investigate the existence and construction of free systems for classes of algebras.

Then we consider certain classes K which are closed under the formation of ultraproducts. For such a class K there exists a free system for any set X and any set F if and only if K is defined by finite disjunctions

$$P_1 \approx Q_1 \vee \ldots \vee P_n \approx Q_n$$

of identities $P_1 \approx Q_1, \ldots, P_n \approx Q_n$ in the following way. For each identity $P_i \approx Q_i$ the terms P_i and Q_i are usual compositions of individual variables and operational symbols. However, the operational symbols are interpreted as operational variables for fundamental operations of an algebra [**5**]. Therefore, it is said that a disjunction *holds in* an algebra $\mathcal{A}$ if whenever the individual variables are replaced by any elements $a \in A$ and the operational symbols are replaced by any fundamental operations of $\mathcal{A}$ of the appropriate arity, then in the disjunction there exist some identity $P_i \approx Q_i$ such that the values of P_i and Q_i are equal.

2. Basic Notions

In this section we introduce some basic notions with respect to the considered algebras [**3**], [**5**].

(a) Let $\mathcal{N}$ be a fixed *arithmetic type*, i.e., a set of natural numbers (greater than zero). A *type* τ_F is a function from a set F of finitary operational symbols into the set of the natural numbers where f is a $\tau_F(f)$-ary operational symbol for $f \in F$. We assume that

$$\mathcal{N} = \{\tau_F(f) \colon f \in F\}$$

and define

$$F_n := \{f \colon f \in F \text{ and } \tau_F(f) = n\}$$

for each $n \in \mathcal{N}$.

(b) In the following we consider algebras $\mathcal{A}$ of a type τ_F (or simply τ_F-*algebras*) such that

$$\mathcal{A} = (A, (f^{\mathcal{A}})_{f \in F})$$

where each $\tau_F(f)$-ary operational symbol f is associated with some $\tau_F(f)$-ary operation $f^{\mathcal{A}}$ on A.

(c) For a type τ_F and a set X of individual variables let

$$\mathcal{T}(X,F) = (\mathrm{T}(X,F), (f^{\mathcal{T}(X,F)})_{f\in F})$$

be the *algebra of terms over* X *and* F of type τ_F where T(X,F) denotes the set of all terms over X and F and each $\tau_F(f)$-ary operational symbol f corresponds with the $\tau_F(f)$-ary operation $f^{\mathcal{T}(X,F)}$ on $\mathrm{T}(X,F)$.

(d) Let $\mathcal{A}$ be a τ_F-algebra and $\mathcal{B}$ be a τ_G-algebra. Then $\mathcal{A}$ is called to be a τ_F*-reduct of* $\mathcal{B}$ if $A = B$ and

$$\{f^{\mathcal{A}} \colon f \in F\} \subseteq \{g^{\mathcal{B}} \colon g \in G\}.$$

(e) A class K of algebras is called to be *closed under the formation of reducts* if for any type τ_F and $\mathcal{B} \in K$ each τ_F-reduct of $\mathcal{B}$ belongs to K.

Proposition 2.1. *Let* $\mathcal{B}$ *be a* τ_G*-algebra. Then for each type* τ_F *there exists a* τ_F*-reduct* $\mathcal{A}$ *of* $\mathcal{B}$*.*

Proof. Let $\mathcal{B}$ be a τ_G-algebra and τ_F be a type. Now, we construct a τ_F-algebra $\mathcal{A}$. For this let $A := B$ and we assume that the set G is well-ordered. If $f \in F$, then let $f^{\mathcal{A}} := g^{\mathcal{B}}$ where g is the least element of the set $G_{\tau_F(f)} = \{g \colon \tau_G(g) = \tau_F(f)\}$ which is not empty. By construction the algebra $\mathcal{A}$ is a τ_F-reduct of $\mathcal{B}$. $\square$

(f) For a class K of algebras let $\Phi_{XF}(K)$ be the set of all homomorphisms φ from $\mathcal{T}(X,F)$ into any τ_F-reduct $\mathcal{A}$ of some $\mathcal{B} \in K$ (because of Proposition 2.1 there exists such a τ_F-reduct).

(g) Let $\mathcal{B}$ be an algebra and $D \subseteq T(X,F) \times T(X,F)$. We say that the *disjunction* (*of identities*)

$$\bigvee_{(P,Q)\in D} P \approx Q$$

holds in $\mathcal{B}$ (in symbols, $\mathcal{B} \models D$) if $D \cap \ker(\varphi) \neq \emptyset$ for all $\varphi \in \Phi_{XF}(\{\mathcal{B}\})$.

(h) K is called to be *defined by disjunctions* if there are a set $\mathbb{X}$ of individual variables, a set $\mathbb{F}$ of operational symbols and a family Δ of sets $D \subseteq T(\mathbb{X},\mathbb{F}) \times T(\mathbb{X},\mathbb{F})$ such that K is equal to the class of all algebras $\mathcal{A}$ with $\mathcal{A} \models D$ for each $D \in \Delta$ (in symbols, $K = \mathrm{MOD}(\Delta)$).

(i) Especially, K is called to be defined by *finite* disjunctions if there are a set $\mathbb{X}$ of individual variables with $|\mathbb{X}| = \aleph_0$, a set $\mathbb{F}$ of operational symbols with $|\mathbb{F}_n| = \aleph_0$ for each $n \in \mathcal{N}$ and a family Δ of finite sets $D \subseteq T(\mathbb{X},\mathbb{F}) \times T(\mathbb{X},\mathbb{F})$ such that $K = \mathrm{MOD}(\Delta)$.

(j) We define $\mathrm{DIS}_{XF}(K)$ to be the family of all sets $D \subseteq T(X,F) \times T(X,F)$ such that $\mathcal{A} \models D$ for each $\mathcal{A} \in K$.

3. Free Systems

We consider the existence and construction of free systems with respect to a set X of individual variables and a set F of operational symbols.

Definition 3.1. Let K be a class of algebras. Then a *free system for* K (*over* X *and* F) is defined to be a set $U \subseteq K$ such that each $\mathcal{C} \in U$ is a τ_F-algebra which is a homomorphic image of $\mathcal{T}(X,F)$ and each homomorphism from $\mathcal{T}(X,F)$ into any τ_F-reduct $\mathcal{A}$ of a $\mathcal{B} \in K$ is the composition of a homomorphism from $\mathcal{T}(X,F)$ onto some $\mathcal{C} \in U$ with a homomorphism from $\mathcal{C}$ into $\mathcal{A}$.

Definition 3.2. For sets U and U' of τ_F-algebras we define $U \prec U'$ (over X and F) if each homomorphism from $\mathcal{T}(X,F)$ onto any $\mathcal{C}' \in U'$ is the composition of a homomorphism from $\mathcal{T}(X,F)$ onto some $\mathcal{C} \in U$ with a homomorphism from $\mathcal{C}$ onto $\mathcal{C}'$.

Let $\mathrm{I}(\mathrm{DIS}_{XF}(K))$ be the set of all $E \subseteq \mathrm{T}(X,F) \times \mathrm{T}(X,F)$ with $E \cap D \neq \emptyset$ for all $D \in \mathrm{DIS}_{XF}(K)$ and $\sigma(E)$ be that congruence relation on the termalgebra $\mathcal{T}(X,F)$ which is generated by some $E \in \mathrm{I}(\mathrm{DIS}_{XF}(K))$.

Proposition 3.3. *Let* K *be a class of algebras and* U *be a set of* τ_F*-algebras. Then the following statements are equivalent.*

(i) U *is a free system for* K (*over* X *and* F).
(ii) $U \prec \{\mathcal{T}(X,F)/\sigma(E) \colon E \in \mathrm{I}(\mathrm{DIS}_{XF}(K))\}$ (*over* X *and* F) *and* $U \subseteq K$.

Proof. (i)$\Longrightarrow$(ii): We assume that U is a free system of algebras for K (over X and F). Now, let α be a homomorphism from $\mathcal{T}(X,F)$ onto some $\mathcal{T}(X,F)/\sigma(E)$ with $E \in \mathrm{I}(\mathrm{DIS}_{XF}(K))$. Then there exists a homomorphism φ from $\mathcal{T}(X,F)$ into some τ_F-reduct $\mathcal{A}$ of a $\mathcal{B} \in K$ such that $\ker(\varphi) \subseteq \sigma(E)$. Otherwise,

$$(\ker(\varphi) \setminus E) \supseteq (\ker(\varphi) \setminus \sigma(E)) \neq \emptyset$$

for each $\varphi \in \Phi_{XF}(K)$. We define $D := \bigcup\{(\ker(\varphi) \setminus E) \colon \varphi \in \Phi_{XF}(K)\}$. Then $D \in \mathrm{DIS}_{XF}(K)$, $E \cap D = \emptyset$. This contradicts the assumption that $E \in \mathrm{I}(\mathrm{DIS}_{XF}(K))$.

The homomorphism φ into $\mathcal{A}$ is also a homomorphism onto a subalgebra $\mathcal{A}'$ of $\mathcal{A}$.

Because of $\ker(\varphi) \subseteq \sigma(E)$ there exists a homomorphism φ' from the algebra $\mathcal{A}'$ onto $\mathcal{T}(X,F)/\sigma(E)$ such that $\alpha = \varphi' \cdot \varphi$.

Since U is a free system for K (over X and F) it follows that the homomorphism φ is the composition $\gamma \cdot \xi$ of a homomorphism ξ from $\mathcal{T}(X,F)$ onto some $\mathcal{C} \in U$ with a homomorphism γ from $\mathcal{C}$ into $\mathcal{A}$. Therefore, α is the composition $(\varphi' \cdot \gamma) \cdot \xi$ of the homomorphism ξ from $\mathcal{T}(X,F)$ onto some $\mathcal{C} \in U$ with the homomorphism $\varphi' \cdot \gamma$ from $\mathcal{C}$ onto $\mathcal{T}(X,F)/\sigma(E)$. Consequently,

$$U \prec \{\mathcal{T}(X,F)/\sigma(E) \colon E \in \mathrm{I}(\mathrm{DIS}_{XF}(K))\}$$

(over X and F). By assumption it is $U \subseteq K$, finally.

(ii)$\Longrightarrow$(i): We assume

$$U \prec \{\mathcal{T}(X,F)/\sigma(E) \colon E \in \mathrm{I}(\mathrm{DIS}_{XF}(K))\}$$

(over X and F) and $U \subseteq K$. Now, let φ be a homomorphism from $\mathcal{T}(X,F)$ into any τ_F-reduct $\mathcal{A}$ of a $\mathcal{B} \in K$. The homomorphism φ is also a homomorphism onto a subalgebra $\mathcal{A}'$ of $\mathcal{A}$ with $\mathcal{A}' \cong \mathcal{T}(X,F)/\ker(\varphi)$. Then there exists some $E \in \mathrm{I}(\mathrm{DIS}_{XF}(K))$ such that

$$\mathcal{T}(X,F)/\ker(\varphi) = \mathcal{T}(X,F)/\sigma(E).$$

Clearly, for $D \in \mathrm{DIS}_{XF}(K)$ it is $\ker(\varphi) \cap D \neq \emptyset$ and $(\ker(\varphi) \cup D) \in \mathrm{DIS}_{XF}(K)$. Consequently, for

$$E := \bigcup\{\ker(\varphi) \cap D \colon D \in \mathrm{DIS}_{XF}(K)\}$$

there hold $E \in \mathrm{I}(\mathrm{DIS}_{XF}(K))$ and $E = \ker(\varphi) = \sigma(E)$.

Therefore, from the assumption it follows that the homomorphism φ is the composition of a homomorphism from $\mathcal{T}(X, F)$ onto some $\mathcal{C} \in U$ with a homomorphism from $\mathcal{C}$ into $\mathcal{A}$. Because of $U \subseteq K$ it is U a free system for K (over X and F). □

Proposition 3.4. *Let K be a class of algebras, U be a free system for K (over X and F) and U' be a set of τ_F-algebras. Then the following statements are equivalent.*

(i) *U' is a free system for K (over X and F).*
(ii) *$U' \prec U$ (over X and F) and $U' \subseteq K$.*

Proof. (i)$\Longrightarrow$(ii): Let U' be a free system for K (over X and F). Then $U' \subseteq K$, consequently and we show $U' \prec U$. For this let α be a homomorphism from $\mathcal{T}(X, F)$ onto any $\mathcal{C} \in U$. Because $\mathcal{C}$ is a τ_F-algebra it is $\mathcal{C}$ a τ_F-reduct of itself. By assumption U' is a free system for K. Therefore, α is the composition of a homomorphism from $\mathcal{T}(X, F)$ onto some $\mathcal{C}' \in U'$ with a homomorphism from $\mathcal{C}'$ onto $\mathcal{C}$, i.e., $U' \prec U$ (over X and F).

(ii)$\Longrightarrow$(i): Let $U' \prec U$ (over X and F), $U' \subseteq K$ and α be a homomorphism from $\mathcal{T}(X, F)$ into any τ_F-reduct $\mathcal{A}$ of a $\mathcal{B} \in K$. Because U is a free system for K (over X and F) there exist a $\mathcal{C} \in U$, a homomorphism β from $\mathcal{T}(X, F)$ onto $\mathcal{C}$ and a homomorphism γ from $\mathcal{C}$ into $\mathcal{A}$ such that $\alpha = \gamma \cdot \beta$. From $U' \prec U$ it follows that there are a $\mathcal{C}' \in U'$, a homomorphism β' from $\mathcal{T}(X, F)$ onto $\mathcal{C}'$ and a homomorphism γ' from $\mathcal{C}'$ onto $\mathcal{C}$ such that $\beta = \gamma' \cdot \beta'$. Consequently, $\alpha = (\gamma \cdot \gamma') \cdot \beta'$ and $\gamma \cdot \gamma'$ is a homomorphism from $\mathcal{C}'$ into $\mathcal{A}$, i.e., U' is a free system for K (over X and F). □

Proposition 3.5. *Let K be a class of algebras which is defined by disjunctions, X be a set of individual variables and F be a set of operational variables. Then there exists a free system U for K (over X and F).*

Proof. By assumption it is $K = \mathrm{Mod}(\Delta)$ and Δ is a family of sets $D \subseteq T(\mathbb{X}, \mathbb{F}) \times T(\mathbb{X}, \mathbb{F})$. Let

$D(\xi)$ be the set of all $(\xi(P), \xi(Q))$ with $(P, Q) \in D$ and a homomorphism ξ from $\mathcal{T}(\mathbb{X}, \mathbb{F})$ into any $\tau_{\mathbb{F}}$-reduct of $\mathcal{T}(X, F)$,

$\mathrm{H}(\Delta)$ be the set of all $D(\xi)$ with respect to all $D \in \Delta$ and to all homomorphisms ξ from $\mathcal{T}(\mathbb{X}, \mathbb{F})$ into any $\tau_{\mathbb{F}}$-reduct $\mathcal{T}(X, F)$,

$\mathrm{I}(\mathrm{H}(\Delta))$ be the set of all $E \subseteq \mathrm{T}(X, F) \times \mathrm{T}(X, F)$ with $E \cap D \neq \emptyset$ for $D \in \mathrm{H}(\Delta)$,

$\sigma(E)$ be that congruence relation on the termalgebra $\mathcal{T}(X, F)$ which is generated by some $E \in \mathrm{I}(\mathrm{H}(\Delta))$.

At first, we define a set U. Let U be the set of all algebras $\mathcal{T}(X, F)/\sigma(E)$ with $E \in \mathrm{I}(\mathrm{H}(\Delta))$.

It holds $U \subseteq K$. For this let $\mathcal{B} \in U$ and $D \in \Delta$. Then there holds $\mathcal{B} = \mathcal{T}(X,F)/\sigma(E)$ for some $E \in \mathrm{I}(\mathrm{H}(\Delta))$. Now, let φ be a homomorphism from $\mathcal{T}(\mathbb{X},\mathbb{F})$ into any $\tau_{\mathbb{F}}$-reduct $(\mathcal{T}(X,F)/\sigma(E))_{\mathbb{F}}$ of $\mathcal{T}(X,F)/\sigma(E)$. Then there exists a homomorphism ξ from $\mathcal{T}(\mathbb{X},\mathbb{F})$ into a appropriate $\tau_{\mathbb{F}}$-reduct $(\mathcal{T}(X,F))_{\mathbb{F}}$ of $\mathcal{T}(X,F)$ such that

$$\varphi(t) = [\xi(t)]_{\sigma(E)} \in (\mathcal{T}(X,F)/\sigma(E))_{\mathbb{F}}$$

for $t \in \mathrm{T}(\mathbb{X},\mathbb{F})$. Because of the definition of E it is a pair $(P,Q) \in D$ such that $(\xi(P),\xi(Q)) \in E \subseteq \sigma(E)$ and therefore

$$\varphi(P) = [\xi(P)]_{\sigma(E)} = [\xi(Q)]_{\sigma(E)} = \varphi(Q).$$

From this it follows $\mathcal{B} \models D$ and $U \subseteq K$.

Then U is a free system of K. Obviously, each $\mathcal{C} \in U$ is a τ_F-algebra which is a homomorphic image of $\mathcal{T}(X,F)$. Now, let $\mathcal{A}$ be a τ_F-reduct of an algebra $\mathcal{B} \in K$ and α be a homomorphism from $\mathcal{T}(X,F)$ into $\mathcal{A}$. We define E to be the family of all sets

$$\{(\xi(P),\xi(Q)) \colon (P,Q) \in D \ \text{ and } \ (\alpha \cdot \xi)(\mathrm{P}) = (\alpha \cdot \xi)(\mathrm{Q})\}$$

with respect to all homomorphisms ξ from $\mathcal{T}(\mathbb{X},\mathbb{F})$ into any $\tau_{\mathbb{F}}$-reduct of $\mathcal{T}(X,F)$ and all $D \in \Delta$. Because of $\mathcal{B} \in K = \mathrm{Mod}(\Delta)$ the elements of E are nonempty sets and therefore $E \in \mathrm{I}(\mathrm{H}(\Delta))$, i.e., $\mathcal{C} := \mathcal{T}(X,F)/\sigma(E) \in U$ and there is a homomorphism β from $\mathcal{T}(X,F)$ onto $\mathcal{C}$ such that $\beta(t) = [t]_{\sigma(E)}$ for $t \in T(X,F)$.

It is easy to check that from $(s,t) \in \sigma(E)$ it follows that $\alpha(s) = \alpha(t)$. Therefore, it exists a homomorphism φ from $\mathcal{C}$ into $\mathcal{A}$ such that $\varphi([t]_{\sigma(E)}) = \alpha(t)$ for $[t]_{\sigma(E)} \in C$.

Consequently, $\alpha(t) = \varphi([t]_{\sigma(E)}) = \varphi(\beta(t))$ for $t \in T(X,F)$, i.e., $\alpha = \varphi \cdot \beta$ and U is a free system for K. $\square$

4. Ultraclosed Classes

In the following section we consider free systems for an ultraclosed class of algebras.

For this let K be a class of algebras. Then K is called to be *ultraclosed* if for each type τ_F, any (not empty) set $\{\mathcal{A}_i \colon i \in I\} \subseteq K$ of τ_F-algebras and any ultrafilter J on I the filtered product $\prod_{i\in I} \mathcal{A}_i/J$ belongs to K. (We assume that the filters are proper, i.e., $\emptyset \notin J$, especially.)

Proposition 4.1. *Let K be a class of algebras which is closed under the formation of reducts and ultraclosed. Then for each $D \subseteq T(X,F) \times T(X,F)$ it exists a finite subset $D' \subseteq D$ such that for each algebra $\mathcal{B} \in K$ from $\mathcal{B} \models D$ it follows that $\mathcal{B} \models D'$.*

Proof. Let K be a class of algebras such that K is closed under the formation of reducts and ultraclosed. Furthermore, let $D \subseteq T(X,F) \times T(X,F)$.

Clearly, there is the least cardinal number λ such that it exists some $D' \subseteq D$ where $|D'| = \lambda$ and for each algebra $\mathcal{B} \in K$ from $\mathcal{B} \models D$ it follows that $\mathcal{B} \models D'$.

Now, it is proved that $\lambda < \aleph_0$: Otherwise, $\lambda \geq \aleph_0$. Let α be the least ordinal number such that $|\{i \colon 0 \leq i < \alpha\}| = \lambda$. Since λ is an infinite cardinal number it follows that α is a limit ordinal number. Then

$$D' = \{(P_i, Q_i) : 0 \leq i < \alpha\}$$

and

$$|\{(P_j, Q_j) : 0 \leq j \leq i\}| < \lambda$$

for $i < \alpha$. Consequently, for each $i < \alpha$ there is a homomorphism φ_i from $\mathcal{T}(X, F)$ into a τ_F-reduct $\mathcal{A}_i$ of some $\mathcal{B} \in K$ such that $\varphi_i(P_j) \neq \varphi_i(Q_j)$ for each $j \leq i$ and $\varphi_i(P_j) = \varphi_i(Q_j)$ for some $j > i$.

Let $I := \{i : 0 \leq i < \alpha\}$ and G be the collection of all $I \setminus M$ with $M \subseteq I$ and $|M| < \lambda$. Because of $\lambda \geq \aleph_0$ it is G a filter on I which is contained in some ultrafilter J on I such that $M \notin J$ for each $M \subseteq I$ with $|M| < \lambda$.

Because K is closed under the formation of reducts it follows $\{\mathcal{A}_i : i \in I\} \subseteq K$. By assumption K is ultraclosed and therefore $\mathcal{C} := \prod_{i \in I} \mathcal{A}_i / J \in K$ and it is $\mathcal{C} \models D'$. Let φ be that homomorphism from $\mathcal{T}(X, F)$ into the τ_F-algebra $\mathcal{C}$ such that

$$\varphi(w) = [(\varphi_i(w) : i \in I)]_J \in \mathcal{C}$$

for each $w \in \mathrm{T}(X, F)$. Consequently,

$$\{(P, Q) : (P, Q) \in D' \text{ and } \varphi(P) = \varphi(Q)\} \neq \emptyset$$

and

$$[(\varphi_i(P_j) : i \in I)]_J = [(\varphi_i(Q_j) : i \in I)]_J$$

for some $j < \alpha$. Let M be the set of all $i \in I$ such that

$$\varphi_i(P_j) = \varphi_i(Q_j).$$

Because of

$$\varphi_i(P_j) \neq \varphi_i(Q_j)$$

for each $j \leq i$ it follows that

$$M \subseteq \{i : 0 \leq i < j\}$$

and $|M| < \lambda$, contradicting $M \in J$, i.e., $\lambda < \aleph_0$. Consequently, for each D there is some $D' \subseteq D$ with $|D'| < \aleph_0$ such that for each algebra $\mathcal{B} \in K$ from $\mathcal{B} \models D$ it follows that $\mathcal{B} \models D'$. $\square$

Now, let $\mathbb{X}$ be a set of individual variables such that $|\mathbb{X}| = \aleph_0$ and $\mathbb{F}$ be a set of operational symbols such that $|\mathbb{F}_n| = \aleph_0$ for each $n \in \mathcal{N}$.

Proposition 4.2. *Let K be a class of algebras which is closed under the formation of reducts and ultraclosed. Then the following implication holds provided that $|X| \geq \aleph_0$ and $|F_n| \geq \aleph_0$ for each $n \in \mathcal{N}$: If U is a free system for K (over X and F), then U is also a free system for* $\mathrm{MOD.DIS}_{\mathbb{XF}}(K)$ *(over X and F).*

Proof. Let U be a free system for K (over X and F). Then by Proposition 3.3 it is $U \prec \{\mathcal{T}(X, F)/\sigma(E) : E \in \mathrm{I}(\mathrm{DIS}_{XF}(K))\}$ and $U \subseteq K$.

Now, it holds $\mathrm{DIS}_{XF}(K) = \mathrm{DIS}_{XF}(\mathrm{MOD.DIS}_{\mathbb{XF}}(K))$. First of all we prove $\mathrm{MOD.DIS}_{XF}(K) = \mathrm{MOD.DIS}_{\mathbb{XF}}(K)$. For this let $\mathcal{A} \in \mathrm{MOD.DIS}_{XF}(K)$ and this is if and only if $\mathcal{A} \models D$ for each $D \in \mathrm{DIS}_{XF}(K)$. $D \in \mathrm{DIS}_{XF}(K)$ means $\mathcal{B} \models D$ for each $\mathcal{B} \in K$. By assumption it is K a class of algebras which is closed under the formation of reducts and ultraclosed. Therefore by Proposition 4.1 it exists a subset $D' \subseteq D$ such that $|D'| < \aleph_0$ and from $\mathcal{B} \models D$ it follows $\mathcal{B} \models D'$ for each $\mathcal{B} \in K$. Consequently, $\mathrm{MOD.DIS}_{XF}(K)$ is the set of all algebras $\mathcal{A}$ such that $\mathcal{A} \models D$ for each $D \in \mathrm{DIS}_{XF}(K)$ with $|D| < \aleph_0$. Especially, $\mathrm{MOD.DIS}_{\mathbb{X}\mathbb{F}}(K)$ is the set of all algebras $\mathcal{A}$ such that $\mathcal{A} \models D$ for each $D \in \mathrm{DIS}_{\mathbb{X}\mathbb{F}}(K)$ with $|D| < \aleph_0$. Since $|X| \geq |\mathbb{X}| = \aleph_0$ and $|F_n| \geq |\mathbb{F}_n| = \aleph_0$ for each $n \in \mathcal{N}$ it follows

$$\mathrm{MOD.DIS}_{XF}(K) = \mathrm{MOD.DIS}_{\mathbb{X}\mathbb{F}}(K)$$

and

$$\mathrm{DIS}_{XF}(\mathrm{MOD.DIS}_{XF}(K)) = \mathrm{DIS}_{XF}(\mathrm{MOD.DIS}_{\mathbb{X}\mathbb{F}}(K)).$$

With respect to the Galois connection of the operators MOD and DIS_{XF} it is $\mathrm{DIS}_{XF}(\mathrm{MOD.DIS}_{XF}(K)) = \mathrm{DIS}_{XF}(K)$ and

$$\mathrm{DIS}_{XF}(K) = \mathrm{DIS}_{XF}(\mathrm{MOD.DIS}_{\mathbb{X}\mathbb{F}}(K)),$$

consequently. Therefore

$$U \prec \{\mathcal{T}(X,F)/\sigma(E) \colon E \in \mathrm{I}(\mathrm{DIS}_{XF}(\mathrm{MOD.DIS}_{\mathbb{X}\mathbb{F}}(K)))\}.$$

With respect to the Galois connection of the operators MOD and $\mathrm{DIS}_{\mathbb{X}\mathbb{F}}$ it is $U \subseteq \mathrm{MOD.DIS}_{\mathbb{X}\mathbb{F}}(K)$ and U is a free system for $\mathrm{MOD.DIS}_{\mathbb{X}\mathbb{F}}(K)$ by Proposition 3.3. □

Proposition 4.3. *A class K of algebras is defined by finite disjunctions if and only if the following statements hold:*

(i) *K is closed under the formation of reducts;*
(ii) *K is closed under the formation of homomorphic images;*
(iii) *for each set X of individual variables and each set F of operational symbols there exists a free system for K (over X and F);*
(iv) *K is ultraclosed.*

Proof. Necessity. Let $K = \mathrm{MOD}(\Delta)$ with a family Δ of finite sets $D \subseteq T(\mathbb{X},\mathbb{F}) \times T(\mathbb{X},\mathbb{F})$.

(i) Let $\mathcal{A} \in K$ be a τ_H-algebra and $\mathcal{A}'$ be a τ_G-reduct of $\mathcal{A}$. Now, let $\mathcal{A}''$ be a $\tau_{\mathbb{F}}$-reduct of $\mathcal{A}'$. Because of $A'' = A' = A$ and

$$\{f^{\mathcal{A}''} \colon f \in \mathbb{F}\} \subseteq \{g^{\mathcal{A}'} \colon g \in G\} \subseteq \{h^{\mathcal{A}} \colon h \in H\}$$

it is $\mathcal{A}''$ a $\tau_{\mathbb{F}}$-reduct of $\mathcal{A}$, too. Therefore, from $\mathcal{A} \in K$, i.e., $\mathcal{A} \models D$ for each $D \in \Delta$ it follows that $\mathcal{A}' \models D$ for each $D \in \Delta$. Consequently, $\mathcal{A}' \in K$.

(ii) Let $\mathcal{A} \in K$ be a τ_G-algebra and $\mathcal{B}$ be a homomorphic image of $\mathcal{A}$ with respect to a homomorphism ψ. Now, let $\mathcal{B}'$ be a $\tau_{\mathbb{F}}$-reduct of $\mathcal{B}$, φ be a homomorphism from $\mathcal{T}(\mathbb{X},\mathbb{F})$ into $\mathcal{B}'$ and $D \in \Delta$.

We construct a $\tau_{\mathbb{F}}$-reduct $\mathcal{A}'$ of $\mathcal{A}$ as follows. For this let us assume that G is well-ordered. If $f \in \mathbb{F}$, then let g_f be the least element of $\{g \colon f^{\mathcal{B}'} = g^{\mathcal{B}}\}$ and $f^{\mathcal{A}'} := g_f^{\mathcal{A}}$. Then $\mathcal{B}'$ is a homomorphic image of $\mathcal{A}'$ with respect to ψ. Since $\mathcal{B}$ is a homomorphic image of $\mathcal{A}$ with respect to ψ it is

$$\psi(g^{\mathcal{A}}(a_1,\ldots,a_n)) = g^{\mathcal{B}}(\psi(a_1),\ldots,\psi(a_n))$$

for $g \in G$, $\tau_G(g) = n$ and $a_1, \ldots, a_n \in A$. For $f \in \mathbb{F}$, $\tau_{\mathbb{F}}(f) = n$ and $a_1, \ldots, a_n \in A$ it holds

$$\psi(f^{\mathcal{A}'}(a_1, \ldots, a_n)) = \psi(g_f^{\mathcal{A}}(a_1, \ldots, a_n))$$

and

$$g_f^{\mathcal{B}}(\psi(a_1), \ldots, \psi(a_n)) = f^{\mathcal{B}'}(\psi(a_1), \ldots, \psi(a_n))$$

for the least element g_f of $\{g\colon f^{\mathcal{B}'} = g^{\mathcal{B}}\}$. Therefore,

$$\psi(f^{\mathcal{A}'}(a_1, \ldots, a_n)) = f^{\mathcal{B}'}(\psi(a_1), \ldots, \psi(a_n))$$

and $\mathcal{B}'$ is a homomorphic image of $\mathcal{A}'$ with respect to ψ.

There exists a homomorphism γ from $\mathcal{T}(\mathbb{X}, \mathbb{F})$ into $\mathcal{A}'$ such that $\varphi(t) = \psi(\gamma(t))$ for each $t \in T(\mathbb{X}, \mathbb{F})$ and $\ker(\gamma) \subseteq \ker(\varphi)$. By (i) it is $\mathcal{A}' \in K$. Therefore, $\mathcal{A}' \models D$ and $D \cap \ker(\gamma) \neq \emptyset$. Consequently, $D \cap \ker(\varphi) \neq \emptyset$ and therefore, $\mathcal{B} \models D$ and $\mathcal{B} \in K$, too.

(iii) By Proposition 3.5 for each set X of individual variables and each set F of operational symbols there exists a free system for K (over X and F).

(iv) Let J be an ultrafilter on a set I and $\{\mathcal{A}_i\colon i \in I\} \subseteq K$ a (not empty) set of τ_F-algebras. Then it follows $\mathcal{C} := \prod_{i\in I} \mathcal{A}_i/J \in K$. For this let φ be a homomorphism from $\mathcal{T}(\mathbb{X}, \mathbb{Y})$ into any $\tau_{\mathbb{F}}$-reduct $\mathcal{C}'$ of $\mathcal{C}$. Similar to the proof of (ii) it follows that there exists a set $\{\mathcal{A}'_i\colon i \in I\} \subseteq K$ of $\tau_{\mathbb{F}}$-reducts $\mathcal{A}'_i$ of $\mathcal{A}_i$ such that $\mathcal{C}' = \prod_{i\in I} \mathcal{A}'_i/J \in K$. Then it exists a system $\{\varphi_i\colon i \in I\}$ of homomorphisms φ_i from $\mathcal{T}(\mathbb{X}, \mathbb{Y})$ into $\mathcal{A}'_i$ such that

$$\varphi(t) = [(\varphi_i(t)\colon i \in I)]_J \in C'$$

for each $t \in \mathrm{T}(\mathbb{X}, \mathbb{Y})$. Let $D \in \Delta$. It follows that

$$\{(P, Q)\colon (P, Q) \in D \ \text{ and } \ \varphi(P) = \varphi(Q)\} \neq \emptyset.$$

Otherwise,

$$[(\varphi_i(P)\colon i \in I)]_J \neq [(\varphi_i(Q)\colon i \in I)]_J$$

for each $(P, Q) \in D$, i.e., $I_{(P,Q)} := \{i\colon \varphi_i(P) = \varphi_i(Q)\} \notin J$ for each $(P, Q) \in D$. Because of J is an ultrafilter on I it follows that $\{I \setminus I_{(P,Q)}\colon (P, Q) \in D\} \subseteq J$. Since $|D| < \aleph_0$ it is $|\{I \setminus I_{(P,Q)}\colon (P, Q) \in D\}| < \aleph_0$, too. J is assumed to be a filter and therefore $\bigcap\{I \setminus I_{(P,Q)}\colon (P, Q) \in D\} \in J$. By assumption it is $\mathcal{A}'_i \models D$ for each $i \in I$ and

$$\{(P, Q)\colon (P, Q) \in D \ \text{ and } \ \varphi_i(P) = \varphi_i(Q)\} \neq \emptyset$$

for $i \in I$. Consequently, for each $i \in I$ there is a $(P, Q) \in D$ such that $\varphi_i(P) = \varphi_i(Q)$ and $i \in I_{(P,Q)}$. Hence, $\bigcap\{I \setminus I_{(P,Q)}\colon (P, Q) \in D\} = \emptyset \in J$. This contradicts the fact that J is a proper filter, i.e., $\emptyset \notin J$, especially. Therefore, K is ultraclosed.

Sufficiency. Let K be a class of algebras such that the statements (i)–(iv) are fulfilled. We will show $K = \mathrm{MOD.DIS}_{\mathbb{XF}}(K)$. Clearly, $K \subseteq \mathrm{MOD.DIS}_{\mathbb{XF}}(K)$. Now, it holds $\mathrm{MOD.DIS}_{\mathbb{XF}}(K) \subseteq K$. For this let

$$\mathcal{A} = (A, (g^{\mathcal{A}})_{g\in G}) \in \mathrm{MOD.DIS}_{\mathbb{XF}}(K),$$

X be a set of individual variables such that

$$|X| = |A| + \aleph_0$$

and F be a set of operational symbols such that

$$|F_n| = |\{g^{\mathcal{A}} : g \in G_n\}| + \aleph_0$$

for each $n \in \mathcal{N}$. Then it is a τ_F-reduct $\mathcal{A}'$ of $\mathcal{A}$ such that $\mathcal{A}$ is a τ_G-reduct of $\mathcal{A}'$.

By (iii) there exists a free system U for K (over X and F), i.e., $U \subseteq K$, especially. Because of $|X| \geq \aleph_0$ and $|F_n| \geq \aleph_0$ for each $n \in \mathcal{N}$ it is U also a free system for $\mathrm{MOD.DIS}_{\mathbb{XF}}(K)$ (over X and F) by Proposition 4.2. Since $|X| \geq |A|$ it exists a homomorphism from $\mathcal{T}(X, F)$ onto $\mathcal{A}'$ and therefore $\mathcal{A}'$ is a homomorphic image of an algebra $\mathcal{B} \in U \subseteq K$ and $\mathcal{A}' \in K$ by (ii). Consequently, $\mathcal{A} \in K$ by (i), i.e., $\mathrm{MOD.DIS}_{\mathbb{XF}}(K) \subseteq K$ and $K = \mathrm{MOD.DIS}_{\mathbb{XF}}(K)$, finally.

Now, let $\Delta := \{D : D \in \mathrm{DIS}_{\mathbb{XF}}(K) \text{ and } |D| < \aleph_0\}$. By (i), (iv) and Proposition 4.1 for each $D \subseteq T(X, F) \times T(X, F)$ it exists a finite subset $D' \subseteq D$ such that for each algebra $\mathcal{B} \in K$ from $\mathcal{B} \models D$ it follows that $\mathcal{B} \models D'$. Therefore, $K = \mathrm{MOD}(\Delta)$ and K is defined by finite disjunctions. $\square$

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