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SOME RESULTS FOR ONE CLASS OF DISCONTINUOUS OPERATORS WITH FIXED POINTS

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ABSTRACT. In 1986, L. Nova ([**11**]) defined a class of operators with fixed points called $D(a, b)$ which includes many classic operators with fixed points. In this paper we give a compilation about existing results in this class. In addition we will prove some results for sequences of operators of class $D(a, b)$, and we will give conditions for this operator class to be closed under sum and composition (or product).

INTRODUCTION

Let A be an arbitrary set and $T : A \longrightarrow A$ a map. The **fixed point theory** consists of finding conditions for A and/or T such that there is at least one point $a \in A$ such that $Ta = a$. If this point exists it is called **fixed point** of T. We consider convenient to indicate some results that have made history in the **fixed point theory**. The topological version of this theory was given in 1912 by L. Brouwer (see, [**8**]) who proved the following result:

Let $f : B[a, r] \subset \mathbb{R}^n \longrightarrow B[a, r]$ *be a continuous function, then there exists* $z \in B[a, r]$ *such that* $f(z) = z$ where $B[a, r]$ is the closed ball with center in a and radius $r > 0$.

The Brouwer's Theorem in the one dimensional case is the Cauchy-Bolzano's Theorem, that states the following:

Let $f : [a, b] \longrightarrow [a, b]$ *be a continuous function, then there exists* $x_0 \in [a, b]$ *such that:*

$$f(x_0) = x_0.$$

The Brouwer's Theorem was generalized for Banach spaces of infinite dimensional by S. Schauder (see, [**8**]) in the following way:

Let $(E, \|.\|)$ *be a Banach space,* $K \subset E$ *a compact and convex subset of* E *and* $T : K \longrightarrow K$ *a continuous map. Then there exists* $z \in K$ *such that* $Tz = z$.

The following result corresponds to the metric version of the Schauder's Theorem.

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Let (M, d) be a complete metric space and $T : M \longrightarrow M$ a map. Then T has a fixed point in M if it satisfies any of the following conditions:

C1. *(Banach, 1922, see* [**8**]*)* T is an α-contraction or Banach contraction, this is:

$$d(Tx, Ty) \leq \alpha\, d(x, y) \quad \forall x, y \in M, \quad 0 \leq \alpha < 1.$$

C2. *(Kannan, 1969, 1971,* [**9, 10**]*)* T satisfies: there is $\alpha \in [0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq \alpha(d(x, Tx) + d(y, Ty)) \;\; \forall x, y \in M.$$

C3. *(Chatterge, 1972,* [**2**]*)* T satisfies the following condition: there is $\alpha \in [0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq \alpha(d(x, Ty) + d(y, Tx)) \;\; \forall x, y \in M.$$

C4. *(Reich, 1971,* [**14, 15**]*)* T satisfies:

$$d(Tx, Ty) \leq a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty)$$
$$\forall x, y \in M, \;\; 0 \leq a_1 + a_2 + a_3 < 1.$$

C5. T satisfies:

$$d(Tx, Ty) \leq a_1 d(x, y) + a_2 d(x, Ty) + a_3 d(y, Tx)$$
$$\forall x, y \in M, \;\; 0 \leq a_1 + a_2 + a_3 < 1.$$

C6. *(Hardy-Rogers, 1973,* [**6**]*)* $\forall x, y \in M$, T satisfies: there are $a_i \geq 0$ such that $A = \sum_{i=1}^{5} a_i < 1$ and

$$d(Tx, Ty) \leq a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty) + a_4 d(x, Ty) + a_5 d(y, Tx).$$

In addition, K. Goebel *et al* [**5**] extended this result to the case $A \leq 1$ for continuous mapping of a nonempty bounded and convex subset K of a uniformly convex Banach space into itself. And J. Lopez-Gomez [**12**] proved that T has a unique fixed point excluding the hypothesis T continuous.

The above conditions are independent among each other in the following sense:

1. All map **C1.** is a continuous map.
2. There is a function that satisfies the condition **C2.** but not the condition **C1.**

$$T : [0, 1] \longrightarrow \mathbb{R} \qquad Tx = \begin{cases} \dfrac{x}{4}, & x \in \left[0, \dfrac{1}{2}\right) \\ \dfrac{x}{5}, & x \in \left[\dfrac{1}{2}, 1\right] \end{cases}.$$

Then T is discontinuous, and therefore, T is not **C1.** and it is easy to see that it satisfies **C2.**

3. There is a function that satisfies **C1.** but not **C2.**

$$T : [0, 1] \longrightarrow [0, 1] \qquad Tx = \frac{x}{3}.$$

It is clear that T is continuous. To see that it is not **C2.** take $y = 0$, $x = 1/3$.

4. There is a function that is neither **C1.** nor **C2.** but that is **C4.**

$$T : [0,1] \longrightarrow [0,1] \qquad Tx = \begin{cases} \dfrac{7}{20}x, & x \in \left[0, \dfrac{1}{2}\right) \\ \dfrac{3}{10}x, & x \in \left[\dfrac{1}{2}, 1\right]. \end{cases}$$

It is clear that T is discontinuous, and therefore is not **C1.** to see that it is not **C2.** take $x = 0$, $y = 1/4$.

5. There is a function that satisfies **C2.** but that does not satisfy **C3.**

$$T : \mathbb{R} \longrightarrow \mathbb{R} \qquad Tx = -\frac{x}{2}$$

To see that T is not **C3.** take $x = 2$ and $y = -2$.

6. There is a function that satisfies **C3.** but not **C2.**

$$T : [0,1] \longrightarrow [0,1] \qquad Tx = \begin{cases} \dfrac{x}{2}, & x \in [0,1) \\ 0, & x = 1. \end{cases}$$

Take $x = 1/2$ and $y = 0$ to see that T is not **C2.**

In [**3**], W. R. Derrick and L. Nova defined the following operator classes:

Let $(E, \|.\|)$ be a Banach space, $K \subset E$ closed and $T : K \longrightarrow K$ an arbitrary operator that satisfies one of the following conditions, for $a, b \geq 0$ and any $x, y \in K$.

(A) $\qquad \|(Tx - Ty) - b\left[(x - Tx) + (y - Ty)\right]\| \leq a\|x - y\|,$

(B) $\qquad \|(Tx - Ty) - b(x - Tx)\| \leq a\|x - y\| + b\|y - Ty\|,$

(C) $\qquad \|(Tx - Ty) - a(x - y)\| \leq \left[\|x - Tx\| + \|y - Ty\|\right],$

(D) $\qquad \|Tx - Ty\| \leq a\|x - y\| + b\left[\|x - Tx\| + \|y - Ty\|\right].$

We shall say T belongs or is of class $A(a,b)$ (respectively $B(a,b)$, $C(a,b)$, $D(a,b)$), when T satisfies the condition (A) (respectively (B), (C), (D)).

Let's note that a mapping satisfying any of the above conditions is a contraction map (**C1.**) when $b = 0$ and $0 < a < 1$. In addition a map $C(0,b)$ is a Kannan's map; this is, (**C2.**).

Kannan proved that T has a unique fixed point if $0 < b < \frac{1}{2}$, he proved the uniqueness of fixed points with $b = \frac{1}{2}$ in uniformly convex spaces under certain restrictions.

Let's see the similarities and contrast these four classes. One similarity is that if T has a fixed point, it is unique whenever $0 \leq a < 1$. Observe, using the triangle inequality, that any map of class $A(a,b)$, $B(a,b)$ or $C(a,b)$ is of class $D(a,b)$.

Moreover, we can notice that no continuity conditions have ever been put on T, therefore, these classes do not exclude discontinuous operators.

In particular, class $D(1,1)$ contains all operators from E onto itself, since

$$\|Tx - Ty\| \leq \|Tx - x\| + \|x - y\| + \|y - Ty\|.$$

Which is a trivial application of the triangle inequality. Since the three first classes are included in the fourth class, of the above, we will restrict our attention on class $D(a,b)$.

The following example due to L. Nova [**11**] show that this class in not empty.

Example 1. Let's consider the following discontinuous operator.

$$Tx = \begin{cases} \gamma x, & 0 \le x < \dfrac{1}{2}, \\ \rho x, & \dfrac{1}{2} \le x \le 1, \end{cases}$$

with $0 < \gamma, \rho < 1,\ \ \gamma \ne \rho$.

Let's remember that class $D(a,b)$ satisfies:

$$\|Tx - Ty\| \le a\|x-y\| + b\left[\|x - Tx\| + \|y - Ty\|\right].$$

Let's see that $T \in D(0, \mu/(1-\mu))$ where $\mu = \max\{\gamma, \rho\}$.
$\forall x \in [0, 1/2)$ we have that

$$\begin{aligned} Tx_i = \gamma x_i \quad &\Longrightarrow \quad x_i - Tx_i = -x_i\gamma + x_i \\ &\Longrightarrow \quad x_i - Tx_i = x_i(1-\gamma) \\ &\Longrightarrow \quad \frac{\gamma}{1-\gamma}(x_i - Tx_i) = \gamma x_i. \end{aligned}$$

From which follows

$$\begin{aligned} |Tx_1 - Tx_2| \le \gamma(x_1 + x_2) \quad &= \frac{\gamma}{1-\gamma}\left\{|x_1 - Tx_1| + |x_2 - Tx_2|\right\} \\ &\le \frac{\mu}{1-\mu}\left\{|x_1 - Tx_1| + |x_2 - Tx_2|\right\}. \end{aligned}$$

In the same way, the inequality is true if $x_i \in [1/2, 1]$. Now, if $x_1 < \frac{1}{2} \le x_2$ we have that:

$$\frac{\gamma}{1-\gamma}(x_1 - Tx_1) = \gamma x_1, \qquad \frac{\rho}{1-\rho}(x_2 - Tx_2) = \rho x_2$$

and

$$|Tx_1 - Tx_2| \le \gamma x_1 + \rho x_2 \le \frac{\mu}{1-\mu}\left\{|x_1 - Tx_1| + |x_2 - Tx_2|\right\}.$$

1. It is clear that this map T has a fixed point.
2. The contraction map is an asymptotically regular operator for any point, this is, $\|T^{n-1}x - T^n x\| \longrightarrow 0$ as $n \longrightarrow \infty$.

   In fact:

$$\frac{\gamma}{1-\gamma}\left(T^n x - T^{n+1}x\right) = \gamma T^n x \le \mu^{n+1}x.$$

Since

$$\begin{array}{lll} Tx = \gamma x & \text{or} & Tx = \rho x \\ T^2x = T(T\gamma x) = \gamma^2 x & \text{or} & T^2 x = \rho^2 x \\ T^3 x = T^2(Tx) = \gamma^3 x & \text{or} & T^3 x = \rho^3 x \\ \vdots & & \vdots \\ T^n x = T(T^{n-1}x) = \gamma^n x & \text{or} & T^n x = \rho^n x. \end{array}$$

Taking $\mu = \max\{\gamma, \rho\}$, in general we have that

$$\gamma T^n x \leq \mu^{n+1} x.$$

So, for n sufficiently large, $x \in [0,1]$ and $0 < \mu < 1$, we have that T is asymptotically regular.

3. Finally we must see that the sequence $x_n = T^n x$ converges to a unique fixed point; in fact, it is clear that $\{x_n - T^n x\}_n \longrightarrow 0$.

1. Some Known Results for $D(a,b)$

In this section we will show some results for class $D(a,b)$. Let's observe that some results are consequences of the result of the value of a, while others depend only on b. First we analyze the properties of the values of a.

Lemma 1 (1989, [**4**]). *Let $T : X \longrightarrow X$ be of class $D(a,b)$ with $0 \leq a < 1$. Then T has at the most one fixed point.*

Lemma 2 (1989, [**4**]). *Let $T : K \longrightarrow K$ be of class $D(a,b)$, $0 \leq a < 1$, and suppose $\inf_k \|x - Tx\| = 0$. Then there exists a convergent sequence $\{x_n\}$ of points in K such that*

$$\|x_n - Tx_n\| \longrightarrow 0 \qquad \text{as} \quad n \longrightarrow \infty.$$

Now we will show three consequences of the condition $0 \leq b < 1$.

Lemma 3 (1989, [**4**]). *Let $T : K \longrightarrow K$ be of class $D(a,b)$, $0 \leq b < 1$.*

(i) *If $\{x_n\}$ converges to a fixed point of T, then $\|x_n - Tx_n\| \longrightarrow 0$ as $n \longrightarrow \infty$.*
(ii) *If $\{x_n\}$ converges and $\|x_n - Tx_n\| \longrightarrow 0$ as $n \longrightarrow \infty$, then T has a fixed point.*
(iii) *If T has a fixed point at p, then T is continuous at p.*

Lemma 4 (1986, [**11**]). *If $T \in D(a,b)$, and $a + 2b < 1$, then*

$$\inf_{x \in K} \|x - Tx\| = 0.$$

Theorem 5 (1989, [**4**]). *Let $T : K \longrightarrow K$ be of class $D(a,b)$ with $0 \leq a, b < 1$. If $\inf_{x \in K} \|x - Tx\| = 0$, then T has a unique fixed point in K.*

Theorem 6 (1989, [**4**]). *Let $T : X \longrightarrow X$, $T \in D(a,b)$, with $a, b \geq 0$, where $a + 2b < 1$. Then*

(i) *T has a unique fixed point $p \in X$.*
(ii) *$\|Tx - p\| < \|x - p\|, \ \forall x \in X, \ x \neq p$.*

L. Nova and W. Derryck give examples where they show that every one of the conditions in the results above are necessary (see [**3, 4, 11**]).

Remark 1. From Lemmas 1, 2, 3 (ii) and 4 we can obtain the following adaptation for $T \in D(a,b)$ of Theorem 2.1 given in [**13**] with $c = 0$ where $0 \leq c < 1$.

Let K be a closed subset of a Banach space X, and let $T \in D(a,b)$ with $a, b \geq 0$ where $a + 2b < 1$. Then for any $x \in K$, $\lim\limits_{n\to\infty} T^n x$ exists and this limit is the unique fixed point of T.

2. Main Results

In this section we will give some results for operators of class $D(a,b)$.

Theorem 7. *Let $\{T_n\}_n$ be a sequence of maps of class $D(a,b)$ defined in a Banach space X or some closed subset $K \subset X$ into itself, such that $\{T_n\}_n$ converges uniformly to T. Then $T \in D(a,b)$, $0 \leq a, b < 1$, moreover the fixed point of T is the limit of the fixed point of T_n.*

Proof. Let $T = \lim\limits_{n\to\infty} T_n$ uniformly,

$$\begin{aligned}
\|Tx - Ty\| &= \|Tx - T_n x + T_n x + T_n y - T_n y + T_y\| \\
&\leq \|T_n x - T_n y\| + \|T_n x - Tx\| + \|Ty - T_n y\| \\
&\leq a\|x - y\| + b\,[\|x - T_n x\| + \|y - T_n y\|] + \|T_n x - Tx\| \\
&\quad + \|Ty - T_n y\|.
\end{aligned}$$

For each n.

For $n \to \infty$

$$\|Tx - Ty\| \leq a\|x - y\| + b\,[\|x - Tx\| + \|y - Ty\|]$$

where we have that $T \in D(a,b)$.
Now, let's see that the fixed point of T is the limit of the fixed point of $\{T_n\}_n$.
Let $x_n = T_n x_n$ and $x_m = T_m x_m$, $m \neq n$. The fixed points are unique because $0 \leq a, b < 1$; thus

$\|x_n - x_m\| = \|T_n x_n - T_m x_m\| < \varepsilon$. Therefore $\{x_n\}_n$ is a Cauchy sequence.

From which exists $\widehat{x}$ such that $x_n \to \widehat{x}$; let's see that $T\widehat{x} = \widehat{x}$.
Since $\|x_n - \widehat{x}\| \to 0$ then $\|T_n x_n - \widehat{x}\| \to 0$. So, as a consequence of Lemma 3 we have that T_n is continuous at x_n, thus

$$\lim_{n\to\infty} \|T_n x_n - \widehat{x}\| \to 0 \;\Rightarrow\; \|\lim_{n\to\infty} T_n x_n - \widehat{x}\| \to 0.$$

Which implies

$$\|T_n(\lim_{n\to\infty} x_n) - \widehat{x}\| \to 0 \;\Rightarrow\; \lim_{n\to\infty} \|T_n \widehat{x} - \widehat{x}\| \to 0$$

and we conclude that $\|T\widehat{x} - \widehat{x}\| = 0$; therefore, $T\widehat{x} = \widehat{x}$. □

Remark 2. If in the previous theorem we change the hypothesis $0 \leq a, b < 1$ by $0 \leq a + 2b < 1$, then from Lemma 4 and Theorem 5 we can to assure that the fixed point to T is in K.

An interesting question is: If $T, S \in D(a,b)$. Is TS of class $D(a,b)$? Let's see the following example.

Example 2. Let's define $T : [0,1] \longrightarrow [0,1]$ as follows

$$Tx = \begin{cases} \dfrac{x}{4}, & 0 \le x < \dfrac{1}{2} \\ \dfrac{x}{8}, & \dfrac{1}{2} \le x \le 1. \end{cases}$$

From example (1) we have $T \in D(0, \frac{1}{3})$, however

$$T^2 x = T(Tx) = \begin{cases} \dfrac{x}{16}, & 0 \le x < \dfrac{1}{2} \\ \dfrac{x}{64}, & \dfrac{1}{2} \le x \le 1. \end{cases}$$

But, as a consequence of example (1) we have $T^2 \in D(0, \frac{1}{15})$.

The above example shows that $D(a,b)$ is not closed under composition, however we will show that under certain conditions we can give any positive answer to the previous question.

Definition 1 (see, [**7**])**.** A norm $\|\cdot\|$ on a Banach space is called strictly convex if whenever $\|x\| = \|y\| = 1$ and $\|x + y\| = 2$ then necessarily $x = y$.

A Banach space X is said to be strictly convex if its norm is strictly convex.

The importance of the previous definition in the next results is that we can assure $\|x + y\| = \|x\| + \|y\|$ if $x = \lambda y$, for any scalar λ.

Theorem 8. *Let X be a strictly convex Banach space, and let $S, T : X \longrightarrow X$. If the following conditions hold*

(i) $T \in D(a,b)$, $b \ge 1$

(ii) $x - Tx = r(Tx - STx)$, *for any scalar r and every $x \in X$*

then $ST \in D(a,b)$.

Proof. Let $x, y \in X$ and $S, T : X \longrightarrow X$

$$\begin{aligned} \|STx - STy\| &= \|STx - Tx - STy + Ty + Tx - Ty\| \\ &\le \|Tx - Ty\| + \|STx - Tx\| + \|STy - Ty\| \\ &\le a\|x - y\| + b\big[\|x - Tx\| + \|y - Ty\|\big] + \|STx - Tx\| \\ &\quad + \|STy - Ty\| \\ &\le a\|x - y\| + b\big[\|x - Tx\| + \|y - Ty\|\big] + b[\|STx - Tx\| \\ &\quad + \|STy - Ty\|]. \end{aligned}$$

From condition (ii) and the fact that X is a strictly convex Banach space, we have $\|x - Tx\| + \|STx - Tx\| = \|x - STx\|$ for all $x \in X$. So

$$\|STx - STy\| \le a\|x - y\| + b\big[\|x - STy\| + \|y - STy\|\big].$$

Therefore, $ST \in D(a,b)$. ☐

By Theorem 8 and mathematical induction for $n \geq 2$, $n \in \mathbb{N}$, we obtain the following theorem.

Theorem 9. *Let X be a strictly convex Banach space, and let $T_1, \ldots, T_n : X \longrightarrow X$ such that the following conditions hold*

(i) $T_n \in D(a,b)$, $b \geq 1$,
(ii) $x - T_n x = r(T_n x - T_1 \cdots T_n x)$ *for any scalar r and every $x \in X$.*

Then $T_1 \cdots T_n \in D(a,b)$.

Proposition 10. *Let X be a strictly convex Banach space, and let $S, T : X \longrightarrow X$ such that*

(i) $T \in D(a,b)$, $b \leq 1$.
(ii) $x - Tx = r(Tx - STx)$, *for any scalar r and every $x \in X$.*

Then, $ST \in D(a,1)$.

Proof. Let $x, y \in X$ and $S, T : X \longrightarrow X$

$$\begin{aligned}
\|STx - STy\| &= \|STx - Tx - STy + Ty + Tx - Ty\| \\
&\leq \|Tx - Ty\| + \|STx - Tx\| + \|STy - Ty\| \\
&\leq a\|x - y\| + b\big[\|x - Tx\| + \|y - Ty\|\big] + \|STx - Tx\| \\
&\quad + \|STy - Ty\| \\
&\leq a\|x - y\| + \|x - Tx\| + \|y - Ty\| + \|STx - Tx\| \\
&\quad + \|STy - Ty\|.
\end{aligned}$$

Condition (ii) and the fact that X is a strictly convex Banach space, allow $\|x - Tx\| + \|STx - Tx\| = \|x - STx\|$ for every $x \in X$. So

$$\|STx - STy\| \leq a\|x - y\| + \|x - STy\| + \|y - STy\|.$$

Hence, $ST \in D(a,1)$. □

By Proposition 10 and mathematical induction for $n \geq 2$, $n \in \mathbb{N}$, we obtain the following theorem.

Theorem 11. *Let X be a strictly convex Banach space, and let $T_1, \ldots, T_n : X \longrightarrow X$. If the following conditions hold*

(i) $T_n \in D(a,b)$, $b \leq 1$,
(ii) $x - T_n x = r(T_n x - T_1 \cdots T_n x)$ *for any scalar r and every $x \in X$.*

then, $T_1 \cdots T_n \in D(a,1)$.

Remark 3.
(i) From Theorem 6 (i) let's note that the uniqueness of the fixed point can't be sure in Theorems 8 and 9 because $b > 1$. And for Proposition 10 and Theorem 11, T (T_i) can has a unique fixed point, however ST ($T_1, \ldots, T_n$) no necessarily has a unique fixed point.
(ii) Let's note that the operation given in Theorems 8, 9, 11, and Proposition 10 does not indicate composition of operators. Thus in the case of product of operators defined in a strictly convex Banach algebra these results are valid.

Moreover, it is not necessary that the operators T, S (or $T_1, \ldots, T_n$) be $D(a,b)$, it's enough that one of these operators belongs to $D(a,b)$.

Another interesting question is: Let $S, T : X \longrightarrow X$ $S, T \in D(a,b)$, is $S+T$ of class $D(a,b)$? The following example shows that in general this is not true.

Example 3. Let $X = [-1,1]$ and let's define the next maps of X into X of class $D(a,b)$ with $0 < a, b < 1$ and $a + 2b < 1$.

$$S(x) = \frac{|x|}{2} \quad \text{and} \quad T(x) = -\frac{x}{2},$$

hence

$$(S+T)(x) = \begin{cases} -x, & \text{if } x \in [-1,0) \\ 0, & \text{if } x \in [0,1]. \end{cases}$$

Let's see that $(S+T) \notin D(a,b)$. This is, let's prove that $(S+T)$ does not satisfy

$$\|Tx - Ty\| \le a\|x-y\| + b\left[\|x - Tx\| + \|y - Ty\|\right]. \tag{1}$$

Let $x \in (0,1]$ and $y = 0$, and suppose that (1) is satisfied

$$\begin{aligned} |-x-0| &\le a|x-0| + b[|-x-x| + |0-0|] = a|x| + b|2x| \\ &= a|x| + 2b|x| = |x|(a+2b) < |x|. \end{aligned}$$

Which is false, thus $(S+T) \notin D(a,b)$.

Therefore $D(a,b)$ is not closed under the sum.

However we prove the following.

Theorem 12. *Let X be a strictly convex Banach space, and let $S, T : B_X \longrightarrow B_X$, where B_X is the open unit ball of X. If the following conditions hold*

(i) $S, T \in D(a,b)$

(ii) $x - Tx = r(x - Sx)$ *for any scalar r and every $x \in B_X$*

then $S + T \in D(a,b)$ for $a + b$ sufficiently small.

Proof. Let $x, y \in B_X$.

$$\begin{aligned} \|Tx - Ty\| &\le a\|x-y\| + b\big[\|x - Tx\| + \|y - Ty\|\big] \\ \|Sx - Sy\| &\le a\|x-y\| + b\big[\|x - Sx\| + \|y - Sy\|\big]. \end{aligned}$$

Then,

$$\begin{aligned} \|Tx - Ty\| + \|Sx - Sy\| &\le 2a\|x-y\| + b[\|x - Tx\| + \|y - Ty\| + \|x - Sx\| \\ &\quad + \|y - Sy\|] \\ \|Tx - Ty + Sx - Sy\| &\le 2a\|x-y\| + b[\|x - Tx\| + \|y - Ty\| + \|x - Sx\| \\ &\quad + \|y - Sy\|] \\ \|(S+T)x - (S+T)y\| &\le 2a\|x-y\| + b[\|x - Tx\| + \|y - Ty\| + \|x - Sx\| \\ &\quad + \|y - Sy\|]. \end{aligned}$$

Condition (ii) and the fact that X is a strictly convex Banach space, imply

$$\|x - Tx\| + \|x - Sx\| = \|2x - (T+S)x\|.$$

From which,

$$\begin{aligned}
&\|(S+T)x-(S+T)y\| \\
&\le 2a\|x-y\| + b\big[\|2x-Tx-Sx\| + \|2y-Ty-Sy\|\big] \\
&= 2a\|x-y\| + b\big[\|2x-(T+S)x\| + \|2y-(S+T)y\|\big] \\
&\le 2a\|x-y\| + b\big[\|x-(S+T)x\| + \|x\| + \|y-(S+T)y\| + \|y\|\big] \\
&= 2a\|x-y\| + b\big[\|x-(S+T)x\| + \|y-(S+T)y\|\big] + b(\|x\|+\|y\|) \\
&= a\|x-y\| + b\big[\|x-(S+T)x\| + \|y-(S+T)y\|\big] + a\|x-y\| + b(\|x\|+\|y\|) \\
&\le a\|x-y\| + b\big[\|x-(S+T)x\| + \|y-(S+T)y\|\big] + (a+b)\|x\| + (a+b)\|y\| \\
&< a\|x-y\| + b\big[\|x-(S+T)x\| + \|y-(S+T)y\|\big] + 2(a+b).
\end{aligned}$$

Since $a+b$ can be as small as we please, and using fact that for each $a, b \in \mathbb{R}$, $a < b+\varepsilon$ for all $\varepsilon > 0$, then $a \le b$. (See [**1**]). We have

$$\|(S+T)x-(S+T)y\| \le a\|x-y\| + b\big[\|x-(S+T)x\| + \|y-(S+T)y\|\big].$$

Hence $S+T \in D(a,b)$. □

Proposition 13. *Let X be a strictly convex Banach space, and suppose that the series $\sum_{i=1}^{\infty} T_i$, where $T_i : B_X \longrightarrow B_X$, for each $i \in \mathbb{N}$, converges. If the following conditions hold*

(i) *$T_i \in D(a,b)$ for each $i \in \mathbb{N}$*
(ii) *$x - T_i x = r(x - T_j x)$ for each $i \neq j$, and moreover $x - T_i x = r(x - \sum_{i=1}^{n} T_i x)$ for all $i = 1, \dots n$, and each value of $n > 1$, r scalar and every $x \in B_X$*

then, $\sum_{i=1}^{\infty} T_i \in D(a,b)$ for $a+b$ sufficiently small.

Proof. Let $x, y \in B_X$ and T_i as in the hypothesis. For $n > 1$ fixed we take $(a+b) = \frac{1}{(n-1)2^{n+1}}$; so

$$\|\sum_{i=1}^{n}(T_i x - T_i y)\| \le \sum_{i=1}^{n}\|T_i x - T_i y\| \le na\|x-y\| + b\Big[\sum_{i=1}^{n}\Big(\|x-T_i x\| + \|y-T_i y\|\Big)\Big].$$

The above is deduced from assuming that each $T_i \in D(a,b)$ and from the sum of these operators n times.

Again, from condition (ii), from the fact that X is a strictly convex Banach space, and applying the reasoning of the previous Theorem we obtain

$$\sum_{i=1}^{n}\|x - T_i x\| = \|nx - \sum_{i=1}^{n} T_i x\|.$$

Hence,

$$\begin{aligned}
\|\sum_{i=1}^{n}(T_ix-T_iy)\| &\leq na\|x-y\|+b\Big[\|nx-\sum_{i=1}^{n}T_ix\|+\|ny-\sum_{i=1}^{n}T_iy\|\Big]\\
&\leq na\|x-y\|+b[\|x-\sum_{i=1}^{n}T_ix\|+(n-1)\|x\|+\|y-\sum_{i=1}^{n}T_iy\|\\
&\quad +(n-1)\|y\|]\\
&\leq na\|x-y\|+b\Big[\|x-\sum_{i=1}^{n}T_ix\|+\|y-\sum_{i=1}^{n}T_iy\|\Big]+2b(n-1)\\
&\leq a\|x-y\|+b\Big[\|x-\sum_{i=1}^{n}T_ix\|+\|y-\sum_{i=1}^{n}T_iy\|\Big]+2b(n-1)\\
&\quad +2a(n-1)\\
&= a\|x-y\|+b\Big[\|x-\sum_{i=1}^{n}T_ix\|+\|y-\sum_{i=1}^{n}T_iy\|\Big]+\frac{1}{2^n}.
\end{aligned}$$

Taking limit $n\to\infty$ we obtain the result. □

The conclusion of the above proposition can be obtained changing the property of the Banach space X and one condition.

Definition 2 (see, [**7**]). A Banach space X is called k-strictly convex iff for any $k+1$ elements $x_0,x_1,\dots,x_k$ of X, the relation

$$\|x_0+x_1+\cdots+x_k\|=\|x_0\|+\|x_1\|+\cdots+\|x_k\|$$

implies that $x_0,x_1,\dots,x_k$ are linearly dependent.
If $k=1$ this definition gives the class of strictly convex spaces.

Theorem 14. *Let X be a k-strictly convex Banach space and suppose that the series $\sum_{i=1}^{\infty}T_i$, where $T_i:B_X\longrightarrow B_X$, for each $i\in\mathbb{N}$, converges. If the following conditions hold*

(i) $T_i\in D(a,b)$
(ii) $x-T_ix\ :\ i=1,\dots k+1$ *are linearly dependent*

then, $\sum_{i=1}^{\infty}T_i\in D(a,b)$ *for $a+b$ sufficiently small.*

Proof. The proof follows as the previous proposition.
From condition (ii) and the fact that X is a k-strictly convex Banach space we have

$$\sum_{i=1}^{k}\|x-T_ix\|=\|kx-\sum_{i=1}^{k}T_ix\|.$$

The rest of the proof is analogue to the previous proposition. □

Remark 4. Let's note that the uniqueness of the fixed point is ensured from Lemma 1 and Theorem 6.

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References

1. Bartle R. and Sherbert D., *Introduction to Real Analysis*, John-Wiley & Sons, New York, 1982.
2. Chatterge S. K., *Fixed Point Theorem*, C. R. Acad Bulgar Sci. **25** (1972), 727–730.
3. Derryck W. R. and Nova L., *Interior Properties and Fixed Points of Certain Discontinuous Operators*, Elsevier Science (1992), 239–245.
4. ______, *Fixed Point Theorems for Discontinuous Operators*, Glasnik Matematički (1989), 339–347.
5. Goebel K., Kirk W. A. and Chimi T. N., *A Fixed Point Theorem in Uniformly Convex Spaces*, Boll. Union. Math. Ital. **7** (1973), 67–75.
6. Hardy G. E. and Rogers T. D., *A Generalization of a Fixed Point Thorem of Reich*, Canad. Math. Bull. **16** (1973), 201–206.
7. Istrăţescu V., *Strictly Convexity and Complex Strictly Convexity: Theory and Applications,* Lecture Notes in Pure and Applied Math **98**, Marcel-Deker (1984).
8. ______, *Fixed Point Theory, an Introduction*, Mathematics and Its Applications **7**, Boston U.S.A, D. Reidel Publishing Company.
9. Kannan R., *Some Results on Fixed Points II*, Amer. Math. Monthly. **76** (1969), 405–408.
10. ______, *Some Results on Fixed Points III*, Fund. Math. **70** (1971), 169–177.
11. Nova L., *Fixed Point Theorems for Some Discontinuous Operators*, Pacific Journal of Maths. (1986), 189–196.
12. Lopez Gomez J., *A Fixed Point Theorems for Discontinuous Operators*, Glasnik Matematički (1988), 115–118.
13. Rashwan R. A., On the Existence of Fixed Points for Some Discontiuos Operators, Math. Japonica. **35**(1) (1990), 97–104.
14. Reich S., *Kannan's Fixed Point Theorems*, Boll. Union. Math. Ital. **4** (1971), 121–128.
15. ______, *Remarks on Fixed Points*, Rend. Accad. Naz. di lincei **52** (1972), 689–697.

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