

RELATIONS BETWEEN THE RECIPROCAL SUM AND THE ALTERNATING SUM FOR GENERALIZED LUCAS NUMBERS

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ABSTRACT. We establish relations between reciprocal and alternating sums involving generalized Lucas numbers.

1. INTRODUCTION

Let p be a nonzero real number. The generalized Fibonacci and Lucas numbers are defined by

$$(1) \quad U_0 = 0, \quad U_1 = 1, \quad U_{n+2} = pU_{n+1} + U_n, \quad (n \geq 0)$$

$$(2) \quad V_0 = 2, \quad V_1 = p, \quad V_{n+2} = pV_{n+1} + V_n, \quad (n \geq 0)$$

respectively. Let α and β be the roots of $x^2 - px - 1 = 0$. Then we have the Binet's formulas:

$$(3) \quad U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta},$$

$$(4) \quad V_n = \alpha^n + \beta^n.$$

For $p = 1$, $\{U_n\}$ and $\{V_n\}$ are the well-known Fibonacci numbers F_n and Lucas numbers L_n , respectively.

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In [2], S. Rabinowitz discusses algorithmic aspects of certain finite reciprocal sums and propose several open problems involving Fibonacci and Lucas numbers. Melham [1] gave several relations between the reciprocal sum and the alternating sum for generalized Fibonacci numbers, which we subsequently extended in [4]. In this note we establish some relations between the reciprocal sum and the alternating sum for generalized Lucas numbers. For U_n and V_n , we have the following well known expansions:

$$(5) \quad U_{n+1} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} p^{n-2k}, \quad (n \geq 0),$$

and

$$(6) \quad V_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-k} \binom{n-k}{k} p^{n-2k}, \quad (n \geq 1).$$

See [3, (2.7),(2.8)]. From (5) and (6), since p is a nonzero real number, it is easy to obtain that $U_n > 0$ ($n \geq 1$) and $V_n > 0$ ($n \geq 0$) for $p > 0$. For $p < 0$, if n is an odd number, then we have $U_{n+1} < 0$ ($n \geq 0$) and $V_n < 0$ ($n \geq 1$). If n is an even number, then we have $U_{n+1} > 0$ ($n \geq 0$) and $V_n > 0$ ($n \geq 0$). Hence we have $U_n \neq 0$ ($n \geq 1$) and $V_n \neq 0$ ($n \geq 0$). Our main result is the following.

Theorem 1.1. *Let m and k be positive integers. Put*

$$(7) \quad \mathcal{S}_k(m) = \sum_{n=1}^{\infty} \frac{1}{V_n V_{n+k} V_{n+2k} \cdots V_{n+mk}}$$

and

$$(8) \quad \mathcal{T}_k(m) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{V_n V_{n+k} V_{n+2k} \cdots V_{n+mk}}.$$

Then

$$(9) \quad \mathcal{S}_k(m) = \frac{1}{1 + (-1)^{(m-1)k} - V_{(m+1)k}} \cdot \left[\sum_{i=1}^k \frac{(-1)^{(m-1)k} V_{i+(m+1)k} - V_{i+2(m+1)k}}{V_i V_{i+k} V_{i+2k} \cdots V_{i+(m+1)k}} + (p^2 + 4) U_{(m+1)k} U_{(m+2)k} \mathcal{T}_k(m+2) \right]$$

and

$$(10) \quad \mathcal{T}_k(m) = \frac{1}{1 + (-1)^{(m-1)k} - (-1)^k V_{(m+1)k}} \left\{ \sum_{i=1}^k \frac{(-1)^{i-1}}{V_i V_{i+k} V_{i+2k} \cdots V_{i+(m+1)k}} \cdot \left[(-1)^{(m-1)k} V_{i+(m+1)k} - (-1)^k V_{i+2(m+1)k} \right] + (p^2 + 4) U_{(m+1)k} U_{(m+2)k} \mathcal{S}_k(m+2) \right\}.$$

When $p = 1$, by special choices of k and m , we have the following interesting results:

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{L_n L_{n+1}} &= \frac{7}{12} - 10 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{L_n L_{n+1} L_{n+2} L_{n+3}}; \\ \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{L_n L_{n+1}} &= \frac{1}{4} + 2 \sum_{n=1}^{\infty} \frac{1}{L_n L_{n+1} L_{n+2} L_{n+3}}; \\ \sum_{n=1}^{\infty} \frac{1}{L_n L_{n+1} L_{n+2}} &= \frac{3}{28} - \frac{15}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{L_n L_{n+1} L_{n+2} L_{n+3} L_{n+4}}; \end{aligned}$$

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{L_n L_{n+1} L_{n+2}} &= \frac{11}{168} + \frac{15}{2} \sum_{n=1}^{\infty} \frac{1}{L_n L_{n+1} L_{n+2} L_{n+3} L_{n+4}}; \\
\sum_{n=1}^{\infty} \frac{1}{L_n L_{n+1} L_{n+2} L_{n+3}} &= \frac{13}{924} - 15 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{L_n L_{n+1} L_{n+2} L_{n+3} L_{n+4} L_{n+5}}; \\
\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{L_n L_{n+1} L_{n+2} L_{n+3}} &= \frac{29}{2772} + \frac{25}{3} \sum_{n=1}^{\infty} \frac{1}{L_n L_{n+1} L_{n+2} L_{n+3} L_{n+4} L_{n+5}}; \\
\sum_{n=1}^{\infty} \frac{1}{L_n L_{n+2}} &= \frac{139}{396} - 24 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{L_n L_{n+2} L_{n+4} L_{n+6}}; \\
\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{L_n L_{n+2}} &= \frac{95}{396} - 242 \sum_{n=1}^{\infty} \frac{1}{L_n L_{n+2} L_{n+4} L_{n+6}}; \\
\sum_{n=1}^{\infty} \frac{1}{L_n L_{n+2} L_{n+4}} &= \frac{123}{16 \cdot 11 \cdot 29} + \frac{199}{84 \cdot 18 \cdot 47} - \frac{105}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{L_n L_{n+2} L_{n+4} L_{n+6} L_{n+8}}; \\
\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{L_n L_{n+2} L_{n+4}} &= \frac{123}{16 \cdot 11 \cdot 29} - \frac{199}{84 \cdot 18 \cdot 47} - \frac{105}{2} \sum_{n=1}^{\infty} \frac{1}{L_n L_{n+2} L_{n+4} L_{n+6} L_{n+8}}.
\end{aligned}$$

2. THE PROOF OF THE RESULTS

Lemma 2.1. *We have*

$$(11) \quad V_{n+k} + (-1)^k V_{n-k} = V_n V_k;$$

$$(12) \quad V_{n+k} = V_n U_{k+1} + V_{n-1} U_k;$$

$$(13) \quad V_{n+k-1} = \frac{U_{k-1} V_{n+k} + (-1)^{k-1} V_n}{U_k};$$

$$(14) \quad U_{2k} = U_k V_k;$$

$$(15) \quad U_{n+k} = U_n U_{k+1} + U_{n-1} U_k;$$

$$(16) \quad U_{(m+1)k}^2 - U_{mk} U_{(m+2)k} = (-1)^{mk} U_k^2;$$

$$(17) \quad V_{n+(m+1)k}^2 - V_{n+mk} V_{n+(m+2)k} = -(-1)^{n+mk} (p^2 + 4) U_k^2;$$

$$(18) \quad U_k V_{n-mk} = (-1)^{mk} (V_n U_{mk+k} - V_{n+k} U_{mk});$$

$$(19) \quad U_{(m+2)k} V_{i+(m+1)k} - (-1)^k U_{(m+1)k} V_{i+mk} = U_k V_{i+2(m+1)k};$$

$$(20) \quad U_{n+k} - (-1)^k U_{n-k} = V_n U_k;$$

$$(21) \quad V_n V_{n+(m+1)k} = -\frac{(-1)^{(m+1)k}}{U_k} U_{(m+1)k} V_{n+(m+1)k} V_{n+(m+2)k} + \frac{(-1)^{(m+1)k}}{U_k} U_{(m+2)k} V_{n+mk} V_{n+(m+2)k} \\ - (-1)^{n+k} (p^2 + 4) U_k U_{(m+2)k};$$

$$\begin{aligned}
(22) \quad V_{n+mk}V_{n+(m+2)k} - V_nV_{n+(m+1)k} &= \frac{(-1)^{(m+1)k}}{U_k}U_{(m+1)k}V_{n+(m+1)k}V_{n+(m+2)k} \\
&+ \left[1 - \frac{(-1)^{(m+1)k}}{U_k}U_{(m+2)k}\right]V_{n+mk}V_{n+(m+2)k} \\
&+ (-1)^{n+k}(p^2 + 4)U_kU_{(m+2)k};
\end{aligned}$$

$$\begin{aligned}
(23) \quad V_{n+mk}V_{n+(m+2)k} + (-1)^{k-1}V_nV_{n+(m+1)k} &= \frac{(-1)^{mk}}{U_k}U_{(m+1)k}V_{n+(m+1)k}V_{n+(m+2)k} \\
&+ \left[1 - \frac{(-1)^{mk}}{U_k}U_{(m+2)k}\right]V_{n+mk}V_{n+(m+2)k} \\
&+ (-1)^n(p^2 + 4)U_kU_{(m+2)k}.
\end{aligned}$$

Proof. Use the Binet formulas (3) and (4) of U_n and V_n . □

Lemma 2.2. *We have*

$$(24) \quad \sum_{n=1}^{\infty} \frac{1}{V_nV_{n+k} \cdots V_{n+(m-1)k}V_{n+(m+1)k}} = \frac{U_{mk} + (-1)^{mk}U_k}{U_{(m+1)k}} \mathcal{S}_k(m) - \frac{(-1)^{mk}U_k}{U_{(m+1)k}} \sum_{i=1}^k \frac{1}{V_iV_{i+k} \cdots V_{i+mk}}$$

and

$$\begin{aligned}
(25) \quad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{V_nV_{n+k} \cdots V_{n+(m-1)k}V_{n+(m+1)k}} &= \frac{U_{mk} + (-1)^{(m+1)k}U_k}{U_{(m+1)k}} \mathcal{T}_k(m) \\
&- \frac{(-1)^{(m+1)k}U_k}{U_{(m+1)k}} \sum_{i=1}^k \frac{(-1)^{i-1}}{V_iV_{i+k} \cdots V_{i+mk}}.
\end{aligned}$$

Proof. We give only the proof of (24), the proof of (25) is similar. Using (11), we have

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{1}{V_n V_{n+k} \cdots V_{n+(m-1)k} V_{n+(m+1)k}} \\
&= \sum_{n=1}^{\infty} \left(\frac{V_{n+(m+1)k} + (-1)^k V_{n+(m-1)k}}{V_n V_{n+k} \cdots V_{n+(m-1)k} V_{n+(m+1)k}} \right) \frac{1}{V_k V_{n+mk}} \\
&= \sum_{n=1}^{\infty} \left(\frac{1}{V_n V_{n+k} \cdots V_{n+(m-1)k}} + \frac{(-1)^k}{V_n V_{n+k} \cdots V_{n+(m-2)k} V_{n+(m+1)k}} \right) \frac{1}{V_k V_{n+mk}} \\
&= \frac{1}{V_k} \mathcal{S}_k(m) + \frac{(-1)^k}{V_k} \sum_{n=1}^{\infty} \frac{1}{V_n V_{n+k} \cdots V_{n+(m-2)k} V_{n+mk} V_{n+(m+1)k}}.
\end{aligned}$$

Using (12), (13) and (15), we have

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{1}{V_n V_{n+k} \cdots V_{n+(m-2)k} V_{n+mk} V_{n+(m+1)k}} \\
&= \sum_{n=1}^{\infty} \left(\frac{V_{n+(m+1)k} + (-1)^k \frac{U_{2k}}{U_k} V_{n+(m-2)k}}{V_n V_{n+k} \cdots V_{n+(m-2)k} V_{n+mk} V_{n+(m+1)k}} \right) \frac{U_k}{U_{3k} V_{n+(m-1)k}} \\
&= \sum_{n=1}^{\infty} \left(\frac{1}{V_n V_{n+k} \cdots V_{n+(m-2)k} V_{n+mk}} \right. \\
&\quad \left. + \frac{(-1)^k \frac{U_{2k}}{U_k}}{V_n V_{n+k} \cdots V_{n+(m-3)k} V_{n+mk} V_{n+(m+1)k}} \right) \cdot \frac{U_k}{U_{3k} V_{n+(m-1)k}}
\end{aligned}$$

$$= \frac{U_k}{U_{3k}} \mathcal{S}_k(m) + \frac{(-1)^k U_{2k}}{U_{3k}} \sum_{n=1}^{\infty} \frac{1}{V_n V_{n+k} \cdots V_{n+(m-3)k} V_{n+(m-1)k} V_{n+mk} V_{n+(m+1)k}}.$$

Similarly, we obtain

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{V_n V_{n+2k} \cdots V_{n+(m+1)k}} \\ &= \frac{U_k}{U_{(m+1)k}} \mathcal{S}_k(m) + \frac{(-1)^k U_{mk}}{U_{(m+1)k}} \sum_{n=1}^{\infty} \frac{1}{V_{n+k} \cdots V_{n+(m+1)k}} \\ &= \frac{U_k}{U_{(m+1)k}} \mathcal{S}_k(m) + \frac{(-1)^k U_{mk}}{U_{(m+1)k}} \mathcal{S}_k(m) - \frac{(-1)^k U_{mk}}{U_{(m+1)k}} \sum_{i=1}^k \frac{1}{V_i V_{i+k} \cdots V_{i+mk}}. \end{aligned}$$

Hence, using (16) and (17), we have

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{V_n V_{n+k} \cdots V_{n+(m-1)k} V_{n+(m+1)k}} \\ &= \frac{1}{V_k} \left\{ 1 + \left[\frac{(-1)^k U_k}{U_{3k}} + \frac{(-1)^{2k} U_k U_{2k}}{U_{3k} U_{4k}} + \cdots + \frac{(-1)^{mk} U_k U_{2k}}{U_{mk} U_{(m+1)k}} \right] \right. \\ & \quad \left. + \frac{(-1)^{mk} U_{2k}}{U_{(m+1)k}} \right\} \mathcal{S}_k(m) - \frac{(-1)^{mk} U_{2k}}{V_k U_{(m+1)k}} \sum_{i=1}^k \frac{1}{V_i V_{i+k} \cdots V_{i+mk}} \\ &= \frac{1}{V_k} \left[1 + \frac{U_k U_{2k}}{U_k^2} \left(-\frac{U_k}{U_{2k}} + \frac{U_{2k}}{U_{3k}} - \frac{U_{2k}}{U_{3k}} + \frac{U_{3k}}{U_{4k}} - \cdots - \frac{U_{(m-1)k}}{U_{mk}} + \frac{U_{mk}}{U_{(m+1)k}} \right) \right] \end{aligned}$$

$$\begin{aligned}
& + \frac{(-1)^{mk}U_{2k}}{U_{(m+1)k}} \Big] \cdot \mathcal{S}_k(m) - \frac{(-1)^{mk}U_k}{U_{(m+1)k}} \sum_{i=1}^k \frac{1}{V_i V_{i+k} \cdots V_{i+mk}} \\
= & \frac{1}{V_k} \left[1 - 1 + \frac{U_{mk}U_{2k}}{U_k U_{(m+1)k}} + \frac{(-1)^{mk}U_{2k}}{U_{(m+1)k}} \right] \mathcal{S}_k(m) \\
& - \frac{(-1)^{mk}U_k}{U_{(m+1)k}} \sum_{i=1}^k \frac{1}{V_i V_{i+k} \cdots V_{i+mk}} \\
= & \frac{U_{mk} + (-1)^{mk}U_k}{U_{(m+1)k}} \mathcal{S}_k(m) - \frac{(-1)^{mk}U_k}{U_{(m+1)k}} \sum_{i=1}^k \frac{1}{V_i V_{i+k} \cdots V_{i+mk}}.
\end{aligned}$$

□

The proof of the theorem: We give only the proof of (9), the proof of (10) is similar. Since

$$\begin{aligned}
(26) \quad & \sum_{n=1}^{\infty} \left(\frac{1}{V_n V_{n+k} \cdots V_{n+(m-1)k} V_{n+(m+1)k}} - \frac{1}{V_{n+k} V_{n+2k} \cdots V_{n+mk} V_{n+(m+2)k}} \right) \\
& = \sum_{i=1}^k \frac{1}{V_i V_{i+k} \cdots V_{i+(m-1)k} V_{i+(m+1)k}},
\end{aligned}$$

using (22) and Lemma 2.1, then the left side of (26) can be written as

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{V_{n+mk} V_{n+(m+2)k} - V_n V_{n+(m+1)k}}{V_n V_{n+k} \cdots V_{n+(m+1)k} V_{n+(m+2)k}} \\
&= \frac{(-1)^{(m+1)k}}{U_k} U_{(m+1)k} \mathcal{S}_k(m) - (-1)^k (p^2 + 4) U_k U_{(m+2)k} \mathcal{T}_k(m+2) \\
&+ \left[1 - \frac{(-1)^{(m+1)k}}{U_k} U_{(m+2)k} \right] \sum_{n=1}^{\infty} \frac{1}{V_n V_{n+k} \cdots V_{n+(m-1)k} V_{n+(m+1)k}} \\
&= \frac{(-1)^{(m+1)k}}{U_k} U_{(m+1)k} \mathcal{S}_k(m) - (-1)^k (p^2 + 4) U_k U_{(m+2)k} \mathcal{T}_k(m+2) \\
&+ \left[1 - \frac{(-1)^{(m+1)k}}{U_k} U_{(m+2)k} \right] \left[\frac{U_{mk} + (-1)^{mk} U_k}{U_{(m+1)k}} \mathcal{S}_k(m) \right. \\
&\quad \left. - \frac{(-1)^{mk} U_k}{U_{(m+1)k}} \sum_{i=1}^k \frac{1}{V_i V_{i+k} \cdots V_{i+mk}} \right] \\
&= \frac{(-1)^k + (-1)^{mk} - (-1)^k V_{(m+1)k}}{U_{(m+1)k}} U_k \mathcal{S}_k(m) - (-1)^k (p^2 + 4) U_k U_{(m+2)k} \mathcal{T}_k(m+2) \\
&- \frac{(-1)^{mk} U_k - (-1)^k U_{(m+2)k}}{U_{(m+1)k}} \sum_{i=1}^k \frac{1}{V_i V_{i+k} \cdots V_{i+mk}}.
\end{aligned} \tag{27}$$

Considering the right sides of (26) and (27), we have

$$\begin{aligned}
 & \frac{(-1)^k + (-1)^{mk} - (-1)^k V_{(m+1)k}}{U_{(m+1)k}} U_k \mathcal{S}_k(m) \\
 = & \sum_{i=1}^k \frac{[(-1)^{mk} U_k - (-1)^k U_{(m+2)k}] V_{i+(m+1)k} + U_{(m+1)k} V_{i+mk}}{U_{(m+1)k} V_i V_{i+k} \cdots V_{i+(m+1)k}} \\
 & + (-1)^k (p^2 + 4) U_k U_{(m+2)k} \mathcal{T}_k(m+2) \\
 = & \sum_{i=1}^k \frac{(-1)^{mk} V_{i+(m+1)k} - (-1)^k V_{i+2(m+1)k}}{U_{(m+1)k} V_i V_{i+k} \cdots V_{i+(m+1)k}} U_k + (-1)^k (p^2 + 4) U_k U_{(m+2)k} \mathcal{T}_k(m+2).
 \end{aligned}$$

Hence

$$\begin{aligned}
 \mathcal{S}_k(m) = & \frac{1}{1 + (-1)^{(m-1)k} - V_{(m+1)k}} \left[\sum_{i=1}^k \frac{(-1)^{(m-1)k} V_{i+(m+1)k} - V_{i+2(m+1)k}}{V_i V_{i+k} V_{i+2k} \cdots V_{i+(m+1)k}} \right. \\
 & \left. + (p^2 + 4) U_{(m+1)k} U_{(m+2)k} \mathcal{T}_k(m+2) \right].
 \end{aligned}$$

The proof of (9) is completed. □

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