

## ON THE HILBERT INEQUALITY

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ABSTRACT. In this paper it is shown that the Hilbert inequality for double series can be improved by introducing a weight function of the form  $\frac{\sqrt{n}}{n+1} \left( \frac{\sqrt{n-1}}{\sqrt{n+1}} - \frac{\ln n}{\pi} \right)$ , where  $n \in N$ . A similar result for the Hilbert integral inequality is also given. As applications, some sharp results of Hardy-Littlewood's theorem and Widder's theorem are obtained.

### 1. INTRODUCTION

Let  $\{a_n\}$  and  $\{b_n\}$  be two sequences of complex numbers. It is all-round known that the inequality

$$(1.1) \quad \left| \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m \bar{b}_n}{m+n} \right|^2 \leq \pi^2 \sum_{n=1}^{\infty} |a_n|^2 \sum_{n=1}^{\infty} |b_n|^2$$

is called the Hilbert inequality for double series, where  $\sum_{n=1}^{\infty} |a_n|^2 < +\infty$  and

$\sum_{n=1}^{\infty} |b_n|^2 < +\infty$ , and that the constant factor  $\pi^2$  in (1.1) is the best possible.

The equality in (1.1) holds if and only if  $\{a_n\}$ , or  $\{b_n\}$  is a zero-sequence (see [2]). The corresponding integral form of (1.1) is that

$$(1.2) \quad \left| \int_0^{\infty} \int_0^{\infty} \frac{f(x) \bar{g}(y)}{x+y} dx dy \right|^2 \leq \pi^2 \left( \int_0^{\infty} |f(x)|^2 dx \right) \left( \int_0^{\infty} |g(x)|^2 dx \right)$$

where  $\int_0^{\infty} |f(x)|^2 dx < +\infty$  and  $\int_0^{\infty} |g(x)|^2 dx < +\infty$ , and that the constant factor

$\pi^2$  in (1.2) is also the best possible. The equality in (1.2) holds if and only if  $f(x) = 0$ , or  $g(x) = 0$ . Recently, various improvements and extensions of (1.1) and (1.2) appeared in a great deal of papers (see [2]). The purpose of the present paper is to build the Hilbert inequality with the weights by means of a monotonic function

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of the form  $\frac{\sqrt{x}}{1+\sqrt{x}}$ , thereby new refinements of (1.1) and (1.2) are established, and then to give some of their important applications.

For convenience, we need the following lemmas.

**Lemma 1.1.** *Let  $n \in \mathbb{N}$ . Then*

$$(1.3) \quad \int_0^{\infty} \frac{dx}{(n+x^2)(1+x)} = \frac{1}{n+1} \left( \frac{\pi}{2\sqrt{n}} + \frac{1}{2} \ln n \right)$$

*Proof.* Let  $a$ ,  $e$  and  $f$  be real numbers. Then

$$\begin{aligned} & \int \frac{dx}{(a^2+x^2)(e+fx)} \\ &= \frac{1}{e^2+a^2f^2} \left\{ f \ln |e+fx| - \frac{1}{2} \ln(a^2+x^2) + \frac{e}{a} \arctan \frac{x}{a} \right\} + C \end{aligned}$$

where  $C$  is an arbitrary constant. This result has been given in the papers (see [3]–[4]). Based on this indefinite integral it is easy to deduce that the equality (1.3) is true.  $\square$

**Lemma 1.2.** *Let  $n \in \mathbb{N}$ ,  $x \in (0, +\infty)$ . Define two functions by*

$$\begin{aligned} f(x) &= \left( \frac{1}{x+n} \left( \frac{n}{x} \right)^{\frac{1}{2}} \right) \left( 1 - \left( \frac{\sqrt{x}}{1+\sqrt{x}} - \frac{\sqrt{n}}{1+\sqrt{n}} \right) \right) \\ g(x) &= \left( \frac{1}{x+n} \left( \frac{n}{x} \right)^{\frac{1}{2}} \right) \left( 1 + \left( \frac{\sqrt{x}}{1+\sqrt{x}} - \frac{\sqrt{n}}{1+\sqrt{n}} \right) \right), \end{aligned}$$

*then  $f(x)$  and  $g(x)$  are monotonously decreasing in  $(0, +\infty)$ , and*

$$(1.4) \quad \int_0^{\infty} f(x) dx = \pi - \pi\omega(n)$$

$$(1.5) \quad \int_0^{\infty} g(x) dx = \pi + \pi\omega(n)$$

*where the weight function  $\omega$  is defined by*

$$(1.6) \quad \omega(n) = \frac{\sqrt{n}}{n+1} \left( \frac{\sqrt{n}-1}{\sqrt{n}+1} - \frac{\ln n}{\pi} \right)$$

*Proof.* At first, notice that  $1 - \frac{\sqrt{x}}{1+\sqrt{x}} = \frac{1}{1+\sqrt{x}}$ , hence we can write  $f(x)$  in form  $f(x) = f_1(x) + f_2(x)$ , where

$$f_1(x) = \left( \frac{1}{(x+n)\sqrt{x}} \right) \left( \frac{n}{1+\sqrt{n}} \right), \quad f_2(x) = \frac{\sqrt{n}}{(x+n)(1+\sqrt{x})\sqrt{x}}.$$

It is obvious that  $f_1(x)$  and  $f_2(x)$  are monotonously decreasing in  $(0, +\infty)$ . Hence  $f(x)$  is monotonously decreasing in  $(0, +\infty)$ . Next, notice that  $1 - \frac{\sqrt{n}}{1+\sqrt{n}} = \frac{1}{1+\sqrt{n}}$ ,

we can write  $g(x)$  in form  $g(x) = g_1(x) + g_2(x)$ , where

$$g_1(x) = \frac{\sqrt{n}}{(1 + \sqrt{n})(x + n)\sqrt{x}}, \quad g_2(x) = \frac{\sqrt{n}}{(x + n)(1 + \sqrt{x})}.$$

It is obvious that  $g_1(x)$  and  $g_2(x)$  are monotonously decreasing in  $(0, +\infty)$ . Hence  $g(x)$  is also monotonously decreasing in  $(0, +\infty)$ . Further we need only to compute two integrals.

$$\begin{aligned} \int_0^\infty f(x) dx &= \int_0^\infty \left( \frac{1}{x+n} \left( \frac{n}{x} \right)^{\frac{1}{2}} \right) \left( 1 + \frac{\sqrt{n}}{1 + \sqrt{n}} - \frac{\sqrt{x}}{1 + \sqrt{x}} \right) dx \\ &= \left( 1 + \frac{\sqrt{n}}{1 + \sqrt{n}} \right) \int_0^\infty \left( \frac{1}{x+n} \left( \frac{n}{x} \right)^{\frac{1}{2}} \right) dx - \int_0^\infty \left( \frac{1}{x+n} \left( \frac{n}{x} \right)^{\frac{1}{2}} \right) \left( \frac{\sqrt{x}}{1 + \sqrt{x}} \right) dx \\ &= \left( 1 + \frac{\sqrt{n}}{1 + \sqrt{n}} \right) \pi - \int_0^\infty \left( \frac{1}{x+n} \left( \frac{n}{x} \right)^{\frac{1}{2}} \right) \left( \frac{\sqrt{x}}{1 + \sqrt{x}} \right) dx \\ &= \pi - \left\{ 2\sqrt{n} \left( \int_0^\infty \frac{1}{(n+t^2)} dt - \int_0^\infty \frac{1}{(n+t^2)(1+t)} dt \right) - \frac{\sqrt{n} \pi}{1 + \sqrt{n}} \right\} \\ &= \pi - \left\{ \pi - 2\sqrt{n} \int_0^\infty \frac{1}{(n+t^2)(1+t)} dt - \frac{\sqrt{n} \pi}{1 + \sqrt{n}} \right\} \end{aligned}$$

By Lemma 1.1, we obtain

$$(1.7) \quad \int_0^\infty f(x) dx = \pi - \left\{ \pi - \left( \frac{\pi}{n+1} + \frac{\sqrt{n} \ln n}{n+1} \right) - \frac{\sqrt{n} \pi}{1 + \sqrt{n}} \right\}$$

The equality (1.4) follows from (1.7) at once after some simple computations and simplifications.

Similarly, the equality (1.5) can be obtained.

□

## 2. MAIN RESULTS

First, we establish a new refinement of (1.1).

**Theorem 2.1.** *Let  $\{a_n\}$  and  $\{b_n\}$  be two sequences of complex numbers. If  $\sum_{n=1}^{\infty} |a_n|^2 < +\infty$  and  $\sum_{n=1}^{\infty} |b_n|^2 < +\infty$ , then*

$$(2.1) \quad \left| \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m \bar{b}_n}{m+n} \right|^4 \leq \pi^4 \left\{ \left( \sum_{n=1}^{\infty} |a_n|^2 \right)^2 - \left( \sum_{n=1}^{\infty} \omega(n) |a_n|^2 \right)^2 \right\} \\ \times \left\{ \left( \sum_{n=1}^{\infty} |b_n|^2 \right)^2 - \left( \sum_{n=1}^{\infty} \omega(n) |b_n|^2 \right)^2 \right\}$$

where the weight function  $\omega(n)$  is defined by (1.6).

*Proof.* Let  $c(x)$  be a real function and satisfy the condition  $1 - c(n) + c(m) \geq 0$ , ( $n, m \in N$ ). Firstly we suppose that  $b_n = a_n$ . Applying Cauchy's inequality we have

$$(2.2) \quad \left| \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m \bar{a}_n}{m+n} \right|^2 = \left| \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m \bar{a}_n}{m+n} (1 - c(n) + c(m)) \right|^2 \\ = \left| \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left( \frac{a_m (1 - c(n) + c(m))^{1/2}}{(m+n)^{1/2}} \left(\frac{m}{n}\right)^{1/4} \right) \right. \\ \left. \times \left( \frac{\bar{a}_n (1 - c(n) + c(m))^{1/2}}{(m+n)^{1/2}} \left(\frac{n}{m}\right)^{1/4} \right) \right|^2 \\ \leq J_1 J_2$$

where

$$J_1 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{|a_m|^2}{m+n} \left(\frac{m}{n}\right)^{\frac{1}{2}} (1 - c(n) + c(m))$$

$$J_2 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{|\bar{a}_n|^2}{m+n} \left(\frac{n}{m}\right)^{\frac{1}{2}} (1 - c(n) + c(m))$$

We can write the double series  $J_1$  in the following form:

$$J_1 = \sum_{n=1}^{\infty} \left( \sum_{m=1}^{\infty} \frac{1}{m+n} \left(\frac{n}{m}\right)^{\frac{1}{2}} (1 - c(m) + c(n)) \right) |a_n|^2.$$

Let  $c(x) = \frac{\sqrt{x}}{1+\sqrt{x}}$ . It is obvious that  $1 - \frac{\sqrt{x}}{1+\sqrt{x}} + \frac{\sqrt{n}}{1+\sqrt{n}} \geq 0$ . It is known from Lemma 1.2 that the function  $f(x)$  is monotonously decreasing. Hence we have

$$J_1 = \sum_{n=1}^{\infty} \left( \sum_{m=1}^{\infty} \frac{1}{m+n} \left(\frac{n}{m}\right)^{\frac{1}{2}} \left( 1 - \frac{\sqrt{m}}{1+\sqrt{m}} + \frac{\sqrt{n}}{1+\sqrt{n}} \right) \right) |a_n|^2 \\ \leq \sum_{n=1}^{\infty} \left\{ \int_0^{\infty} \left( \frac{1}{x+n} \left(\frac{n}{x}\right)^{\frac{1}{2}} \right) \left( 1 - \left( \frac{\sqrt{x}}{1+\sqrt{x}} - \frac{\sqrt{n}}{1+\sqrt{n}} \right) \right) dx \right\} |a_n|^2 \\ = \pi \sum_{n=1}^{\infty} |a_n|^2 - \pi \sum_{n=1}^{\infty} \omega(n) |a_n|^2$$

where the weight function  $\omega(n)$  is defined by (1.6).

Similarly,

$$J_2 \leq \sum_{n=1}^{\infty} \left\{ \int_0^{\infty} \frac{1}{x+n} \left(\frac{n}{x}\right)^{\frac{1}{2}} \left(1 + \left(\frac{\sqrt{x}}{1+\sqrt{x}} - \frac{\sqrt{n}}{1+\sqrt{n}}\right)\right) dx \right\} |\bar{a}_n|^2$$

$$= \pi \sum_{n=1}^{\infty} |a_n|^2 + \pi \sum_{n=1}^{\infty} \omega(n) |a_n|^2.$$

Whence  $J_1 J_2 \leq \pi^2 \left\{ \left(\sum_{n=1}^{\infty} |a_n|^2\right)^2 - \left(\sum_{n=1}^{\infty} \omega(n) |a_n|^2\right)^2 \right\}$ .

Consequently, we have

$$(2.3) \quad \left| \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m \bar{a}_n}{m+n} \right|^2 \leq \pi^2 \left\{ \left(\sum_{n=1}^{\infty} |a_n|^2\right)^2 - \left(\sum_{n=1}^{\infty} \omega(n) |a_n|^2\right)^2 \right\}$$

where the weight function  $\omega(n)$  is defined by (1.6).

If  $b_n \neq a_n$ , then we can apply Schwarz's inequality to estimate the right-hand side of (2.1) as follows:

$$(2.4) \quad \left| \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m \bar{b}_n}{m+n} \right|^4 = \left\{ \left| \int_0^1 \left(\sum_{m=1}^{\infty} a_m t^{m-\frac{1}{2}}\right) \left(\sum_{n=1}^{\infty} \bar{b}_n t^{n-\frac{1}{2}}\right) dt \right|^2 \right\}^2$$

$$\leq \left| \int_0^1 \left(\sum_{m=1}^{\infty} |a_m| t^{m-\frac{1}{2}}\right)^2 dt \int_0^1 \left(\sum_{n=1}^{\infty} |b_n| t^{n-\frac{1}{2}}\right)^2 dt \right|^2$$

$$= \left| \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m \bar{a}_n}{m+n} \right|^2 \left| \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{b_m \bar{b}_n}{m+n} \right|^2$$

And then by using the relation (2.3), from (2.4) and the inequality (2.1), we obtain at once. □

Similarly, we can establish a new refinement of (1.2).

**Theorem 2.2.** *Let  $f(x)$  and  $g(x)$  be two functions in complex number field.*

*If  $\int_0^{\infty} |f(x)|^2 dx < +\infty$ ,  $\int_0^{\infty} |g(x)|^2 dx < +\infty$ , then*

$$(2.5) \quad \left| \int_0^{\infty} \int_0^{\infty} \frac{f(x) \bar{g}(x)}{x+y} dx dy \right|^4 \leq \pi^4 \left\{ \left(\int_0^{\infty} |f(x)|^2 dx\right)^2 - \left(\int_0^{\infty} \omega(x) |f(x)|^2 dx\right)^2 \right\}$$

$$\times \left\{ \left(\int_0^{\infty} |g(x)|^2 dx\right)^2 - \left(\int_0^{\infty} \omega(x) |g(x)|^2 dx\right)^2 \right\}$$

where the weight function  $\omega$  is defined by

$$(2.6) \quad \omega(x) = \begin{cases} 0 & x = 0 \\ \frac{\sqrt{x}}{x+1} \left( \frac{\sqrt{x}-1}{\sqrt{x}+1} - \frac{\ln x}{\pi} \right) & x > 0 \end{cases}$$

Its proof is similar to that of Theorem 2.1, it is omitted here.

For the convenience of the applications, we list the following result.

**Corollary 2.3.** *Let  $f(x)$  be a function in complex number field. If  $\int_0^\infty |f(x)|^2 dx < +\infty$ , then*

$$(2.7) \quad \left| \int_0^\infty \int_0^\infty \frac{f(x)\bar{f}(y)}{x+y} dx dy \right|^2 \leq \pi^2 \left\{ \left( \int_0^\infty |f(x)|^2 dx \right)^2 - \left( \int_0^\infty \omega(x) |f(x)|^2 dx \right)^2 \right\}$$

where the weight function  $\omega$  is defined by (2.6).

### 3. APPLICATIONS

As applications, we shall give some new refinements of Hardy-Littlewood's theorem and Widder's theorem.

Let  $f(x) \in L^2(0,1)$  and  $f(x) \neq 0$  for all  $x$ . Define a sequence  $\{a_n\}$  by  $a_n = \int_0^1 x^n f(x) dx$ ,  $n = 0, 1, 2, \dots$ . Hardy-Littlewood ([1]) proved that

$$(3.1) \quad \sum_{n=0}^{\infty} a_n^2 < \pi \int_0^1 f^2(x) dx,$$

where  $\pi$  is the best constant that the inequality (3.1) keeps valid.

**Theorem 3.1.** *Let  $f(x) \in L^2(0,1)$  and  $f(x) \neq 0$  for all  $x$ . Define a sequence  $\{a_n\}$  by  $a_n = \int_0^1 x^{n-1/2} f(x) dx$   $n = 1, 2, \dots$ . Then*

$$(3.2) \quad \left( \sum_{n=1}^{\infty} a_n^2 \right)^2 \leq \pi \left\{ \left( \sum_{n=1}^{\infty} a_n^2 \right)^2 - \left( \sum_{n=1}^{\infty} \omega(n) a_n^2 \right)^2 \right\}^{\frac{1}{2}} \int_0^1 f^2(x) dx$$

where  $\omega(n)$  is defined by (1.6).

*Proof.* By our assumptions, we may write  $a_n^2$  in the form

$$a_n^2 = \int_0^1 a_n x^{n-1/2} f(x) dx.$$

Applying Cauchy-Schwarz's inequality we estimate the right hand side of (3.2) as follows

$$\begin{aligned}
 \left(\sum_{n=1}^{\infty} a_n^2\right)^2 &= \left(\sum_{n=1}^{\infty} \int_0^1 a_n x^{n-1/2} f(x) dx\right)^2 = \left\{ \int_0^1 \left(\sum_{n=1}^{\infty} a_n x^{n-1/2}\right) f(x) dx \right\}^2 \\
 &\leq \int_0^1 \left(\sum_{n=1}^{\infty} a_n x^{n-1/2}\right)^2 dx \int_0^1 f^2(x) dx \\
 &= \int_0^1 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_m a_n x^{m+n-1} dx \int_0^1 f^2(x) dx \\
 (3.3) \quad &= \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m a_n}{m+n}\right) \int_0^1 f^2(x) dx
 \end{aligned}$$

It is known from (2.3) and (3.3) that the inequality (3.2) is valid. Therefore the theorem is proved.  $\square$

Let  $a_n \geq 0$  ( $n = 0, 1, 2, \dots$ ),  $A(x) = \sum_{n=0}^{\infty} a_n x^n$ ,  $A^*(x) = \sum_{n=0}^{\infty} \frac{a_n x^n}{n!}$ . Then

$$(3.4) \quad \int_0^1 A^2(x) dx \leq \pi \int_0^{\infty} (e^{-x} A^*(x))^2 dx$$

This is Widder's theorem (see [1]).

**Theorem 3.2.** *With the assumptions as the above-mentioned, it yields*

$$\begin{aligned}
 (3.5) \quad &\left(\int_0^1 A^2(x) dx\right)^2 \\
 &\leq \pi^2 \left\{ \left(\int_0^{\infty} (e^{-x} A^*(x))^2 dx\right)^2 - \left(\int_0^{\infty} \omega(x) (e^{-x} A^*(x))^2 dx\right)^2 \right\}
 \end{aligned}$$

where  $\omega(x)$  is defined by (2.6).

*Proof.* At first we have the following relation:

$$\begin{aligned}
 \int_0^{\infty} e^{-t} A^*(tx) dt &= \int_0^{\infty} e^{-t} \sum_{n=0}^{\infty} \frac{a_n (xt)^n}{n!} dt \\
 &= \sum_{n=0}^{\infty} \frac{a_n x^n}{n!} \int_0^{\infty} t^n e^{-t} dt = \sum_{n=0}^{\infty} a_n x^n = A(x)
 \end{aligned}$$

Let  $tx = s$ . Then we have

$$\begin{aligned}
 \int_0^1 A^2(x) dx &= \int_0^1 \left\{ \int_0^\infty e^{-t} A^*(tx) dt \right\}^2 dx = \int_0^1 \left( \int_0^\infty e^{-\frac{s}{x}} A^*(s) ds \right)^2 \frac{1}{x^2} dx \\
 &= \int_1^\infty \left( \int_0^\infty e^{-sy} A^*(s) ds \right)^2 dy = \int_0^\infty \left( \int_0^\infty e^{-s(u+1)} A^*(s) ds \right)^2 du \\
 (3.6) \quad &= \int_0^\infty \left( \int_0^\infty e^{-su} f(s) ds \right)^2 du = \int_0^\infty \int_0^\infty \frac{f(s)f(t)}{s+t} ds dt
 \end{aligned}$$

where  $f(x) = e^{-x} A^*(x)$ . By Corollary 2.3, the inequality (3.5) follows from (3.6) at once.  $\square$

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