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MAXIMAL OPERATORS OF THE FEJÉR MEANS OF THE TWO DIMENSIONAL CHARACTER SYSTEM OF THE p-SERIES FIELD IN THE KACZMARZ REARRANGEMENT

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Abstract. The main aim of this paper is to prove that the maximal operator σ^* of the Fejér means of the two dimensional character system of the p-series field in the Kaczmarz rearrangement is bounded from the Hardy space H_α to the space L_α for $\alpha > 1/2$, provided that the supremum in the maximal operator is taken over a positive cone. We also prove that the maximal operator σ_0^* of Fejér means of the two dimensional character system of the p-series field in the Kaczmarz rearrangement is not bounded from the Hardy space $H_{1/2}$ to the space weak-$L_{1/2}$.

1. Introduction

The first result with respect to the a.e. convergence of the Walsh-Fejér means $\sigma_n f$ is due to Fine [**1**]. Later, Schipp [**9**] showed that the maximal operator $\sigma^* f$ is of weak type $(1,1)$, from which the a. e. convergence follows on standard argument. Schipp result implies also the boundedness of $\sigma^* : L_\alpha \to L_\alpha$ $(1 < \alpha \leq \infty)$ by interpolation. This fails to hold for $\alpha = 1$ but Fujii [**2**] proved that σ^* is bounded from the dyadic Hardy space H_1 to the space L_1 (see also Simon [**13**]). Fujii's theorem was extened by Weisz [**15**]. Namely, he proved that the maximal operator of the Fejér means of the one-dimensional Walsh-Fourier series is bounded from the martingale Hardy space H_α to the space L_α for $\alpha > 1/2$. Simon [**11**] gave a counterexample, which shows that this boundedness does not hold for $0 < \alpha < 1/2$. In the endpoint case $\alpha = 1/2$ Weisz [**17**] proved that σ^* is bounded from the Hardy space $H_{1/2}(G_2)$ to the space weak-$L_{1/2}(G_2)$. The author [**6**] proved that σ^* is not bounded from the Hardy space $H_{1/2}(G_2)$ to the space $L_{1/2}(G_2)$.

If the Walsh system is taken in the Kaczmarz ordening, the analogous to the statement of Schipp [**9**] is due to Gát [**3**]. Moreover he proved an (H_1, L_1)-type estimation. Gát result was extended to the Hardy space by Simon [**12**], who proved

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that σ^* is of type (H_α, L_α) for $\alpha > 1/2$. Weisz [**17**] showed that in endpoint case $\alpha = 1/2$ the maximal operator is of weak type $\left(H_{1/2}, L_{1/2}\right)$.

Gát and Nagy [**4**] proved the a. e. convergence $\sigma_n f \to f$ $(n \to \infty)$ for an integrable function $f \in L_1(G_p)$, where $\sigma_n f$ is the Fejér means of the function f with respect to the character system in the Kaczmarz rearrangement. This result was generalized by the author [**7**] and it is proved that the maximal operator σ^* of the Fejér means of the one dimensional character system of the p-series field in the Kaczmarz rearrangement is bounded from the Hardy space $H_{1/2}(G_p)$ to the space weak-$L_{1/2}(G_p)$. By interpolation it follows that σ^* is of type (H_α, L_α) for $\alpha > 1/2$. We also prove that the assumption $\alpha > 1/2$ is essentiall, in particular, it is proved that the maximal operator σ^* is not bounded from the Hardy space $H_{1/2}(G_p)$ to the space $L_{1/2}(G_p)$. By interpolation it follows that σ^* is not of type $(H_\alpha, \text{weak-}L_\alpha)$ for $0 < \alpha < 1/2$.

The aim of this paper is to prove that the maximal operator of Fejér means of the two dimensional character system of the p-series field in the Kaczmarz rearrangement is bounded from the Hardy space $H_\alpha(G_p \times G_p)$ to the space $L_\alpha(G_p \times G_p)$ for $\alpha > 1/2$ and is of weak type $(1,1)$ provided that the supremum in the maximal operator is taken over a positive cone. So we obtain that the Fejer means of a function $f \in L_1(G_p \times G_p)$ converge a. e. to the function in the question, provided again that the limit is taken over a positive cone. We also proved that the maximal operator σ_0^* of Fejér means of the two dimensional character system of the p-series field in the Kaczmarz rearrangement is not bounded from the Hardy space $H_{1/2}(G_p \times G_p)$ to the space weak-$L_{1/2}(G_p \times G_p)$. Thus, in the question of boundedness of the maximal operator σ_0^* the case of two dimensional character system of the p-series field in the Kaczmarz rearrangement differs from that one-dimensional character system of the p-series field in the Kaczmarz rearrangement. By Theorem 2 and interpolation it follows that σ_0^* is not bounded from $H_\alpha(G_p \times G_p)$ to the space weak-$L_\alpha(G_p \times G_p)$ for $0 < \alpha \leq 1/2$. In particular, from Theorem 2 we have that in Theorem 1 the assumption $\alpha > 1/2$ is essential.

2. Definitions and Notation

Let $\mathbf{P}$ denote the set of positive integers, $\mathbf{N} := \mathbf{P} \cup \{0\}$. Let $2 \leq p \in \mathbf{N}$ and denote by $\mathbf{Z}_p$ the pth cyclic group, that is, $\mathbf{Z}_p$ can be represented by the set $\{0, 1, \ldots, p-1\}$, where the group operation is the mod p addition and every subset is open. The Haar measure on $\mathbf{Z}_p$ is given in the way that

$$\mu_k(\{j\}) := \frac{1}{j} \qquad (j \in \mathbf{Z}).$$

The group operation on G_p is the coordinate-wise addition, the normalized Haar measure μ is the product measure. The topology on G_p is the product topology, a base for the neighborhoods of G_p can be given in the following way:

$$I_0(x) := G_p,$$
$$I_n(x) := \{y \in G_p : y = (x_0, \ldots, x_{n-1}, y_n, y_{n+1}, \ldots)\}, \qquad (x \in G_p, n \in \mathbf{N}).$$

Let $0 = (0 : i \in \mathbf{N}) \in G_p$ denote the null element of G_p, $I_n := I_n(0)$ $(n \in \mathbf{N})$, $\overline{I}_n := G_p \setminus I_n$. Let

$$\Delta := \{I_n(x) : x \in G_p,\ n \in \mathbf{N}\}.$$

The elements of Δ are intervals of G_p. Set $e_i := (0, \ldots, 0, 1, 0, \ldots) \in G_p$ whose i-th coordinate is 1, the rest are zeros.

The norm (or quasinorm) of the space $L_\alpha(G_p \times G_p)$ is defined by

$$\|f\|_\alpha := \left(\int\limits_{G_p \times G_p} \left|f\left(x^1, x^2\right)\right|^\alpha \mathrm{d}\mu\left(x^1, x^2\right) \right)^{1/\alpha}, \qquad (0 < \alpha < +\infty).$$

Let $\Gamma(p)$ denote the character group of G_p. We arrange the elements of $\Gamma(p)$ as follows. For $k \in \mathbf{N}$ and $x \in G_p$ denote by r_k the k-th generalized Rademacher function

$$r_k(x) := \exp\left(\frac{2\pi \,\mathrm{i}\, x_k}{p}\right) \qquad \left(\mathrm{i} := \sqrt{-1},\quad x \in G_p,\quad k \in \mathbf{N}\right).$$

Let $n \in \mathbf{N}$. Then

$$n = \sum_{i=0}^{\infty} n_i p^i, \qquad \text{where } \ 0 \le n_i < p \quad (n_i, i \in \mathbf{N}),$$

n is expressed in the number system with base p. Denote by

$$|n| := \max(j \in \mathbf{N} : n_j \neq 0) \qquad \text{i.e., } \ p^{|n|} \le n < p^{|n|+1}.$$

Now, we define the sequence of functions $\psi := (\psi_n : n \in \mathbf{N})$ by

$$\psi_n(x) := \prod_{k=0}^{\infty} (r_k(x))^{n_k} \qquad (x \in G_p,\ n \in \mathbf{N}).$$

We remark that $\Gamma(p) = \{\psi_n : n \in \mathbf{N}\}$ is a complete orthogonal system with respect to the normalized Haar measure on G_p.

The character group $\Gamma(p)$ can be given in the Kaczmarz rearrangement as follows: $\Gamma(p) = \{\chi_n : n \in \mathbf{N}\}$, where

$$\chi_n(x) := r_{|n|}^{n_{|n|}}(x) \prod_{k=0}^{|n|-1} (r_{|n|-1-k}(x))^{n_k} \qquad (x \in G_p, n \in \mathbf{P}),$$

$$\chi_0(x) = 1 \qquad (x \in G_p).$$

Let the transformation $\tau_A : G_p \to G_p$ be defined as follows:

$$\tau_A(x) := (x_{A-1}, x_{A-2}, \ldots, x_0, x_A, x_{A+1}, \ldots).$$

The transformation is measure-preservingand and $\tau_A(\tau_A(x)) = x$. By the definition of τ_A , we have

$$\chi_n(x) = r_{|n|}^{n_{|n|}}(x)\psi_{n - n_{|n|}p^n}(\tau_{|n|}(x)) \qquad (n \in \mathbf{N},\ x \in G_p).$$

The rectangular partial sums of the double Fourier series are defined as follows:

$$S_{M,N}(f;x^1,x^2) := \sum_{i=0}^{M-1}\sum_{j=0}^{N-1} \widehat{f}\,(i,j)\,\chi_i\left(x^1\right)\chi_j\left(x^2\right),$$

where the number

$$\widehat{f}\,(i,j) = \int\limits_{G_p\times G_p} f\left(x^1,x^2\right)\overline{\chi}_i\left(x^1\right)\overline{\chi}_j\left(x^2\right)\mathrm{d}\mu\left(x^1,x^2\right)$$

is said to be the (i,j)-th Fourier coefficient of the function f. Let

$$I_{n,n}\left(x^1,x^2\right) := I_n\left(x^1\right)\times I_n\left(x^2\right).$$

The σ-algebra generated by the dyadic rectangles

$$\left\{I_{n,n}\left(x^1,x^2\right) : \left(x^1,x^2\right)\in G_p\times G_p\right\}$$

will be denoted by $F_{n,n}$ $(n\in\mathbf{N})$.

Denote by $f = \left(f^{(n,n)}, n\in\mathbf{N}\right)$ martingale with respect to $(F_{n,n}, n\in\mathbf{N})$ (for details see, e. g. [**14, 16**]

The diagonal maximal function of a martingale f is defined by

$$f^* = \sup_{n\in\mathbf{N}}\left|f^{(n,n)}\right|.$$

In case $f\in L_1(G_p\times G_p)$, diagonal maximal function can also be given by

$$f^*\left(x^1,x^2\right) = \sup_{n\in\mathbf{N}}\frac{1}{\mu\left(I_{n,n}(x^1,x^2)\right)}\left|\int\limits_{I_{n,n}(x^1,x^2)} f\left(u^1,u^2\right)\mathrm{d}\mu\left(u^1,u^2\right)\right|,$$

$$\left(x^1,x^2\right)\in G_p\times G_p.$$

For $0<p<\infty$ the Hardy martingale space $H_p(G_p\times G_p)$ consists of all martingales for which

$$\|f\|_{H_p} := \|f^*\|_p < \infty.$$

If $f\in L_1(G_p\times G_p)$ then it is easy to show that the sequence $(S_{p^n,p^n}(f) : n\in\mathbf{N})$ is a martingale. If f is a martingale, that is $f = (f^{(n,n)} : n\in\mathbf{N})$, then the Fourier coefficients must be defined in a little bit different way:

$$\widehat{f}\,(i,j) = \lim_{k\to\infty}\int\limits_{G\times G} f^{(k,k)}\left(x^1,x^2\right)\overline{\chi}_i\left(x^1\right)\overline{\chi}_j\left(x^2\right)\mathrm{d}\mu\left(x^1,x^2\right).$$

The Fourier coefficients of $f\in L_1(G_p\times G_p)$ are the same as the ones of the martingale $(S_{p^n,p^n}(f) : n\in\mathbf{N})$ obtained from f.

For $n,m\in\mathbf{P}$ and a martingale f the Fejér means of order (n,m) of the two-dimensional character system of the p-series field in the Kaczmarz rearrangement of the martingale f is given by

$$\sigma_{n,m}(f;x^1,x^2) = \frac{1}{nm}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} S_{i,j}(f;x^1,x^2).$$

For the martingale f, the restricted maximal operator of the Fejér means is defined by

$$\sigma_{\lambda}^{*} f\left(x^{1}, x^{2}\right)=\sup_{p^{-\lambda} \leq n / m \leq p^{\lambda}}\left|\sigma_{n, m}\left(f ; x^{1}, x^{2}\right)\right|, \qquad \lambda>0.$$

The Dirichlet kernels and Fejér kernels are defined as follows

$$D_{n}^{\gamma}(x):=\sum_{j=0}^{n-1} \gamma_{j}(x), \qquad K_{n}^{\gamma}(x):=\sum_{j=0}^{n-1} D_{j}^{\gamma}(x),$$

where γ is either ψ or χ.

The p^{n}th Dirichlet kernels have a closed form:

$$D_{p^{n}}^{\psi}(x)=D_{p^{n}}^{\chi}(x)=\begin{cases} p^{n} & \text{if } x \in I_{n}, \\ 0 & \text{if } x \notin I_{n},\end{cases} \qquad \text{where } x \in G_{p}. \tag{1}$$

3. Formulation of Main Results

Theorem 1. *Let $\alpha>1/2$. Then the maximal operator σ_{λ}^{*} is bounded from the Hardy space $H_{\alpha}(G_{p} \times G_{p})$ to the space $L_{\alpha}(G_{p} \times G_{p})$. Especialy, if $f \in L_{1}(G_{p} \times G_{p})$ then*

$$\mu\left(\sigma_{\lambda}^{*}>y\right) \leq \frac{c}{y}\|f\|_{1}.$$

Corollary 1. *If $f \in L_{1}(G_{p} \times G_{p})$, then*

$$\sigma_{n, m} f\left(x^{1}, x^{2}\right) \rightarrow f\left(x^{1}, x^{2}\right) \qquad \text{a. e.}$$

as $\min(n, m) \rightarrow \infty$ *and* $p^{-\lambda} \leq n / m \leq p^{\lambda}$ $(\lambda>0)$.

Theorem 2. *The maximal operator σ_{0}^{*} is not bounded from the Hardy space $H_{1/2}(G_{p} \times G_{p})$ to the space weak-$L_{1/2}(G_{p} \times G_{p})$.*

4. Auxiliary Propositions

We shall need the following lemmas

Lemma 1 (Gát, Nagy [**4**])**.** *Let $A \in \mathbf{N}$ and $n:=n_{A} p^{A}+n_{A-1} p^{A-1}+\cdots+n_{0} p^{0}$. Then*

$$\begin{aligned} n K_{n}^{\chi}(x)= & 1+\sum_{j=0}^{A-1} \sum_{i=1}^{p-1} r_{j}^{i}(x) p^{j} K_{p^{j}}^{\psi}\left(\tau_{j}(x)\right)+\sum_{j=0}^{A-1} p^{j} D_{p^{j}}^{\psi}(x) \sum_{l=1}^{p-1} \sum_{i=0}^{l-1} r_{j}^{i}(x) \\ & +p^{A} \sum_{l=1}^{n_{A}-1} r_{A}^{l}(x) K_{p^{A}}^{\psi}\left(\tau_{A}(x)\right)+r_{A}^{n_{A}}(x)\left(n-n_{A} p^{A}\right) K_{n-n_{A} p^{A}}^{\psi}\left(\tau_{A}(x)\right) \\ & +\left(n-n_{A} p^{A}\right) \sum_{i=0}^{n_{A}-1} r_{A}^{i}(x) D_{p^{A}}^{\psi}(x)++p^{A} \sum_{j=1}^{n_{A}-1} \sum_{i=0}^{j-1} r_{A}^{i}(x) D_{p^{A}}^{\psi}(x). \end{aligned}$$

Lemma 2 (Gát, Nagy [**4**])**.** *Let $A,\ l \in \mathbf{N}$, $A > l$ and $x \in I_l \setminus I_{l+1}$. Then*

$$K_{p^A}^{\psi}(x) = \begin{cases} 0, & \text{if } \ x - x_l e_l \notin I_A, \\ \dfrac{p^l}{1 - r_l(x)} & \text{if } \ x - x_l e_l \in I_A. \end{cases}$$

Lemma 3 ([**7**])**.** *Let $n < p^{A+1}$, $A > N$ and $x \in I_N(x_0, \ldots, x_{m-1}, x_m \neq 0, 0, \ldots, 0, x_l \neq 0, 0, \ldots, 0)$ $m = -1, 0, \ldots, l-1$, $l = 0, \ldots, N$. Then*

$$\int_{I_N} n \left| K_n^{\psi}(\tau_A(x - t)) \right| \mathrm{d}\mu(t) \leq \frac{cp^A}{p^{m+l}},$$

where

$$I_N(x_0, \ldots, x_{m-1}, x_m \neq 0, 0, \ldots, 0, x_l \neq 0, 0, \ldots, 0)$$
$$:= I_N(0, \ldots, 0, x_l \neq 0, 0, \ldots, 0) \qquad \text{for } \ m = -1,$$

and

$$I_N(x_0, \ldots, x_{m-1}, x_m \neq 0, 0, \ldots, 0, x_l \neq 0, 0, \ldots, 0)$$
$$:= I_N(x_0, \ldots, x_{m-1}, x_m \neq 0, 0, \ldots, 0), \qquad \text{for } \ l = N.$$

Lemma 4 ([**5**])**.** *Let $A \in \mathbf{N}$ and $n_A := p^{2A} + p^{2A-2} + \ldots + p^2 + p^0$. Then*

$$n_{A-1}|K_{n_{A-1}}(x)| \geq cp^{2k+2s}$$

for $x \in I_{2A}(0, \ldots, 0, x_{2k} \neq 0, 0, \ldots, 0, x_{2s} \neq 0, x_{2s+1}, \ldots, x_{2A-1})$, $k = 0, 1, \ldots, A-3$, $s = k+2, k+3, \ldots, A-1$.

Lemma 5. *Let $x \in \overline{I}_N$ and $n \geq p^N$. Then*

$$\int_{I_N} |K_n^{\chi}(x - t)| \mathrm{d}\mu(t)$$
$$\leq c \left\{ \sum_{l=0}^{N} \sum_{m=-1}^{l-1} \frac{1}{p^{m+l}} \mathbf{1}_{I_N(x_0, \ldots, x_{m-1}, x_m \neq 0, 0, \ldots, 0, x_l \neq 0, 0, \ldots, 0)}(x) \right.$$
$$\left. + \frac{1}{p^{2N}} \sum_{j=1}^{N} p^{2j} \sum_{l=0}^{j-1} \frac{1}{p^l} \mathbf{1}_{I_N(0, \ldots, 0, x_l \neq 0, 0, \ldots, 0,, x_j, \ldots, x_{N-1})}(x) \right\}.$$

Proof. From Lemma 1 we write

$$n\left|K_n^{\chi}(x)\right| \leq c \left\{ 1 + \sum_{j=0}^{A} p^j \left| K_{p^j}^{\psi}(\tau_j(x)) \right| \right.$$
$$\left. + \sum_{j=0}^{A} p^j \left| D_{p^j}^{\psi}(x) \right| + \left(n - n_A p^A\right) \left| K_{n - n_A p^A}^{\psi}(\tau_A(x)) \right| \right\}. \tag{2}$$

Using Lemma 3 we obtain

$$\frac{1}{n}\int_{I_N}\left(n-n_Ap^A\right)\left|K^{\psi}_{n-n_Ap^A}\left(\tau_A\left(x-t\right)\right)\right|\mathrm{d}\mu\left(t\right)$$

$$\leq c\left\{\sum_{l=0}^{N}\sum_{m=-1}^{l-1}\frac{1}{p^{m+l}}\mathbf{1}_{I_N\left(x_0,\dots,x_{m-1},x_m\neq0,0,\dots,0,x_l\neq0,0,\dots,0\right)}\left(x\right)\right\}. \tag{3}$$

Let $x\in I_N\left(x_0,\dots,x_{m-1},x_m\neq0,0,\dots,0,x_l\neq0,0,\dots,0\right)$ for some $m=-1,\dots,$ $\dots,l-1,\ l=0,\dots,N$. Then using Lemma 2 $K^{\psi}_{p^j}\left(\tau_j\left(x-t\right)\right)\neq0\ \ (j>N)$ implies

$$t\in I_j\left(0,\dots,0,x_N,\dots,x_{j-1}\right),\qquad m=-1.$$

Consequently, we can write

$$\begin{aligned}\int_{I_N}p^j\left|K^{\psi}_{p^j}\left(\tau_j\left(x-t\right)\right)\right|\mathrm{d}\mu\left(t\right)&\leq\frac{cp^j}{p^j}p^{j-l}\mathbf{1}_{I_N\left(0,\dots,0,x_l\neq0,0,\dots,0\right)}\left(x\right)\\&=\frac{cp^j}{p^l}\mathbf{1}_{I_N\left(0,\dots,0,x_l\neq0,0,\dots,0\right)}\left(x\right).\end{aligned}\tag{4}$$

Let $j<N$. Then using Lemma 2 $K^{\psi}_{p^j}\left(\tau_j\left(x-t\right)\right)\neq0$ implies

$$x\in I_N\left(0,\dots,0,x_l\neq0,0,\dots,0,x_j,\dots,x_{N-1}\right),l=-1,0,\dots,j-1.$$

Hence we have

$$\begin{aligned}\int_{I_N}p^j\left|K^{\psi}_{p^j}\left(\tau_j\left(x-t\right)\right)\right|\mathrm{d}\mu\left(t\right)&\leq\frac{cp^j}{p^N}\sum_{l=0}^{j-1}p^{j-l}\mathbf{1}_{I_N\left(0,\dots,0,x_l\neq0,0,\dots,0,x_j,\dots,x_{N-1}\right)}\left(x\right)\\&=\frac{cp^{2j}}{p^N}\sum_{l=0}^{j-1}p^{-l}\mathbf{1}_{I_N\left(0,\dots,0,x_l\neq0,0,\dots,0,x_j,\dots,x_{N-1}\right)}\left(x\right).\end{aligned}\tag{5}$$

From (1) we can write

$$\begin{aligned}\sum_{j=0}^{A}p^j\int_{I_N}\left|D^{\psi}_{p^j}\left(x-t\right)\right|\mathrm{d}\mu\left(t\right)&\leq\frac{c}{p^N}\sum_{j=0}^{N-1}p^j\left|D^{\psi}_{p^j}\left(x\right)\right|\\&\leq\frac{c}{p^N}\sum_{j=0}^{N-1}p^{2j}\mathbf{1}_{I_N\left(0,\dots,0,x_j,\dots,x_{N-1}\right)}\left(x\right).\end{aligned}\tag{6}$$

Combining (2)–(6) we complete the proof of Lemma 5. □

5. Proofs of Main results

Proof of Theorem 1. In order to prove Theorem 1 it is enough to show that (see Simon [**11**], Theorem 1)

$$\int_{\overline{I}_N}\left(\sup_{n\geq 2^N}\int_{I_N}\left|K_n^{\chi}(x-t)\right|\mathrm{d}\mu(t)\right)^{\alpha}\mathrm{d}\mu(x)\leq c_{\alpha}p^{-N},\qquad \text{for } 1/2<\alpha\leq 1.$$

Applying the inequality

$$\left(\sum_{k=0}^{\infty}a_k\right)^{\alpha}\leq\sum_{k=0}^{\infty}a_k^{\alpha}\qquad\left(a_k\geq 0,\quad 0<\alpha\leq 1\right),$$

from Lemma 5 we can write

$$\int_{\overline{I}_N}\left(\sup_{n\geq 2^N}\int_{I_N}\left|K_n^{\chi}(x-t)\right|\mathrm{d}\mu(t)\right)^{\alpha}\mathrm{d}\mu(t)$$

$$\leq c_{\alpha}\left\{\sum_{l=0}^{N}\sum_{m=-1}^{l-1}\frac{1}{p^{\alpha(m+l)}}\int_{G}\mathbf{1}_{I_N(x_0,\ldots,x_{m-1},x_m\neq 0,0,\ldots,0,x_l\neq 0,0,\ldots,0)}(x)\,\mathrm{d}\mu(x)\right.$$

$$\left.+\frac{1}{p^{2\alpha N}}\sum_{j=1}^{N}p^{2j\alpha}\sum_{l=0}^{j-1}\frac{1}{p^{l\alpha}}\int_{G}\mathbf{1}_{I_N(0,\ldots,0,x_l\neq 0,0,\ldots,0,,x_j,\ldots,x_{N-1})}(x)\,\mathrm{d}\mu(x)\right\}$$

$$\leq c_{\alpha}\left\{\frac{1}{p^N}\sum_{l=0}^{N}\sum_{m=-1}^{l-1}\frac{p^m}{p^{\alpha(m+l)}}+\frac{1}{p^Np^{2\alpha N}}\sum_{j=1}^{N}p^{2j\alpha}\sum_{l=0}^{j-1}\frac{p^{N-j}}{p^{l\alpha}}\right\}\leq cp^{-N}.$$

The proof of Theorem 1 is complete. □

Proof of Theorem 2. Let $A\in\mathbf{P}$ and

$$f_A\left(x^1,x^2\right):=\left(D_{p^{2A+1}}\left(x^1\right)-D_{p^{2A}}\left(x^1\right)\right)\left(D_{p^{2A+1}}\left(x^2\right)-D_{p^{2A+1}}\left(x^2\right)\right).$$

It is simple to calculate

$$\widehat{f}_A^{\psi}(i,k)=\begin{cases}1, & \text{if } i,k=p^{2A},\ldots,p^{2A+1}-1,\\ 0, & \text{otherwise.}\end{cases}$$

and

$$S_{i,j}^{\psi}\left(f_A;x^1,x^2\right)=\begin{cases}\left(D_i^{\psi}\left(x^1\right)-D_{p^{2A}}\left(x^1\right)\right)\left(D_j^{\psi}\left(x^2\right)-D_{p^{2A}}\left(x^2\right)\right), & \\ \qquad\qquad \text{if } i,j=p^{2A}+1,\ldots,p^{2A+1}-1, & \\ f_A\left(x^1,x^2\right), & \\ \qquad\qquad \text{if } i,j\geq p^{2A+1}, & \\ 0, & \\ \qquad\qquad \text{otherwise.} & \end{cases}\tag{7}$$

Since

$$f_A^*\left(x^1,x^2\right)=\sup_{n\in\mathbf{N}}\left|S_{p^n,p^n}\left(f_A;x^1,x^2\right)\right|=\left|f_A\left(x^1,x^2\right)\right|,$$

from (1) we get

$$\left\|f_A\right\|_{H_\alpha}=\left\|f_A^*\right\|_\alpha=\left\|D_{p^{2A}}\right\|_\alpha^2=p^{4A(1-1/\alpha)}. \tag{8}$$

Since

$$D^\chi_{k+p^{2A}}(x)-D^\chi_{p^{2A}}(x)=r_{2A}(x)\,D_k\left(\tau_{2A}(x)\right),\qquad k=1,2,\ldots,p^{2A},$$

from (7) we obtain

$$\begin{aligned}
&\sigma_0^{\chi*}f_A\left(x^1,x^2\right)\\
&=\sup_{n\in\mathbf{N}}\left|\sigma_{n,n}f_A\left(x^1,x^2\right)\right|\geq\left|\sigma_{n_A,n_A}f_A\left(x^1,x^2\right)\right|\\
&=\frac{1}{(n_A)^2}\left|\sum_{i=0}^{n_A-1}\sum_{j=0}^{n_A-1}S^\chi_{i,j}f_A\left(x^1,x^2\right)\right|\\
&=\frac{1}{(n_A)^2}\left|\sum_{i=p^{2A}+1}^{n_A-1}\sum_{j=p^{2A}+1}^{n_A-1}\left(D^\chi_i(x^1)-D_{p^{2A}}\left(x^1\right)\right)\left(D^\chi_j(x^2)-D_{p^{2A}}(x^2)\right)\right|\\
&=\frac{1}{(n_A)^2}\left|\sum_{i=1}^{n_{A-1}-1}\sum_{j=1}^{n_{A-1}-1}\left(D^\chi_{i+p^{2A}}(x^1)-D_{p^{2A}}(x^1)\right)\left(D^\chi_{j+p^{2A}}(x^2)-D_{p^{2A}}(x^2)\right)\right|\\
&=\frac{1}{(n_A)^2}\left|r_{2A}\left(x^1\right)r_{2A}\left(x^2\right)\sum_{i=1}^{n_{A-1}-1}\sum_{j=1}^{n_{A-1}-1}D^\psi_i\left(\tau_{2A}\left(x^1\right)\right)D^\psi_j\left(\tau_{2A}\left(x^2\right)\right)\right|\\
&=\frac{n^2_{A-1}}{n^2_A}\left|K^\psi_{n_{A-1}}\left(\tau_{2A}\left(x^1\right)\right)\right|\left|K^\psi_{n_{A-1}}\left(\tau_{2A}\left(x^2\right)\right)\right|.
\end{aligned}\tag{9}$$

Denote

$$J_{2A}^{m,s}(x):=I_{2A}\left(x_0,x_1,\ldots,x_{2A-2s-2},x_{2A-2s-1}=1,0,\ldots,x_{2A-2m-1}=1,0,\ldots,0\right)$$

and let

$$\left(x^1,x^2\right)\in J_{2A}^{k_l^1,k_l^1+1}\left(x^1\right)\times J_{2A}^{k_l^2,k_l^2+1}\left(x^2\right),$$

where

$$\begin{aligned}
k_l^1&:=\left[\frac{A}{2}\right]+\left[\frac{1}{8}\log_p A\right]-l,\\
k_l^2&:=\left[\frac{A}{2}\right]+\left[\frac{1}{8}\log_p A\right]+l\qquad l=0,1,\ldots,\left[\frac{1}{8}\log_p A\right].
\end{aligned}$$

Then from Lemma 4 and (9) we obtain

$$\sigma_0^* f_A\left(x^1, x^2\right) \geq c\frac{p^{4k_l^1+4k_l^2}}{p^{4A}} \geq \frac{p^{2A+\log_p\sqrt{A}-4l}p^{2A+\log_p\sqrt{A}+4l}}{p^{4A}} \geq cA.$$

On the other hand,

$$\mu\left\{\left(x^1, x^2\right) \in G_p \times G_p : \left|\sigma_0^{\chi *} f_A\left(x^1, x^2\right)\right| \geq cA\right\}$$
$$\geq c \sum_{l=1}^{[\frac{1}{8}\log_q\sqrt{A}]} \sum_{x_0^1=0}^{p-1} \cdots \sum_{x^1_{2A-2k_l^1-2}=0}^{p-1} \sum_{x_0^2=0}^{p-1} \cdots \sum_{x^2_{2A-2k_l^1-2}=0}^{p-1} \mu\left(J_{2A}^{k_l^1, k_l^1+1}\left(x^1\right) \times J_{2A}^{k_l^2, k_l^2+1}\left(x^2\right)\right)$$
$$\geq c \sum_{l=1}^{[\frac{1}{8}\log_q\sqrt{A}]} \frac{p^{2A-2k_l^1}p^{2A-2k_l^2}}{p^{4A}}$$
$$= c \sum_{l=1}^{[\frac{1}{8}\log_q\sqrt{A}]} \frac{1}{p^{2k_l^1}p^{2k_l^2}}$$
$$= c \sum_{l=1}^{[\frac{1}{8}\log_q\sqrt{A}]} \frac{1}{p^{A+\log_p\sqrt[4]{A}-2l}p^{A+\log_p\sqrt[4]{A}+2l}}$$
$$\geq c\frac{\log_p A}{p^{2A+\log_p\sqrt{A}}}$$
$$= c\frac{\log_p A}{\sqrt{A}p^{2A}}.$$

Then from (8) we obtain

$$\frac{cA\left(\mu\left\{\left(x^1, x^2\right) \in G_p \times G_p : \left|\sigma_0^{\chi *} f_A\left(x^1, x^2\right)\right| \geq cA\right\}\right)^2}{\|f_A\|_{H_{1/2}}}$$
$$\geq \frac{cA\log_p^2 A}{p^{-4A}p^{4A}A} \geq c\log_p^2 A \to \infty \qquad \text{as} \qquad A \to \infty.$$

Theorem 2 is proved. □

We remark that in the case $p = 2$ Theorem 2 is due to Goginava and Nagy [**8**].

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