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COMPARISON THEOREMS FOR HALF-LINEAR DIFFERENTIAL EQUATIONS OF THE FOURTH ORDER

JAROŠ J.

Abstract. An identity of the Picone type for fourth-order half-linear ordinary differential operators of the form

$$l_\alpha[x] \equiv (p\varphi(x''))'' - (r\varphi(x'))' + q\varphi(x)$$

and

$$L_\alpha[y] \equiv (P\varphi(y''))'' - (R\varphi(y'))' + Q\varphi(y).$$

where $\varphi(u) := |u|^{\alpha-1}u, \alpha > 0, u \in R$, and p, q, r, P, Q and R are continuous functions on a given interval I is derived and then Sturmian comparison theory for the corresponding fourth-order equations $l_\alpha[x] = 0$ and $L_\alpha[y] = 0$ based on this identity is developed.

1. Introduction

The classical Picone identity (see [**10**]) associated with a pair of Sturm-Liouville differential equations of the form

$$(p(t)u')' + q(t)u = 0 \tag{1}$$

and

$$(P(t)v')' + Q(t)v = 0 \tag{2}$$

where p, q, P and Q are continuous functions on a given interval I with $p(t) > 0$ and $P(t) > 0$ on I, says that if u and v satisfy (1) and (2), respectively, and $v(t) \neq 0$ on I, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{u}{v}(pu'v - Pv'u)\right] = (Q-q)u^2 + (p-P)u'^2 + P\left(u' - u\frac{v'}{v}\right)^2. \tag{3}$$

The Sturm-Picone comparison theorem readily follows from (3). Indeed, if we assume that Eq. (1) has a nontrivial solution u with consecutive zeros a and b, $a < b$, and

$$p(t) \geq P(t), \qquad Q(t) \geq q(t) \tag{4}$$

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on $[a,b]$, then integrating (3) on $[a,b]$ we get that Eq. (2) cannot possess a solution v which is nonzero in (a,b), except in the special case where $p(t)\equiv P(t)$ and $q(t)\equiv Q(t)$ and v is a constant multiple of u on $[a,b]$.

In [**3**] (see also [**4**]), the identity (3) was generalized to the case of the half-linear differential equations

$$(p(t)\varphi(u'))' + q(t)\varphi(u) = 0 \tag{5}$$

and

$$(P(t)\varphi(v'))' + Q(t)\varphi(v) = 0, \tag{6}$$

where $\varphi(u) := |u|^{\alpha-1}, u\in R, \alpha>0$, and p, q, P and Q are continuous functions on an interval I with $p(t)>0$ and $P(t)>0$ on I.

If u and v satisfy (5) and (6), respectively, with $v(t)\neq 0$ on I, then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}&\left\{\frac{u}{\varphi(v)}\left[\varphi(v)p\varphi(u') - \varphi(u)P\varphi(v')\right]\right\}\\ &= (Q-q)|u|^{\alpha+1} + (p-P)|u'|^{\alpha+1}\\ &\quad + P\left[|u'|^{\alpha+1} + \alpha\left|\frac{uv'}{v}\right|^{\alpha+1} - (\alpha+1)u'\varphi\left(\frac{uv'}{v}\right)\right].\end{aligned} \tag{7}$$

The half-linear generalization of Sturm-Picone comparison principle obtained previously in [**1**], [**9**] and [**11**] by different methods, now easily follows from (7) if we assume that the inequalities (4) hold on $[a,b]$, where a and b are consecutive zeros of u, and use the Young inequality to show that the last expression in (7) is nonnegative with the equality holding if and only if u and v are proportional on $[a,b]$. Actually, the following more general result is true.

Theorem A (Leighton-type comparison)**.** *If there exists a nontrivial solution u of* (5) *such that $u(a)=u(b)=0$ and*

$$\int_a^b \left[(p(t)-P(t))|u'(t)|^{\alpha+1} + (Q(t)-q(t))|u(t)|^{\alpha+1}\right]\mathrm{d}t \ge 0, \tag{8}$$

then every solution v of (7) *has at least one zero in (a,b) except in the special case when $p(t)\equiv P(t), q(t)\equiv Q(t)$ and $u(t)=cv(t)$ on $[a,b]$ for some constant c.*

The situation in the case of fourth-order linear differential equations of the form

$$(p(t)u'')'' + q(t)u = 0 \tag{9}$$

and

$$(P(t)v'')'' + Q(t)v = 0 \tag{10}$$

is more complicated. If u is a nontrivial solution of [**9**] on an interval $[a,b]$ satisfying

$$u(a)=u'(a)=u(b)=u'(b)=0 \tag{11}$$

and if

$$p(t)\ge P(t), \quad q(t)\ge Q(t) \quad \text{for} \quad t\in[a,b] \tag{12}$$

then, in general, it is not true that an arbitrary solution v of [**10**] (or any of its derivatives) has a zero in $[a,b]$. This is the consequence of the result of Leighton and Nehari (see [**8**]) which asserts that if $Q(t) < 0$ for $t \ge a$ and v is a solution of [**10**] generated by the initial conditions

$$v(a) \ge 0, \quad v'(a) \ge 0, \quad v''(a) \ge 0 \quad \text{and} \quad (Pv'')'(a) \ge 0$$

(but not all zero), then

$$v(t) > 0, \quad v'(t) > 0, \quad v''(t) > 0 \quad \text{and} \quad (Pv'')'(t) > 0$$

for all $t > a$. Thus, neither the solution v itself nor any of its derivatives v', v'' and $(Pv'')'$ can vanish at the point greater than a.

However, a sort of the Sturm-Picone comparison result can be obtained for [**9**] and [**10**] if we consider only solutions v of [**10**] for which v' and $(Pv'')'$ have opposite signs.

Theorem B. *Let u be a nontrivial solution of* [**9**] *satisfying* (11). *If v is a solution of* [**10**] *for which v' and $(Pv'')'$ have opposite signs and if the inequalities* (12) *hold on $[a,b]$, then v, v' or $(Pv'')'$ has a zero in $[a,b]$.*

(See [**5**].) The key tool in proving the above theorem was the Picone-type identity which asserts that if u and v are solutions of [**9**] and [**10**], respectively, and none of v and v' vanish in I, then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}&\left\{\frac{u'}{v'}\big[v'pu''-u'Pv''\big] - \frac{u}{v}\big[v(pu'')' - u(Pv'')'\big]\right\} \\ &= (p-P)u''^2 + (q-Q)u^2 - v'(Pv'')'\left(\frac{u'}{v'} - \frac{u}{v}\right)^2 \\ &\quad + P\left(u'' - \frac{u'v''}{v'}\right)^2.\end{aligned} \tag{13}$$

The following comparison theorem of the Leighton type concerning the more general fourth-order linear differential equations

$$(p(t)u'')'' - (r(t)u')' + q(t)u = 0 \tag{14}$$

and

$$(P(t)v'')'' - (R(t)v')' + Q(t)v = 0 \tag{15}$$

can be obtained as a special case of the results in [**7**].

Theorem C. *Suppose that there exists a nontrivial solution of* (14) *which satisfies* (12) *and*

$$\int_a^b \big[(p-P)u^2 + (r-R)u'^2 + (q-Q)u''^2\big]\mathrm{d}t \ge 0. \tag{16}$$

If v satisfies (15) *with $P(t) \ge 0$ in (a,b),*

$$v'[R(t)v' - (P(t)v'')'] \ge 0 \quad \text{and} \quad R(t)v' - (P(t)v'')' \ne 0 \quad \text{in} \quad (a,b) \tag{17}$$

then at least one of v and v' has a zero in $[a,b]$.

The purpose of this paper is to generalize the identity (13) to the case of half-linear differential equations of the fourth order and use it in proving comparison theorems of the Sturm-Picone and Leighton type.

For related results concerning the linear case see also [**6**] and [**12**].

2. Main results

Consider the operators

$$l_\alpha[x] \equiv (p(t)\varphi(x''))'' - (r(t)\varphi(x'))' + q(t)\varphi(x) \tag{18}$$

and

$$L_\alpha[y] \equiv (P(t)\varphi(y''))'' - (R(t)\varphi(y'))' + Q(t)\varphi(y) \tag{19}$$

where p, r, q, P, R and Q are continuous functions defined on $[a,b] \subset I$ and $\varphi[u] := |u|^\alpha \mathrm{sgn}\, u, \alpha > 0$, as before.

Let $D_{l_\alpha}(I)$ (resp. $D_{L_\alpha}(I)$) denote the set of all continuous functions x (resp. y) defined on I such that x (resp. y) is two times continuously differentiable on I and also $(r\varphi(x'))'$ and $(p\varphi(x''))''$ (resp. $(R\varphi(y'))'$ and $(P\varphi(y''))''$) exist and are continuous on I.

Denote by Φ_α the form defined for $u, v \in \mathbb{R}$ and $\alpha > 0$ by

$$\Phi_\alpha(u,v) := u\varphi(u) + \alpha v\varphi(v) - (\alpha+1)u\varphi(v). \tag{20}$$

It follows from the Young inequality that $\Phi_\alpha(u,v) \geq 0$ for all $u, v \in \mathbb{R}$ and the equality holds if and only if $u = v$.

The following lemma can be verified by a direct computation.

Lemma. *If $x \in D_{l_\alpha}(I)$ and $y \in D_{L_\alpha}(I)$ on an interval I and if none of y and y' vanish in I, then*

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big\{ \frac{x'}{\varphi(y')}\big[\varphi(y')p\varphi(x'') - \varphi(x')P\varphi(y'')\big] \\
&\qquad - \frac{x}{\varphi(y)}\big[\varphi(y)(p\varphi(x''))' - \varphi(x)(P\varphi(y''))'\big] \\
&\qquad - \frac{x}{\varphi(y)}\big[\varphi(y)r\varphi(x') - \varphi(x)R\varphi(y')\big]\Big\} \\
&= \frac{x}{\varphi(y)}\big\{\varphi(x)L_\alpha[y] - \varphi(y)l_\alpha[x]\big\} \\
&\qquad + (q-Q)|x|^{\alpha+1} + (r-R)|x'|^{\alpha+1} + (p-P)|x''|^{\alpha+1} \\
&\qquad + P\Phi_\alpha\left(x'', \frac{x'y''}{y'}\right) + y'\big[R\varphi(y') - (P\varphi(y''))'\big]\Phi_\alpha\left(\frac{x'}{y'}, \frac{x}{y}\right).
\end{aligned} \tag{21}$$

Theorem 1 (Leighton-type comparison)**.** *If there exists a nontrivial $u \in D_{l_\alpha}([a,b])$ such that*

$$\int_a^b u l_\alpha[u]\mathrm{d}t \leq 0, \tag{22}$$

$$u(a) = u'(a) = u(b) = u'(b) = 0 \tag{23}$$

and

$$\mathrm{V}_\alpha[u] \equiv \int_a^b \left[(p-P)|u''|^{\alpha+1} + (r-R)|u'|^{\alpha+1} + (q-Q)|u|^{\alpha+1}\right]\mathrm{d}t \geq 0, \tag{24}$$

then for any $v \in D_{L_\alpha}([a,b])$ *satisfying*

$$vL_\alpha[v] \geq 0 \quad in \quad (a,b), \quad P(t) \geq 0, \tag{25}$$

$$\begin{aligned} &v'\left[R(t)\varphi(v') - (P(t)\varphi(v''))'\right] \geq 0, \\ &R(t)\varphi(v') - (P(t)\varphi(v''))' \neq 0 \quad in \quad (a,b), \end{aligned} \tag{26}$$

v *or* v' *has a zero in* $[a,b]$.

Proof. Suppose to the contrary that there exists a function $v \in D_{L_\alpha}([a,b])$ satisfying the inequality (25) in (a,b) such that $v(t) \neq 0$ and $v'(t) \neq$ in $[a,b]$. Integrating the identity (21) where $x = u$ and $y = v$ on $[a,b]$, we obtain

$$0 \geq V_\alpha[u] + \int_a^b v'\left[R(t)\varphi(v') - (P(t)\varphi(v''))'\right]\Phi_\alpha\left(\frac{u'}{v'}, \frac{u}{v}\right)\mathrm{d}t \geq 0. \tag{27}$$

Thus, we get

$$\int_a^b v'\left[R(t)\varphi(v') - (P(t)\varphi(v''))'\right]\Phi_\alpha\left(\frac{u'}{v'}, \frac{u}{v}\right)\mathrm{d}t = 0.$$

The assumption (26) implies that $\Phi_\alpha(u'/v', u/v) \equiv 0$ in (a,b) which means that $u = cv$ on $[a,b]$ for some nonzero constant c. Since $u(a) = u(b) = 0$ and $v(t) \neq 0$ on $[a,b]$, this leads to a contradiction. The proof is complete. □

Corollary (Sturm-Picone comparison)**.** *If*

$$p(t) \geq P(t) > 0, \quad r(t) \geq R(t) \quad and \quad q(t) \geq Q(t) \tag{28}$$

on $[a,b]$ *and there exists a nontrivial solution* u *of*

$$(p(t)\varphi(u''))'' - (r(t)\varphi(u'))' + q(t)\varphi(u) = 0, \quad a < t < b, \tag{29}$$

satisfying (23)*, then for any solution* v *of the majorant equation*

$$(P(t)\varphi(v''))'' - (R(t)\varphi(v'))' + Q(t)\varphi(v) = 0, \quad a < t < b, \tag{30}$$

satisfying (26) *in* (a,b)*,* v *or* v' *must have a zero in* $[a,b]$.

3. Disconjugacy criterion

Consider Eq. (29) in an interval I. Two points $a, b \in I$ are called *conjugate with respect to* (29) if there exists a nontrivial solution $u \in D_{l_\alpha}([a,b])$ satisfying (23). Eq. (29) is called *disconjugate on* I if no two points of I are conjugate with respect to (29).

The following disconjugacy criterion for Eq. (29) is an immediate consequence of Theorem 1.

Theorem 2. *Eq.* (29) *is disconjugate on I if there exist a half-linear differential operator L_α defined by* (19) *and a function $v \in D_{L_\alpha}(I)$ satisfying*

$$p(t) \geq P(t) \geq 0, \quad r(t) \geq R(t) \quad and \quad q(t) \geq Q(t) \quad in \quad I, \tag{31}$$

$$vL_\alpha[v] \geq 0 \quad in \quad I, \quad v(t) \neq 0 \quad in \quad I, \tag{32}$$

and

$$v'\big[R(t)\varphi(v') - (P(t)\varphi(v''))'\big] > 0 \quad in \quad I. \tag{33}$$

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Jaroš J., Faculty of Mathematics, Physics and Informatics, Comenius University Mlynská dolina, 842 48 Bratislava, Slovak Republic, *e-mail*: `Jaroslav.Jaros@fmph.uniba.sk`