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BLOW UP VERSUS GLOBAL BOUNDEDNESS OF SOLUTIONS OF REACTION DIFFUSION EQUATIONS WITH NONLINEAR BOUNDARY CONDITIONS*

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Abstract. In this paper we analyze the behavior of solutions of reaction-diffusion equations with nonlinear boundary conditions of the type (1.1). We show that if $f(x,u)=-\beta_0 u^p$ and $g(x,u)=u^q$ in a neighborhood of a point $x_0\in\Gamma_N$, then

i) for the case $q>1$, if $p+1<2q$ or if $p+1=2q$ and $\beta_0<q$, then blow up in finite time at x_0 occurs.

ii) for the case $p>1$ if $p+1>2q$ or if $p+1=2q$ and $\beta_0>q$ then any solution is globally bounded around the point x_0.

Key words. reaction-diffusion, nonlinear boundary conditions, blow-up, boundedness

1. Introduction. We consider the following reaction diffusion equation with nonlinear boundary conditions in a smooth C^2 domain $\Omega\subset\mathbb{R}^N$,

$$\begin{cases} u_t-\Delta u=f(x,u) & \text{in } \Omega\\ u=0 & \text{on } \Gamma_D\\ \dfrac{\partial u}{\partial \vec{n}}=g(x,u) & \text{on } \Gamma_N\\ u(0,x)=u_0(x)\geq 0 & \text{in } \Omega \end{cases} \tag{1.1}$$

where $\Gamma=\partial\Omega=\Gamma_D\cup\Gamma_N$ is a regular disjoint partition of the boundary of Ω and f and g are suitably smooth functions of (x,u). The subindices D and N on Γ indicate the part of the boundary with Dirichlet and Neumann type condition, respectively. We are interested in nonnegative solutions of (1.1) so we will assume

$$f(x,0)\geq 0, \text{ for all } x\in\Omega, \quad g(x,0)\geq 0 \text{ for all } x\in\Gamma_N$$

We want to obtain local conditions on the nonlinearities f and g, which will be imposed in a neighborhood of a point $x_0\in\Gamma_N$, that guarantee that either i) there exists initial conditions with support in a neighborhood of x_0 such that the "proper solution" starting at this initial condition blows up at x_0 or that ii) for all initial data $u_0\in L^\infty(\Omega)$ the "proper solution" starting at u_0 is bounded in a neighborhood of x_0 for all times $t\geq 0$. We refer to [4, 8, 9] for the concept of proper solution.

Notice that if $f(x,u)$ behaves like u^p locally around certain point $z\in\Omega$ and $p>1$, then, by comparison with the Dirichlet problem in a neighborhood of z and using that the superlinear nonlinearity u^p is explosive we get that, regardless of the behavior of g, we have initial conditions that blow-up in finite time. On the other hand, if $f(x,u)$ behaves like $-u^p$ and $g(x,u)$ behaves like $-u^q$ throughout the whole domain, then both

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nonlinearities are dissipative and we have global exitence and boundedness of solutions. The most interesting case is when $f(x,u)$ is a dissipative nonlinearity of the form $-\beta_0 u^p$ and $g(x,u)$ is an explosive nonlinearity of the form u^q. This two mechanisms are in competition and it seems clear that the relative size of p, q and β_0 will determine the relative strength of both mechanisms.

Actually, in the pioneer work of [6] they treated the one dimensional case, say $\Omega = (0,1)$, with $f(x,u) = -\beta_0 u^p$, $g(x,u) = u^q$ and $\Gamma_D = \emptyset$ and they already obtained that the critical relations are $p+1$ vs. $2q$ and if $p+1 = 2q$ then β_0 vs. q, in the sense that if $p+1 < 2q$ or $p+1 = 2q$ and $\beta_0 < q$ then blow-up is produced and if $p+1 > 2q$ or $p+1 = 2q$ and $\beta_0 > q$ then the solutions are globally bounded. They also treated the very delicate case where $p+1 = 2q$ and $\beta_0 = q$. They actually showed that the solutions were defined for all timet $t > 0$ but the phenomenon of infinite time blow-up was present.

Later on, in [13, 14], they treated the case of arbitrary dimension and obtained that if $\Gamma_D = \emptyset$ and the nonlinearities f and g that behave for u large as $f \sim -\beta_0 u^p$ and $g \sim u^q$, then blow-up is produced if $p+1 < 2q$ or if $p+1 = 2q$ and $\beta_0 < q$. Also, they showed that if $p+1 > 2q$ or if $p+1 = 2q$ and β_0 is large enough, then the solutions are globally bounded. Also, in [1] they studied the porous medium equation in any dimension and as a particular case they considered the equation (1.1) with $\Gamma_D = \emptyset$, $f(x,u) = -\beta_0 u^p$ and $g(x,u) = u^q$. They showed that if $p+1 < 2q$ or $p+1 = 2q$ and $\beta_0 < q$ then blow-up is produced and if $p+1 > 2q$ of $p+1 = 2q$ and $\beta_0 > q$ then the solutions are globally bounded.

With all these works it is clear that the critical relations that mark the line between blow-up and boundedness are given by $p+1$ vs. $2q$ and in case $p+1 = 2q$, β_0 vs. q. These works have a common characteristic and it is that the nonlinear boundary condition is imposed in the whole domain, $\Gamma_D = \emptyset$ and the construction of sub or super solutions is done for the whole domain. Hence, the balances between f and g need to hold throughout the domain to obtain the result and both, the blow-up and the boundedness result are global in space. In particular, none of them can treat the case as in the equation (4.1) where $p+1 = 2q$ but in some part of the boundary the relation is $\beta_0 > q$ and in other part the relation is $\beta_0 < q$ or even when $\Gamma_D \neq \emptyset$.

In this paper we will prove that both mechanisms (dissipativeness vs. blow-up) compete at a local level. Actually, we will show that if $f(x,u) = -\beta_0 u^p$ and $g(x,u) = u^q$ in a neighborhood of a point $x_0 \in \Gamma_N$, then

i) for the case $q > 1$, if $p+1 < 2q$ or if $p+1 = 2q$ and $\beta_0 < q$, then blow up in finite time at x_0 occurs, see Section 2.

ii) for the case $p > 1$ if $p+1 > 2q$ or if $p+1 = 2q$ and $\beta_0 > q$ then any solution is globally bounded around the point x_0, see Section 3.

In Section 2 we analyze the first case and we refer to [3] for details. In Section 3 we consider the case ii) and we announce the results of [2]. In Section 4 we consider several important remarks and comments.

2. Localization of blow-up. In terms of characterizing the sizes of p, q and β_0 that will produce blow-up we have:

PROPOSITION 2.1. *Let $x_0 \in \Gamma_N$, $p \geq 1$, $q > 1$ and let $R_0 > 0$, $M_0 > 0$ such that*

$$\begin{aligned} &f(x,u) \geq -\beta_0 u^p, && x \in B(x_0,R_0) \cap \Omega, && u \geq M_0,\\ &g(x,u) \geq u^q, && x \in B(x_0,R_0) \cap \partial\Omega, && u \geq M_0. \end{aligned} \tag{2.1}$$

If one of the two following conditions holds

i) $p+1 < 2q$ *or*

ii) $p+1=2q$ *and* $\beta_0<q$,
then, there exists an initial condition $0\leq u_0\in L^\infty(\Omega)$ *with support in a neighborhood of* x_0 *such that the proper minimal solution of* (1.1) *starting at* u_0 *blows up in finite time at the point* x_0.

Proof. Let us provide a proof of ii). Actually this case is more critical than i).

In order to simplify, consider that $x_0=0\in\Gamma_N$ and that the outward normal vector at $x_0=0$ is given by $\vec{n}(0)=(0,\ldots,0,-1)$. Let R, $\delta>0$ be small numbers and $y_R=x_0+R\vec{n}(x_0)=(0,\ldots,0,-R)$ with the property that $B(y_R,R)\cap\bar{\Omega}=\emptyset$ and that $B(y_R,R+\delta)\subset B(0,R_0/2)$. The fact that the domain has a C^2 boundary, guarantees that this construction can be done. See FIG. 2.1.

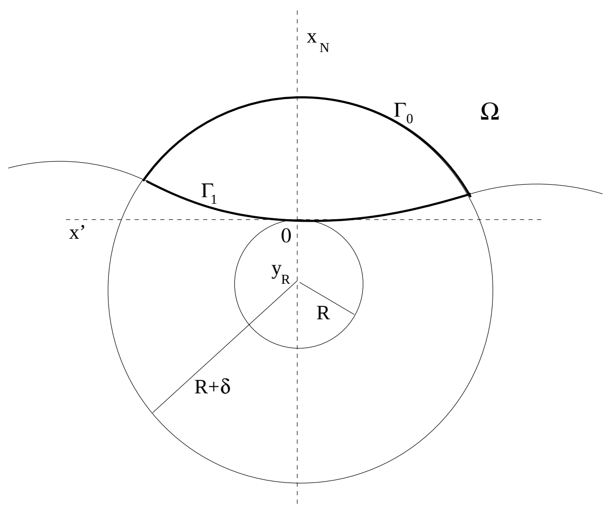

FIG. 2.1. *The domain* Ω *near* x_0.

We will construct a function $z(t,x)$ which will be radially symmetric around y_R, increasing in time and that it will be a subsolution of (1.1) locally around the point x_0. For this, define for $a\geq 1$, the function $\psi_a(t)$ as the solution of the problem

$$\begin{cases} \psi'=\psi^q, \\ \psi(0)=a. \end{cases} \tag{2.2}$$

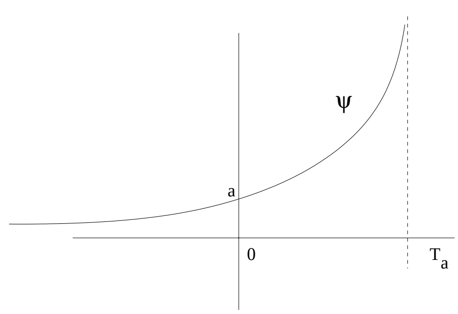

FIG. 2.2. *The solution of Equation* (2.2).

Solving this equation, we get that $\psi_a(t) = \dfrac{E}{(T_a - t)^{\frac{1}{q-1}}}$ for $-\infty < t < T_a$ with $E = \dfrac{1}{(q-1)^{\frac{1}{q-1}}}$ and $T_a = \dfrac{1}{(q-1)a^{q-1}}$. Observe that, since $a \geq 1$ and $q > 1$, $T_a \leq 1/(q-1)$ and that $T_a \to 0$ as $a \to +\infty$. Notice also that $\psi_a(t) \leq E/(-t)^{1/(q-1)}$ for any $t < 0$ and any $a \geq 1$.

We define $z_a(t,x) = \psi_a(t + R - |x - y_R|)$ for $x \in \mathbb{R}^N \setminus B(y_R, R)$, $0 \leq t < T_a$, see Fig. 2.3.

Direct computations show that $\frac{\partial z_a}{\partial n} \leq z_a^q$ for $x \in \Gamma_1$ and $0 < t < T_a$ and $\frac{\partial z_a}{\partial t} - \Delta z_a \leq (1 + \frac{N-1}{R} - q z_a^{q-1}) z_a^q$ for $x \in \Omega \cap B(y_R, R+\delta)$ and $t \in (0, T_a)$. Notice that z_a is increasing in time and that $z_a(t,x) \geq z_a(0,x) = \psi_a(R - |x - y_R|) = \psi_a(-\delta) = \frac{E}{(T_a+\delta)^{\frac{1}{q-1}}} \to +\infty$ as $a \to +\infty$ and $\delta \to 0$, for $x \in \Omega \cap B(y_R, R+\delta)$. Hence, choosing a_0 large enough and δ_0 small enough, we can guarantee, since $\beta_0 < q$, that for $a \geq a_0$ and $0 < \delta < \delta_0$, that $1 + \frac{N-1}{R} - q z_a^{q-1} \leq -\beta_0 z_a^{2q-1} = -\beta_0 z_a^p$ as long as $x \in \Omega \cap B(y_R, R+\delta)$ and $0 \leq t < T_a$.

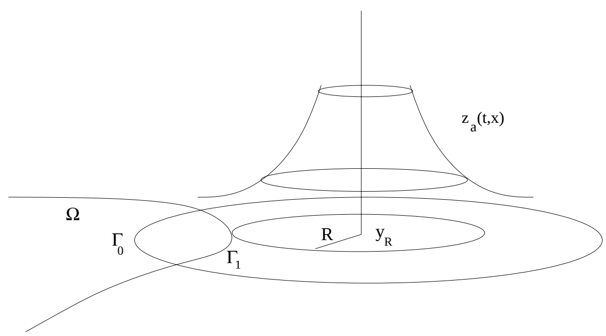

Fig. 2.3. *The function z_a.*

In particular, we get

$$\begin{cases} \dfrac{\partial z_a}{\partial t} - \Delta z_a \leq -\beta_0 z_a^p, & x \in \Omega \cap B(y_R, R+\delta),\ t \in (0, T_a), \\ \dfrac{\partial z_a}{\partial n} \leq z_a^q, & x \in \Gamma_1 = \partial\Omega \cap B(y_R, R+\delta),\ t \in (0, T_a). \end{cases} \tag{2.3}$$

Consider now a smooth initial condition $v_0 \in C^\infty(\Omega)$ such that $v_0 \equiv 0$ in $\Omega \setminus B(0, R_0)$ and $u_0 \geq \frac{2E}{\delta^{\frac{1}{q-1}}}$ in $\Omega \cap B(y_R, R+\delta)$. The solution of (1.1) starting at u_0 will satisfy that for a small time T we will have that $u(x,t,v_0) \geq \frac{E}{\delta^{\frac{1}{q-1}}}$ for $x \in \Gamma_0 \equiv \Omega \cap \partial B(y_R, R+\delta)$, $0 \leq t < T$. By monotonicity, for any $u_0 \geq v_0$ in Ω, we will also have that the proper solution starting at u_0 will satisfy, $u(x,t,u_0) \geq \frac{E}{\delta^{\frac{1}{q-1}}}$ for $x \in \Gamma_0 \equiv \Omega \cap \partial B(y_R, R+\delta)$, $0 \leq t < T$.

In particular, let us choose $a > a_0$ with the property that $0 < T_a < T$ and let us choose u_0 such that $u_0(x) \geq v_0(x)$ and $u_0(x) \geq \psi_a(-R) \geq z_a(0,x)$ for $x \in \Omega \cap B(y_R, R+\delta)$. Hence, for $0 \leq t < T_a$ we have $z_a(t,x) \leq \frac{E}{\delta^{\frac{1}{q-1}}} \leq u(x,t,u_0)$ for $x \in \Gamma_0$ and $z_a(0,x) \leq u_0(x)$

for $x \in \Omega \cap B(y_R, R+\delta)$. That is, z_a satisfies,

$$\begin{cases} \dfrac{\partial z_a}{\partial t} - \Delta z_a \le -\beta_0 z_a^p, & x \in \Omega \cap B(y_R, R+\delta),\ t \in (0, T_a), \\ \dfrac{\partial z_a}{\partial n} \le z_a^q, & x \in \Gamma_1 = \partial\Omega \cap B(y_R, R+\delta),\ t \in (0, T_a), \\ z_a(t,x) \le u(x,t,u_0), & x \in \Gamma_0,\ t \in (0, T_a), \\ z_a(0,x) \le u_0, & x \in \Omega \cap B(y_R, R+\delta), \end{cases} \tag{2.4}$$

which implies that $z_a(t,x) \le u(t,x,u_0)$ for all $x \in \Omega \cap B(y_R, R+\delta)$ and $t \in (0, T_a)$. The fact that $z_a(T_a, x)$ blows up at $x = 0$ proves the result. □

REMARKS. i) The time T_a does not need to be the classical blow-up time, that is, the time T_∞ for which the solution is classical for $0 < t < T_\infty$ and such that $\|u(t,\cdot,u_0)\|_{L^\infty(\Omega)} \to +\infty$ as $t \nearrow T_\infty$. We just can assure that $T_\infty \le T_a$.

ii) Observe that if for $\alpha \in (0, T - T_a)$ we define the function $w_\alpha(t,x) = z_a(t-\alpha, x)$ defined for $x \in \Omega \cap B(y_R, R+\delta)$ and $t \in (\alpha, T_a + \alpha)$, then, we easily obtain that w_α satisfies

$$\begin{cases} \dfrac{\partial w_\alpha}{\partial t} - \Delta w_\alpha \le -\beta_0 w_\alpha^p, & x \in \Omega \cap B(y_R, R+\delta),\ t \in (\alpha, T_a + \alpha), \\ \dfrac{\partial w_\alpha}{\partial n} \le w_\alpha^q, & x \in \Gamma_1 = \partial\Omega \cap B(y_R, R+\delta),\ t \in (\alpha, \alpha + T_a), \\ w_\alpha(t,x) \le u(x,t,u_0), & x \in \Gamma_0,\ t \in (\alpha, \alpha + T_a), \\ w_\alpha(\alpha,x) \le u_0, & x \in \Omega \cap B(y_R, R+\delta). \end{cases} \tag{2.5}$$

The third inequality is obtained since for $x \in \Gamma_0$ we have $w_\alpha(t,x) \le \frac{E}{\delta^{\frac{1}{q-1}}} \le u(x,t,u_0)$

From (2.5) we obtain that $w_\alpha(t,x) = z_a(t-\alpha, x) \le u(t,x,u_0)$ for all $\alpha \in (0, T - T_a)$. This implies that for $t \in (T_a, T)$ we have $z_a(T_a, x) \le u(t,x,u_0)$ which means that the solution u is "pinned" to the value ∞ during the time $T_a \le t \le T$.

iii) With some extra effort, see [3] for details, it is possible to show that the construction of PROPOSITION 2.1 can be performed in a neighborhood of $x_0 \in \partial\Omega$. As a matter of fact the parameters, R, δ, a_0, δ_0, and the initial condition u_0 can be chosen the same for a small neighborhood $\partial\Omega \cap B(x_0, \eta)$ for $\eta > 0$ small. This means that the proper solution $u(t,x,u_0)$ will blow up, not only at x_0 but at $B(x_0, \eta') \cap \partial\Omega$ for some small $\eta' > 0$, and it will remain "pinned" to the value ∞ for a period of time $T_a \le t \le T$.

3. Localization of global boundedness. In this section we present the results of [2] that, roughly speaking, say that if the complementary conditions of PROPOSITION 2.1 hold, also near a point $x_0 \in \partial\Omega$, then the proper solution is bounded globally in time around this point x_0. As a matter of fact, we have

PROPOSITION 3.1. *Let $x_0 \in \Gamma_N$, $p > 1$, $q \ge 1$ and let $R_0 > 0$, $M_0 > 0$ such that*

$$\begin{aligned} f(x,u) &\le -\beta_0 u^p, && x \in B(x_0, R_0) \cap \Omega, && u \ge M_0, \\ g(x,u) &\le u^q, && x \in B(x_0, R_0) \cap \partial\Omega, && u \ge M_0. \end{aligned} \tag{3.1}$$

If one of the two following conditions holds

i) $p + 1 > 2q$ *and* $\beta_0 > 0$ *or*

ii) $p + 1 = 2q$ *and* $\beta_0 > q$*,*

then, for any initial condition $0 \le u_0 \in L^\infty(\Omega)$ *the proper solution of* (1.1) *starting at* u_0 *is bounded in a neighborhood of* x_0 *in* $\bar{\Omega}$*, for all* $t > 0$*. That is, there exist* $\delta, M > 0$ *such that*

$$\sup_{0 \le t < \infty,\, x \in B(x_0,\delta)\cap\bar{\Omega}} u(t,x,u_0) \le M. \tag{3.2}$$

To prove the result, we construct appropriate super solutions locally around the point $x_0 \in \Gamma_N$. As a matter of fact we extensively use the singular solutions of the following elliptic problem

$$\begin{cases} -\Delta z + \beta z^p = 0 \quad \text{in } B(0,R), \\ z(R) = +\infty, \end{cases} \tag{3.3}$$

and the fact that the asymptotics of this radial solution as $r \to R$ is well understood, see [5, 12].

We refer to [2] for details on the proof of this result.

4. Concluding Remarks. We present in this section several important comments and remarks.

i) Both results are local in nature: if the conditions of Proposition 2.1 (resp. Proposition 3.1) hold in a neighborhood of certain point $x_0 \in \partial\Omega$, then, independently of the behavior of the nonlinearities outside this neighborhood, we will have that blow-up (resp. global boundedness of solutions) occurs in the neighborhood of x_0. In particular, from the control theory point of view it turns out that it is impossible to prevent blow-up (resp. to produce blow-up) in a neighborhood of a point of the boundary of the domain by modifying the equation somehow away from this point.

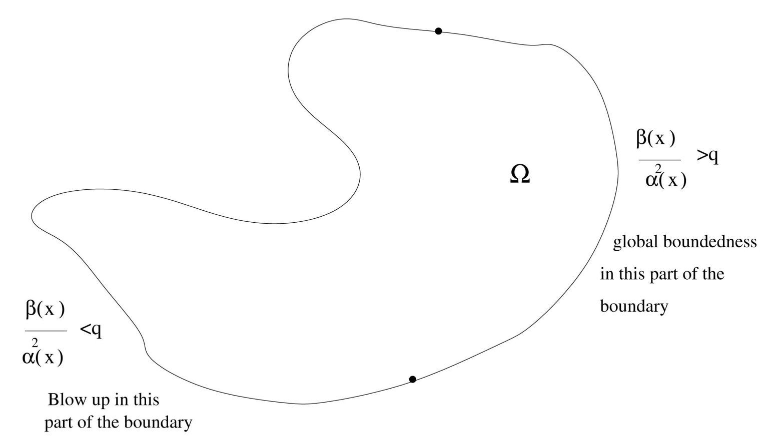

Fig. 4.1. *The domain of the example.*

ii) With an appropriate rescaling it is not difficult to see that if the local conditions of the nonlinearities f and g in Proposition 2.1 and Proposition 3.1 are of the type $f(x,u) \sim -\beta_0 u^p$, $g(x,u) \sim \alpha_0 u^q$, for $x \in B(x_0,R_0)\cap\partial\Omega, u \ge M_0$, then, the condition $\beta_0 < q$ (resp. $\beta_0 > q$) should be changed to $\beta_0 > q\alpha_0^2$, (resp. $\beta_0 < q\alpha_0^2$).

iii) It is important to mention that the balances obtained for p, q and β_0 are independent of the dimension of the space and even of the geometry of the domain.

iv) As an example, consider for instance the problem

$$\begin{aligned} &u_t - \Delta u = -\beta(x)u^p \quad \text{in } \Omega, \\ &\frac{\partial u}{\partial \vec{n}} = \alpha(x)u^q \qquad \text{on } \partial\Omega, \\ &u(0,x) = u_0(x) \geq 0 \quad \text{in } \Omega, \end{aligned} \tag{4.1}$$

with β and α continuous functions, $\beta(x) > 0$ in $\bar{\Omega}$ and $\alpha(x) > 0$ in $\partial\Omega$, see Fig. 4.1.

Then if $p+1 = 2q > 2$ and $x_0 \in \partial\Omega$ with $\frac{\beta(x_0)}{\alpha(x_0)^2} < q$ then from [3], there are initial conditions where blow up is produced near x_0, while if $\frac{\beta(x_0)}{\alpha(x_0)^2} > q$, then from Theorem 2.1 above, for any initial condition $u_0 \in L^\infty(\Omega)$ the proper minimal solution is bounded near x_0. Hence, we have the situation as in Fig. 4.1

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