Proceedings of Equadiff-11
2005, pp. 1–7

BLOW UP VERSUS GLOBAL BOUNDEDNESS OF SOLUTIONS OF REACTION DIFFUSION EQUATIONS WITH NONLINEAR BOUNDARY CONDITIONS*

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Abstract. In this paper we analyze the behavior of solutions of reaction-diffusion equations with nonlinear boundary conditions of the type (1.1). We show that if $f(x,u) = -\beta_0 u^p$ and $g(x,u) = u^q$ in a neighborhood of a point $x_0 \in \Gamma_N$, then

i) for the case $q > 1$, if $p + 1 < 2q$ or if $p + 1 = 2q$ and $\beta_0 < q$, then blow up in finite time at x_0 occurs.
ii) for the case $p > 1$ if $p + 1 > 2q$ or if $p + 1 = 2q$ and $\beta_0 > q$ then any solution is globally bounded around the point x_0.

Key words. reaction-diffusion, nonlinear boundary conditions, blow-up, boundedness

1. Introduction.

We consider the following reaction diffusion equation with nonlinear boundary conditions in a smooth C^2 domain $\Omega \subset \mathbb{R}^N$,

$$\begin{cases} u_t - \Delta u = f(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_D \\ \dfrac{\partial u}{\partial \vec{n}} = g(x,u) & \text{on } \Gamma_N \\ u(0,x) = u_0(x) \geq 0 & \text{in } \Omega \end{cases} \tag{1.1}$$

*Partially supported by Project BFM2003-03810

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where $\Gamma = \partial\Omega = \Gamma_D \cup \Gamma_N$ is a regular disjoint partition of the boundary of Ω and f and g are suitably smooth functions of (x, u). The subindices D and N on Γ indicate the part of the boundary with Dirichlet and Neumann type condition, respectively. We are interested in nonnegative solutions of (1.1) so we will assume

$$f(x, 0) \geq 0, \text{ for all } x \in \Omega, \quad g(x, 0) \geq 0 \text{ for all } x \in \Gamma_N$$

We want to obtain local conditions on the nonlinearities f and g, which will be imposed in a neighborhood of a point $x_0 \in \Gamma_N$, that guarantee that either i) there exists initial conditions with support in a neighborhood of x_0 such that the "proper solution" starting at this initial condition blows up at x_0 or that ii) for all initial data $u_0 \in L^\infty(\Omega)$ the "proper solution" starting at u_0 is bounded in a neighborhood of x_0 for all times $t \geq 0$. We refer to [4, 8, 9] for the concept of proper solution.

Notice that if $f(x, u)$ behaves like u^p locally around certain point $z \in \Omega$ and $p > 1$, then, by comparison with the Dirichlet problem in a neighborhood of z and using that the superlinear nonlinearity u^p is explosive we get that, regardless of the behavior of g, we have initial conditions that blow-up in finite time. On the other hand, if $f(x, u)$ behaves like $-u^p$ and $g(x, u)$ behaves like $-u^q$ throughout the whole domain, then both nonlinearities are dissipative and we have global exitence and boundedness of solutions. The most interesting case is when $f(x, u)$ is a dissipative nonlinearity of the form $-\beta_0 u^p$ and $g(x, u)$ is an explosive nonlinearity of the form u^q. This two mechanisms are in competition and it seems clear that the relative size of p, q and β_0 will determine the relative strength of both mechanisms.

Actually, in the pioneer work of [6] they treated the one dimensional case, say $\Omega = (0, 1)$, with $f(x, u) = -\beta_0 u^p$, $g(x, u) = u^q$ and $\Gamma_D = \emptyset$ and they already obtained that the critical relations are $p + 1$ vs. $2q$ and if $p + 1 = 2q$ then β_0 vs. q, in the sense that if $p + 1 < 2q$ or $p + 1 = 2q$ and $\beta_0 < q$ then blow-up is produced and if $p + 1 > 2q$ or $p + 1 = 2q$ and $\beta_0 > q$ then the solutions are globally bounded. They also treated the very delicate case where $p + 1 = 2q$ and $\beta_0 = q$. They actually showed that the solutions were defined for all timet $t > 0$ but the phenomenon of infinite time blow-up was present.

Later on, in [13, 14], they treated the case of arbitrary dimension and obtained that if

$\Gamma_D = \emptyset$ and the nonlinearities f and g that behave for u large as $f \sim -\beta_0 u^p$ and $g \sim u^q$, then blow-up is produced if $p+1 < 2q$ or if $p+1 = 2q$ and $\beta_0 < q$. Also, they showed that if $p+1 > 2q$ or if $p+1 = 2q$ and β_0 is large enough, then the solutions are globally bounded. Also, in [1] they studied the porous medium equation in any dimension and as a particular case they considered the equation (1.1) with $\Gamma_D = \emptyset$, $f(x,u) = -\beta_0 u^p$ and $g(x,u) = u^q$. They showed that if $p+1 < 2q$ or $p+1 = 2q$ and $\beta_0 < q$ then blow-up is produced and if $p+1 > 2q$ of $p+1 = 2q$ and $\beta_0 > q$ then the solutions are globally bounded.

With all these works it is clear that the critical relations that mark the line between blow-up and boundedness are given by $p+1$ vs. $2q$ and in case $p+1 = 2q$, β_0 vs. q. These works have a common characteristic and it is that the nonlinear boundary condition is imposed in the whole domain, $\Gamma_D = \emptyset$ and the construction of sub or super solutions is done for the whole domain. Hence, the balances between f and g need to hold throughout the domain to obtain the result and both, the blow-up and the boundedness result are global in space. In particular, none of them can treat the case as in the equation (4.1) where $p+1 = 2q$ but in some part of the boundary the relation is $\beta_0 > q$ and in other part the relation is $\beta_0 < q$ or even when $\Gamma_D \neq \emptyset$.

In this paper we will prove that both mechanisms (dissipativeness vs. blow-up) compete at a local level. Actually, we will show that if $f(x,u) = -\beta_0 u^p$ and $g(x,u) = u^q$ in a neighborhood of a point $x_0 \in \Gamma_N$, then

- i) for the case $q > 1$, if $p+1 < 2q$ or if $p+1 = 2q$ and $\beta_0 < q$, then blow up in finite time at x_0 occurs, see Section 2.
- ii) for the case $p > 1$ if $p+1 > 2q$ or if $p+1 = 2q$ and $\beta_0 > q$ then any solution is globally bounded around the point x_0, see Section 3.

In Section 2 we analyze the first case and we refer to [3] for details. In Section 3 we consider the case ii) and we announce the results of [2]. In Section 4 we consider several important remarks and comments.

2. Localization of blow-up. In terms of characterizing the sizes of p, q and β_0 that will produce blow-up we have:

PROPOSITION 2.1. *Let $x_0 \in \Gamma_N$, $p \geq 1$, $q > 1$ and let $R_0 > 0$, $M_0 > 0$ such that*

$$\begin{aligned} f(x,u) &\geq -\beta_0 u^p, \qquad x \in B(x_0, R_0) \cap \Omega, \qquad u \geq M_0, \\ g(x,u) &\geq u^q, \qquad x \in B(x_0, R_0) \cap \partial\Omega, \qquad u \geq M_0. \end{aligned} \tag{2.1}$$

If one of the two following conditions holds

i) $p + 1 < 2q$ or

ii) $p + 1 = 2q$ and $\beta_0 < q$,

then, there exists an initial condition $0 \leq u_0 \in L^\infty(\Omega)$ with support in a neighborhood of x_0 such that the proper minimal solution of (1.1) starting at u_0 blows up in finite time at the point x_0.

Proof. Let us provide a proof of ii). Actually this case is more critical than i).

In order to simplify, consider that $x_0 = 0 \in \Gamma_N$ and that the outward normal vector at $x_0 = 0$ is given by $\vec{n}(0) = (0, \dots, 0, -1)$. Let R, $\delta > 0$ be small numbers and $y_R = x_0 + R\vec{n}(x_0) = (0, \dots, 0, -R)$ with the property that $B(y_R, R) \cap \bar{\Omega} = \emptyset$ and that $B(y_R, R+\delta) \subset B(0, R_0/2)$. The fact that the domain has a C^2 boundary, guarantees that this construction can be done. See FIG. 2.1.

We will construct a function $z(t, x)$ which will be radially symmetric around y_R, increasing in time and that it will be a subsolution of (1.1) locally around the point x_0. For this, define for $a \geq 1$, the function $\psi_a(t)$ as the solution of the problem

$$\begin{cases} \psi' = \psi^q, \\ \psi(0) = a. \end{cases} \tag{2.2}$$

Solving this equation, we get that $\psi_a(t) = \dfrac{E}{(T_a - t)^{\frac{1}{q-1}}}$ for $-\infty < t < T_a$ with $E = \dfrac{1}{(q-1)^{\frac{1}{q-1}}}$ and $T_a = \dfrac{1}{(q-1)a^{q-1}}$. Observe that, since $a \geq 1$ and $q > 1$, $T_a \leq 1/(q-1)$ and that $T_a \to 0$ as $a \to +\infty$. Notice also that $\psi_a(t) \leq E/(-t)^{1/(q-1)}$ for any $t < 0$ and any $a \geq 1$.

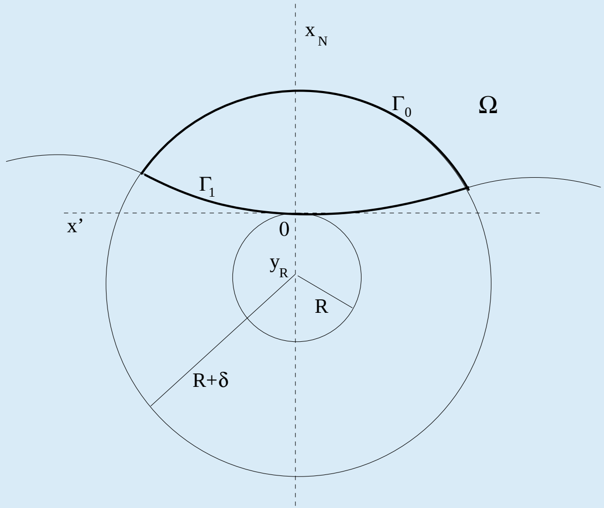

FIG. 2.1. *The domain* Ω *near* x_0.

We define $z_a(t,x) = \psi_a(t+R-|x-y_R|)$ for $x \in \mathbb{R}^N \setminus B(y_R,R)$, $0 \le t < T_a$, see FIG. 2.3.

Direct computations show that $\frac{\partial z_a}{\partial n} \le z_a^q$ for $x \in \Gamma_1$ and $0 < t < T_a$ and $\frac{\partial z_a}{\partial t} - \Delta z_a \le (1 + \frac{N-1}{R} - q z_a^{q-1}) z_a^q$ for $x \in \Omega \cap B(y_R, R+\delta)$ and $t \in (0,T_a)$. Notice that z_a is increasing in time and that $z_a(t,x) \ge z_a(0,x) = \psi_a(R-|x-y_R|) = \psi_a(-\delta) = \frac{E}{(T_a+\delta)^{\frac{1}{q-1}}} \to +\infty$ as $a \to +\infty$ and $\delta \to 0$, for $x \in \Omega \cap B(y_R, R+\delta)$. Hence, choosing a_0 large enough and δ_0 small enough, we can guarantee, since $\beta_0 < q$, that for $a \ge a_0$ and $0 < \delta < \delta_0$, that $1 + \frac{N-1}{R} - q z_a^{q-1} \le -\beta_0 z_a^{2q-1} = -\beta_0 z_a^p$ as long as $x \in \Omega \cap B(y_R, R+\delta)$ and $0 \le t < T_a$.

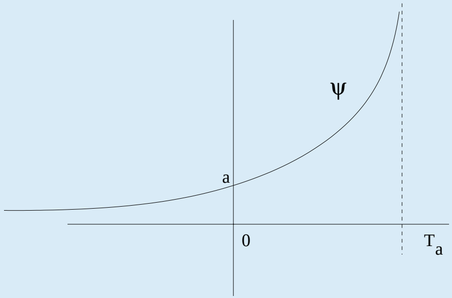

FIG. 2.2. *The solution of Equation* (2.2).

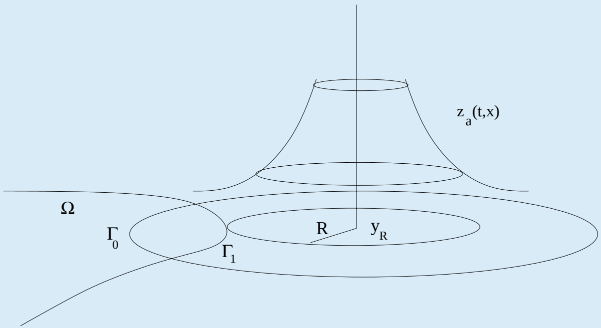

FIG. 2.3. *The function* z_a.

In particular, we get

$$\begin{cases} \dfrac{\partial z_a}{\partial t} - \Delta z_a \leq -\beta_0 z_a^p, & x \in \Omega \cap B(y_R, R+\delta),\ t \in (0, T_a), \\ \dfrac{\partial z_a}{\partial n} \leq z_a^q, & x \in \Gamma_1 = \partial\Omega \cap B(y_R, R+\delta),\ t \in (0, T_a). \end{cases} \tag{2.3}$$

Consider now a smooth initial condition $v_0 \in C^\infty(\Omega)$ such that $v_0 \equiv 0$ in $\Omega\backslash B(0,R_0)$ and $u_0 \geq \frac{2E}{\delta^{\frac{1}{q-1}}}$ in $\Omega \cap B(y_R, R+\delta)$. The solution of (1.1) starting at u_0 will satisfy that for a small time T we will have that $u(x,t,v_0) \geq \frac{E}{\delta^{\frac{1}{q-1}}}$ for $x \in \Gamma_0 \equiv \Omega \cap \partial B(y_R, R+\delta)$, $0 \leq t < T$. By monotonicity, for any $u_0 \geq v_0$ in Ω, we will also have that the proper solution starting at u_0 will satisfy, $u(x,t,u_0) \geq \frac{E}{\delta^{\frac{1}{q-1}}}$ for $x \in \Gamma_0 \equiv \Omega \cap \partial B(y_R, R+\delta)$, $0 \leq t < T$.

In particular, let us choose $a > a_0$ with the property that $0 < T_a < T$ and let us choose u_0 such that $u_0(x) \geq v_0(x)$ and $u_0(x) \geq \psi_a(-R) \geq z_a(0,x)$ for $x \in \Omega \cap B(y_R, R+\delta)$. Hence, for $0 \leq t < T_a$ we have $z_a(t,x) \leq \frac{E}{\delta^{\frac{1}{q-1}}} \leq u(x,t,u_0)$ for $x \in \Gamma_0$ and $z_a(0,x) \leq u_0(x)$ for $x \in \Omega \cap B(y_R, R+\delta)$. That is, z_a satisfies,

$$\begin{cases} \dfrac{\partial z_a}{\partial t} - \Delta z_a \leq -\beta_0 z_a^p, & x \in \Omega \cap B(y_R, R+\delta),\ t \in (0,T_a), \\ \dfrac{\partial z_a}{\partial n} \leq z_a^q, & x \in \Gamma_1 = \partial\Omega \cap B(y_R, R+\delta),\ t \in (0,T_a), \\ z_a(t,x) \leq u(x,t,u_0), & x \in \Gamma_0,\ t \in (0,T_a), \\ z_a(0,x) \leq u_0, & x \in \Omega \cap B(y_R, R+\delta), \end{cases} \tag{2.4}$$

which implies that $z_a(t,x) \leq u(t,x,u_0)$ for all $x \in \Omega \cap B(y_R, R+\delta)$ and $t \in (0,T_a)$. The fact that $z_a(T_a,x)$ blows up at $x = 0$ proves the result. □

Remarks. i) The time T_a does not need to be the classical blow-up time, that is, the time T_∞ for which the solution is classical for $0 < t < T_\infty$ and such that $\|u(t,\cdot,u_0)\|_{L^\infty(\Omega)} \to +\infty$ as $t \nearrow T_\infty$. We just can assure that $T_\infty \leq T_a$.

ii) Observe that if for $\alpha \in (0, T - T_a)$ we define the function $w_\alpha(t,x) = z_a(t-\alpha, x)$

defined for $x \in \Omega \cap B(y_R, R+\delta)$ and $t \in (\alpha, T_a+\alpha)$, then, we easily obtain that w_α satisfies

$$\begin{cases} \dfrac{\partial w_\alpha}{\partial t} - \Delta w_\alpha \le -\beta_0 w_\alpha^p, & x \in \Omega \cap B(y_R, R+\delta),\ t \in (\alpha, T_a+\alpha), \\ \dfrac{\partial w_\alpha}{\partial n} \le w_\alpha^q, & x \in \Gamma_1 = \partial\Omega \cap B(y_R, R+\delta),\ t \in (\alpha, \alpha+T_a), \\ w_\alpha(t,x) \le u(x,t,u_0), & x \in \Gamma_0,\ t \in (\alpha, \alpha+T_a), \\ w_\alpha(\alpha,x) \le u_0, & x \in \Omega \cap B(y_R, R+\delta). \end{cases} \tag{2.5}$$

The third inequality is obtained since for $x \in \Gamma_0$ we have $w_\alpha(t,x) \le \frac{E}{\delta^{\frac{1}{q-1}}} \le u(x,t,u_0)$

From (2.5) we obtain that $w_\alpha(t,x) = z_a(t-\alpha,x) \le u(t,x,u_0)$ for all $\alpha \in (0, T-T_a)$. This implies that for $t \in (T_a, T)$ we have $z_a(T_a,x) \le u(t,x,u_0)$ which means that the solution u is "pinned" to the value ∞ during the time $T_a \le t \le T$.

iii) With some extra effort, see [3] for details, it is possible to show that the construction of Proposition 2.1 can be performed in a neighborhood of $x_0 \in \partial\Omega$. As a matter of fact the parameters, R, δ, a_0, δ_0, and the initial condition u_0 can be chosen the same for a small neighborhood $\partial\Omega \cap B(x_0,\eta)$ for $\eta > 0$ small. This means that the proper solution $u(t,x,u_0)$ will blow up, not only at x_0 but at $B(x_0,\eta') \cap \partial\Omega$ for some small $\eta' > 0$, and it will remain "pinned" to the value ∞ for a period of time $T_a \le t \le T$.

3. Localization of global boundedness. In this section we present the results of [2] that, roughly speaking, say that if the complementary conditions of Proposition 2.1 hold, also near a point $x_0 \in \partial\Omega$, then the proper solution is bounded globally in time around this point x_0. As a matter of fact, we have

Proposition 3.1. *Let $x_0 \in \Gamma_N$, $p > 1$, $q \ge 1$ and let $R_0 > 0$, $M_0 > 0$ such that*

$$\begin{aligned} f(x,u) &\le -\beta_0 u^p, & x \in B(x_0,R_0) \cap \Omega, \quad & u \ge M_0, \\ g(x,u) &\le u^q, & x \in B(x_0,R_0) \cap \partial\Omega, \quad & u \ge M_0. \end{aligned} \tag{3.1}$$

If one of the two following conditions holds
i) $p+1 > 2q$ *and* $\beta_0 > 0$ *or*
ii) $p+1 = 2q$ *and* $\beta_0 > q$,
then, for any initial condition $0 \le u_0 \in L^\infty(\Omega)$ *the proper solution of* (1.1) *starting at* u_0 *is bounded in a neighborhood of* x_0 *in* $\bar{\Omega}$, *for all* $t > 0$. *That is, there exist* $\delta, M > 0$ *such that*

$$\sup_{0\le t<\infty,\, x\in B(x_0,\delta)\cap\bar{\Omega}} u(t,x,u_0) \le M. \tag{3.2}$$

To prove the result, we construct appropriate super solutions locally around the point $x_0 \in \Gamma_N$. As a matter of fact we extensively use the singular solutions of the following elliptic problem

$$\begin{cases} -\Delta z + \beta z^p = 0 & \text{in } B(0,R), \\ z(R) = +\infty, \end{cases} \tag{3.3}$$

and the fact that the asymptotics of this radial solution as $r \to R$ is well understood, see [5, 12].

We refer to [2] for details on the proof of this result.

4. Concluding Remarks. We present in this section several important comments and remarks.

i) Both results are local in nature: if the conditions of Proposition 2.1 (resp. Proposition 3.1) hold in a neighborhood of certain point $x_0 \in \partial\Omega$, then, independently of the behavior of the nonlinearities outside this neighborhood, we will have that blow-up (resp. global boundedness of solutions) occurs in the neighborhood of x_0. In particular, from the control theory point of view it turns out that it is impossible to prevent blow-up (resp. to produce blow-up) in a neighborhood of a point of the boundary of the domain by modifying the equation somehow away from this point.

ii) With an appropriate rescaling it is not difficult to see that if the local conditions of the nonlinearities f and g in Proposition 2.1 and Proposition 3.1 are of the

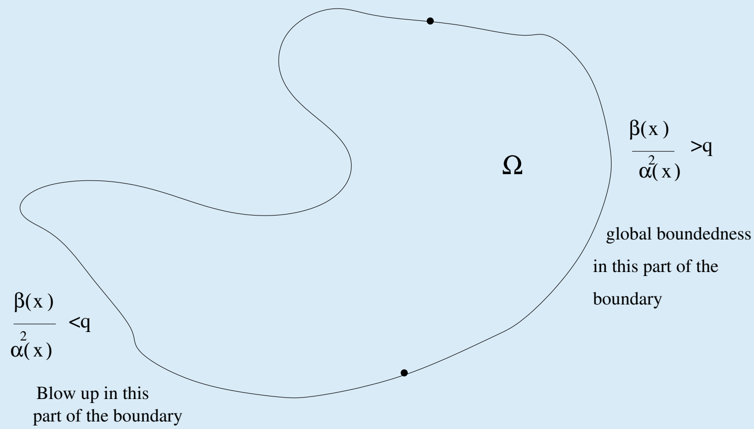

FIG. 4.1. *The domain of the example.*

type $f(x,u) \sim -\beta_0 u^p$, $g(x,u) \sim \alpha_0 u^q$, for $x \in B(x_0, R_0) \cap \partial\Omega, u \geq M_0$, then, the condition $\beta_0 < q$ (resp. $\beta_0 > q$) should be changed to $\beta_0 > q\alpha_0^2$, (resp. $\beta_0 < q\alpha_0^2$).

iii) It is important to mention that the balances obtained for p, q and β_0 are independent of the dimension of the space and even of the geometry of the domain.

iv) As an example, consider for instance the problem

$$\begin{aligned} u_t - \Delta u &= -\beta(x) u^p \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \vec{n}} &= \alpha(x) u^q \quad \text{on } \partial\Omega, \\ u(0,x) &= u_0(x) \geq 0 \quad \text{in } \Omega, \end{aligned} \tag{4.1}$$

with β and α continuous functions, $\beta(x) > 0$ in $\bar{\Omega}$ and $\alpha(x) > 0$ in $\partial\Omega$, see FIG. 4.1.

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Then if $p+1 = 2q > 2$ and $x_0 \in \partial\Omega$ with $\frac{\beta(x_0)}{\alpha(x_0)^2} < q$ then from [3], there are initial conditions where blow up is produced near x_0, while if $\frac{\beta(x_0)}{\alpha(x_0)^2} > q$, then from THEOREM 2.1 above, for any initial condition $u_0 \in L^\infty(\Omega)$ the proper minimal solution is bounded near x_0. Hence, we have the situation as in FIG. 4.1

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