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PROPERTIES OF EIGENFUNCTIONS OF NON-LOCAL OPERATORS

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Abstract. In this paper we consider the spectral properties of a class of non-local operators, with particular emphasis on properties of the associated eigenfunctions. The operators studied here are bounded perturbations of linear (local) differential operators. The non-local perturbation is in the form of an integral term. It is shown here that the spectral properties of these non-local operators can differ considerably from those of their local counterpart. The eigenfunctions of these non-local operators are studied and new oscillation results are presented. These results highlight problems with certain similar oscillation results and provide an alternative formulation.

Key words. Non-local, Eigenvalues, Eigenfunctions, Oscillation Theorem

AMS subject classifications. 34L05, 34L10, 47A75, 47G20

1. Introduction. This paper studies the spectral properties of a class of linear integro-differential operators of the form

$$[L_\varepsilon u](x) := u''(x) + a(x)u(x) + \varepsilon c(x) \int_{-1}^{1} d(x)u(x)\,\mathrm{d}x, \qquad x \in (-1,1). \tag{1.1}$$

The functions a, c and d are taken to be continuous, real valued, and are defined on the closed interval $[-1,1]$. The operator L_ε is defined on a domain that incorporates homogeneous Dirichlet boundary conditions. By varying the real parameter ε, the non-local operator can be viewed as a continuous, bounded perturbation of the (local) differential operator,

$$[Au](x) = u''(x) + a(x)u(x). \tag{1.2}$$

In this paper this structure will be exploited to study the spectral properties of L_ε and the nodal properties of the associated eigenfunctions. Results are not restricted to small ε; rather, ε should be viewed as a homotopy parameter from the local operator A to the general form L_ε.

The spectral properties of (1.2) are well-known, and in [6], certain corresponding properties for the non-local operator L_ε are derived using the perturbation theory of linear operators (see e.g. [9]). As is shown in [6], the presence of the non-local term in L_ε gives a much wider variety of possible behaviour of the spectrum, than that of the corresponding local operator.

In the following section we will detail a notation consistent with that used in [6] and state some basic results that will be required later. In Section 3, new results regarding the spectral properties of L_ε are presented. Some knowledge of the multiplicities of the eigenvalues is a prerequisite to studying nodal properties of eigenfunctions. Section 4 details how the multiplicities of the eigenvalues change with ε, and nodal properties of the corresponding eigenfunctions are discussed in Section 5. Oscillation theorem-type results

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for non-local operators similar to the form considered here have been recorded previously, albeit with Neumann boundary conditions, (see [3, 4]). However, a counterexample given in [8] shows that the proof of the main oscillation theorem in [4] contains gaps. This counterexample suggests that the main problem with the oscillation theorem in [4] is that this theorem overlooks the possibility of complex eigenvalues of L_ε. However we present a further counterexample in Section 5 which shows that even if L_ε is self-adjoint, i.e. even if the eigenvalues of L_ε remain real $\forall \varepsilon \in \mathbb{R}$, then the oscillation theorem in [4] does not hold. Using a different method of proof, we are able to establish a number of oscillation results pertaining to the problem studied here, including a result regarding the non-local equivalent of a principal eigenvalue. Certain of these results also hold in the Neumann boundary condition case.

Throughout this paper we consider the problem augmented with homogeneous Dirichlet boundary conditions. However, it is straightforward to show that most of the following results still hold for boundary conditions of the form

$$\begin{aligned} \alpha_1 u(-1) + \alpha_2 u'(-1) &= 0, \\ \alpha_3 u(1) + \alpha_4 u'(1) &= 0, \end{aligned}$$

for $\alpha_i \in \mathbb{R}$, $i = 1 \dots 4$ where α_1 and α_2 are not both equal to 0, and α_3 and α_4 are not both equal to 0. In this case the domain of L_ε is modified to incorporate these new boundary conditions.

2. Preliminaries. Let $A, B, L_\varepsilon : H^2(-1,1) \cap H_0^1(-1,1) \subset L^2(-1,1) \to L^2(-1,1)$ be defined by

$$Au = u'' + a(x)u, \quad Bu = c(x) \int_{-1}^{1} d(x)u(x)\,\mathrm{d}x, \quad \text{and} \quad L_\varepsilon = A + \varepsilon B, \tag{2.1}$$

where $a, c, d \in C[-1,1]$; $c, d \not\equiv 0$ and $\varepsilon \in \mathbb{R}$.

Then A is a densely defined, closed, self-adjoint operator with compact resolvent. Its spectrum is real, bounded above and consists entirely of isolated, simple eigenvalues. Denote these eigenvalues by γ_k, $k = 1, 2, 3, \dots$ and the corresponding eigenfunctions by v_k. Then it is well-known that $\gamma_1 > \gamma_2 > \dots > \gamma_k > \gamma_{k+1} > \dots$ and $\gamma_k \to -\infty$ as $k \to \infty$. Moreover, the eigenfunction, v_k corresponding to γ_k has exactly $(k-1)$ zeros on $(-1,1)$, which are all simple, and $\{v_k\}_{k=1}^{\infty}$ forms an orthonormal basis for $L^2(-1,1)$.

Clearly B is a bounded linear operator, and therefore it can be shown that for each fixed ε, L_ε is a densely defined, closed operator with compact resolvent. Hence, for each fixed ε, the spectrum, $\sigma(L_\varepsilon)$, consists entirely of isolated eigenvalues. Denote these eigenvalues by $\lambda_k(\varepsilon)$ and for consistency, let $\lambda_k(0) = \gamma_k$ for each $k \in \mathbb{N}$. Denote the corresponding eigenfunctions by $u_k(\varepsilon)$. Then in this way, we generate a set of functions, $\Sigma := \{\lambda_k(\varepsilon)\}_{k=1}^{\infty}$, which we shall also refer to as eigenvalues of L_ε. Similarly, the functions of ε, $u_k(\varepsilon)$ will be referred to as eigenfunctions. Then we may deduce the following from the results contained in Sections II-1, III-6.4, IV-3.5 and VII-1.3 of [9]:

Lemma 2.1.

(a) *For each k, $\lambda_k(\varepsilon)$ is a continuous function of ε, $\forall \varepsilon \in \mathbb{R}$.*
(b) *Fix k. If $\lambda_k(\varepsilon) \neq \lambda_j(\varepsilon)$ for all $j \neq k$ and $\forall \varepsilon \in (\varepsilon_1, \varepsilon_2)$, then $\lambda_k(\varepsilon)$ is an analytic function of ε $\forall \varepsilon \in (\varepsilon_1, \varepsilon_2)$, and the eigenprojection corresponding to $\lambda_k(\varepsilon)$ is an analytic function of ε $\forall \varepsilon \in (\varepsilon_1, \varepsilon_2)$.*
(c) *Let $S \subset \Sigma$ be a finite dimensional set. If $\lambda_k(\varepsilon) \neq \lambda_j(\varepsilon)$ for any $\lambda_k(\varepsilon) \in S$ and $\lambda_j(\varepsilon) \in \Sigma \backslash S$, $\forall \varepsilon \in (\varepsilon_1, \varepsilon_2)$, then the sum of the eigenvalues in S is an analytic*

function of ε $\forall \varepsilon \in (\varepsilon_1, \varepsilon_2)$. *Furthermore, the total eigenprojection corresponding to all the eigenvalues in* S *is an analytic function of* ε $\forall \varepsilon \in (\varepsilon_1, \varepsilon_2)$.

As shown in [6], in certain cases, the paths of different eigenvalues of L_ε intersect, i.e. for some $\varepsilon^* \neq 0$, $\lambda_k(\varepsilon^*) = \lambda_j(\varepsilon^*)$, for $k \neq j$. In this instance the definition of the k^{th} eigenvalue contains some ambiguity: there is a choice of assigning indices to the eigenvalues which have just intersected. However, in the case where an eigenvalue $\lambda_j(\varepsilon)$ is constant with respect to ε $\forall \varepsilon \in \mathbb{R}$, the same index shall be assigned to this eigenvalue $\forall \varepsilon \in \mathbb{R}$ i.e. $\lambda_j(\varepsilon) \equiv \gamma_j$. Following [6]:

DEFINITION 2.2. We call $\lambda_k(\varepsilon)$ a *fixed eigenvalue iff* $\lambda_k(\varepsilon) \equiv \gamma_k$. *If* $\lambda_k(\varepsilon)$ *is not fixed, then it is referred to as a moving eigenvalue.*

REMARK 2.3. *Note that an eigenfunction* $u_k(\varepsilon)$ *corresponding to a fixed eigenvalue* $\lambda_k(\varepsilon)$, *may or may not vary with* ε. *If the latter holds, i.e.* $u_k(\varepsilon) \equiv v_k$, *then we refer to such an eigenfunction as being fixed.*

Finally, the adjoint of L_ε, denoted L_ε^* is defined by

$$L_\varepsilon^* u = Au + \varepsilon B^* u, \quad \varepsilon \in \mathbb{R}, \quad u \in H^2(-1,1) \cap H_0^1(-1,1)$$

where

$$B^* u = d(x) \int_{-1}^{1} c(x) u(x) \, \mathrm{d}x.$$

As already noted, A is self-adjoint, i.e. L_0 is self-adjoint. Moreover, L_ε is self-adjoint iff $c \equiv d$, and clearly if L_{ε^*} is self-adjoint for some $\varepsilon^* \neq 0$, then L_ε is self-adjoint for all $\varepsilon \in \mathbb{R}$.

Propositions 3.1, 3.3, 3.5 and 3.7 in [6] are central to many arguments in this paper, and hence are summarized in the following Lemma for ease of reference.

LEMMA 2.1 ([6]).

(a) $\lambda_k(\varepsilon) \equiv \gamma_k$ *iff either* $Bv_k \equiv 0$ *or* $B^* v_k \equiv 0$. *If* $Bv_k \equiv 0$, *then* $u_k(\varepsilon) \equiv v_k$.

(b) $\lambda_i(\varepsilon_1) = \lambda_j(\varepsilon_2)$ *for any* $i, j \in \mathbb{N}$ *and any* $\varepsilon_1 \neq \varepsilon_2$ *only if* $\lambda_i(\varepsilon_1) = \lambda_j(\varepsilon_2) \equiv \gamma_k$ *for some* $k \in \mathbb{N}$.

(c)

$$\lambda_k'(0) = \frac{\int_{-1}^{1} c(x) v_k(x) \, \mathrm{d}x \int_{-1}^{1} d(x) v_k(x) \, \mathrm{d}x}{\int_{-1}^{1} \left(v_k(x)\right)^2 \mathrm{d}x}.$$

3. Spectral Properties of L_ε. In this section, general spectral properties of L_ε are considered in more detail and new results concerning the eigenvalues and eigenfunctions are given.

As noted above, in general, $\lambda_k(\varepsilon)$ will change with ε. However, it is possible that any number of the eigenvalues remain fixed. Indeed, it is possible for all of the eigenvalues of L_ε to be fixed, as demonstrated by the following example.

In the definition of L_ε, let a and c be even functions, and let d be an odd function. Then it is straightforward to show that since a is even, v_k is an even function if k is odd and vice-versa. Therefore $Bv_k \equiv 0$ if k is odd, and $B^* v_k \equiv 0$ if k is even. Hence by LEMMA 2.1(a), all the eigenvalues of L_ε are fixed.

However, in the case where $\sigma(L_\varepsilon) = \sigma(A)$, $\forall \varepsilon \in \mathbb{R}$, varying ε affects the corresponding eigenfunctions as is now shown.

THEOREM 3.1. *If the eigenvalues of L_ε are all fixed, then the eigenfunctions of L_ε are not all fixed.*

Proof. Suppose that $\lambda_k(\varepsilon) \equiv \gamma_k$, $\forall k \in \mathbb{N}$. As stated above, $\{v_k\}_{k=1}^\infty$ forms a basis for $L^2(-1,1)$, and hence $\exists\, v_j \in \{v_k\}_{k=1}^\infty$ such that $\int_{-1}^1 d(x)v_j(x)\,\mathrm{d}x \neq 0$, i.e. $Bv_j \not\equiv 0$. But by assumption $\lambda_j(\varepsilon) \equiv \gamma_j$ and so it follows from the equation

$$Au_j(\varepsilon) + \varepsilon B u_j(\varepsilon) = \lambda_j(\varepsilon)u_j(\varepsilon),$$

that $u_j(\varepsilon) \not\equiv v_j$. ∎

As stated above, the eigenfunctions of A form an orthonormal basis for $L^2(-1,1)$. The possible variation of the eigenfunctions of L_ε means that in general, $\{u_k(\varepsilon)\}_{k=1}^\infty$ may no longer form a basis for $L^2(-1,1)$. However,

THEOREM 3.2. *If L_ε is self-adjoint, then the eigenfunctions of L_ε form an orthonormal basis for $L^2(-1,1)$, for each $\varepsilon \in \mathbb{R}$.*

Proof. Fix ε and assume without loss of generality that 0 is not an eigenvalue of L_ε. (If 0 is an eigenvalue then simply consider the operator $L_\varepsilon + KI$ for some constant K suitably chosen.) The only spectral values of L_ε are eigenvalues, therefore L_ε^{-1} exists. Furthermore, $L_\varepsilon = L_\varepsilon^*$ implies $L_\varepsilon^{-1} = (L_\varepsilon^{-1})^*$. It follows that L_ε^{-1} is defined on a dense subset of $L^2(-1,1)$, and is compact. Also, 0 is not an eigenvalue of L_ε^{-1}. Hence

$$\ker L_\varepsilon^{-1} = \{0\}.$$

Applying [10, Corollary 6.35] and using the equivalence of the eigenfunctions of L_ε and L_ε^{-1} concludes the proof. ∎

Returning to the general case, by LEMMA 2.1(b) it is known that for each fixed real number, $\lambda \neq \gamma_k\ \forall k \in \mathbb{N}$, there exists at most one value of $\varepsilon = \varepsilon^*$ say such that $\lambda_{k^*}(\varepsilon^*) = \lambda$, for some $k^* \in \mathbb{N}$. i.e. the graphs of the solutions $\varepsilon(\lambda)$ of $\lambda_k(\varepsilon) = \lambda$, do not *overlap*. The following lemma gives an expression for this value ε^*, when it exists. Let

$$c(x) = \sum_{i=1}^\infty c_i v_i(x), \quad \text{and} \quad d(x) = \sum_{i=1}^\infty d_i v_i(x). \tag{3.1}$$

THEOREM 3.1. *Suppose that $\lambda \neq \gamma_k$ for any $k \in \mathbb{N}$. If the solution $\varepsilon^*(\lambda)$ of the equation $\lambda = \lambda_{k^*}(\varepsilon^*)$ exists, then it is unique and is given by*

$$\varepsilon^*(\lambda) = \left(\sum_{i=1}^\infty \frac{c_i d_i}{(\lambda - \gamma_i)} \right)^{-1}.$$

Proof. The uniqueness of the value ε^* follows from the arguments above. Let

$$u(x) = \sum_{i=1}^\infty \beta_i v_i(x).$$

Substituting the above expressions for c, d and u into the equation $L_\varepsilon u = \lambda u$ and comparing the coefficients of v_i for each $i \in \mathbb{N}$ gives

$$\beta_i(\gamma_i - \lambda) + \varepsilon c_i \int_{-1}^1 d(x)u(x)\,\mathrm{d}x = 0.$$

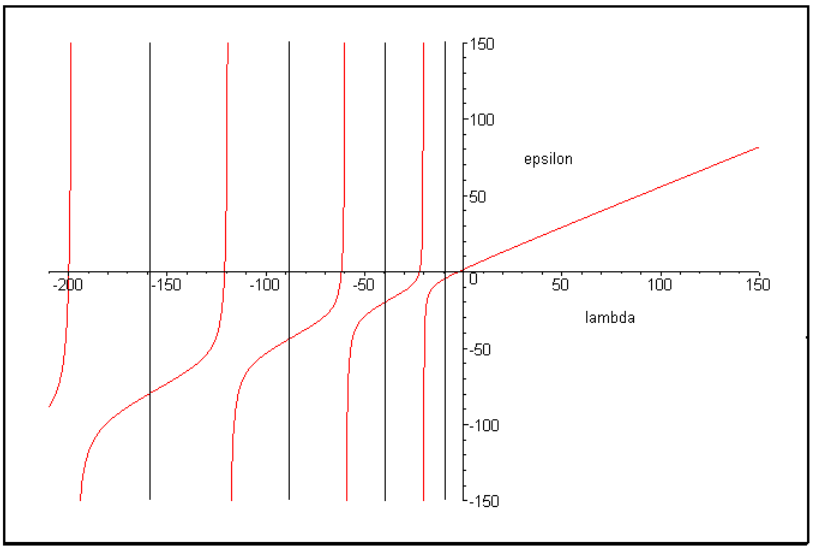

FIG. 3.1. *Paths of moving eigenvalues, λ of L_ε, as ε varies, where* $[L_\varepsilon u](x) = u''(x) + \varepsilon \int_{-1}^{1} u(x)\,\mathrm{d}x$.

Hence, either $c_i = \beta_i = 0$ or

$$\varepsilon = \frac{\beta_i(\lambda - \gamma_i)}{c_i \int_{-1}^{1} d(x)u(x)\,\mathrm{d}x}.$$

But this will hold for all i such that $c_i \neq 0$, and so in this case

$$\frac{\beta_i(\lambda - \gamma_i)}{c_i} = K, \tag{3.2}$$

for some constant K independent of i, and without loss of generality we take $K = 1$. Hence,

$$u(x) = \sum_{i=1}^{\infty} \frac{c_i}{(\lambda - \gamma_i)} v_i(x), \tag{3.3}$$

and it follows directly from (3.2) and (3.3) that

$$\varepsilon = \frac{1}{\int_{-1}^{1} d(x)u(x)\,\mathrm{d}x} = \left(\sum_{i=1}^{\infty} \frac{c_i d_i}{(\lambda - \gamma_i)} \right)^{-1}. \tag{3.4}$$

□

The expression (3.4) allows the paths of the eigenvalues $\lambda_k(\varepsilon)$ to be computed. For practical purposes, it may be sufficient to use a truncation of the series or, if $u(x)$ can be computed explicitly, the first expression in (3.4) can be used. FIG. 3.1 was plotted using the second method, whilst FIG. 3.2 used a truncated Fourier series approximation.

Note that in both of these figures, the fixed eigenvalues shown have been added by hand, as this method only finds the moving eigenvalues. Note also that in FIG. 3.2 there are points where as ε increases (or decreases), the paths of 2 eigenvalues join, and then disappear from the plot. This corresponds to those eigenvalues becoming complex.

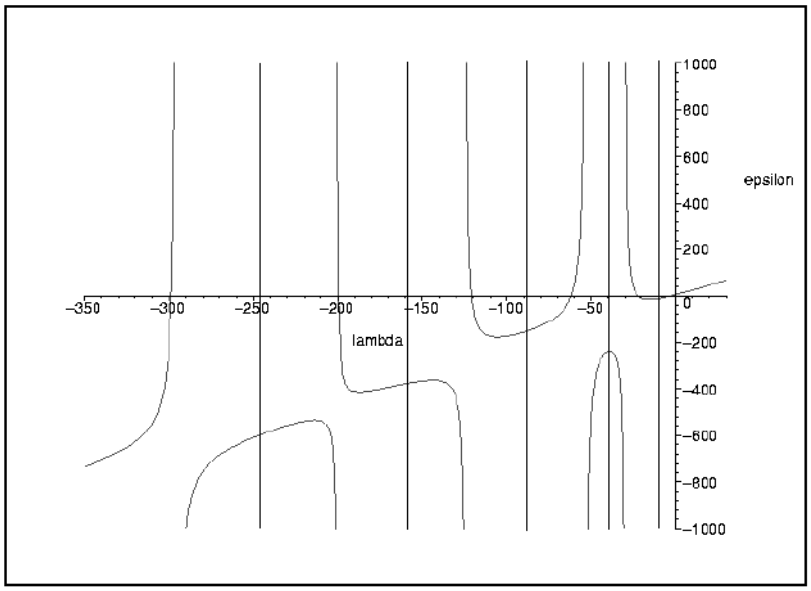

FIG. 3.2. *Paths of real moving eigenvalues of L_ε, as ε varies, where $[L_\varepsilon u](x) = u''(x) + \varepsilon c(x) \int_{-1}^{1} u(x)\,\mathrm{d}x$, and $c(x) = 0$ for $|x| > \frac{1}{3}$, $1 - 9x^2$ for $|x| \leq \frac{1}{3}$.*

4. Algebraic and Geometric Multiplicity . As was noted above, knowledge of the multiplicities of the eigenvalues is of importance in establishing conditions for results on nodal properties of eigenfunctions. Hence, we now consider whether the multiplicities of the eigenvalues $\lambda_k(\varepsilon)$ change as the parameter ε is varied. Geometric multiplicity of an eigenvalue λ of L_ε, can be defined in the usual way, i.e. $\dim(N(L_\varepsilon - \lambda I))$. Algebraic multiplicity however, is usually only defined for compact operators and therefore we must check that an appropriate definition can be derived here, similar to that for the local differential operator. From standard theory, it is known that for a compact operator T, with eigenvalue λ, $\exists k \in \mathbb{N}$ such that

$$N((T - \lambda I)^{(k-1)}) \subset N((T - \lambda I)^k) = N((T - \lambda I)^{(k+1)}) = ...$$

The algebraic multiplicity of λ is defined to be the dimension of $N((T - \lambda I)^k)$. The operator L_ε is in general not compact. However, if L_ε^{-1} exists, then it is compact. We assume without loss of generality that L_ε^{-1} does exist. (If L_ε is not invertible, then consider $L_\varepsilon + KI$ for an appropriate constant, K.) Then since L_ε is a closed linear operator with compact resolvant, the algebraic multiplicity of an eigenvalue, λ of L_ε can be defined to be the algebraic multiplicity of the eigenvalue, $1/\lambda$ of L_ε^{-1}. A *simple eigenvalue* is defined to be an eigenvalue of algebraic multiplicity 1, (see e.g. [2]).

4.1. Algebraic Multiplicity. The following theorem can be deduced from [9, Section IV-3.5].

THEOREM 4.1. *Let $S \subset \Sigma$ be a finite-dimensional set. If $\lambda_k(\varepsilon) \neq \lambda_j(\varepsilon)$ for any $\lambda_k(\varepsilon) \in S$ and $\lambda_j(\varepsilon) \in \Sigma \backslash S$ $\forall \varepsilon \in (\varepsilon_1, \varepsilon_2)$, then the sum of the algebraic multiplicities of the eigenvalues in S is constant with respect to ε, $\forall \varepsilon \in (\varepsilon_1, \varepsilon_2)$.*

COROLLARY 4.2. *If*

$$|\varepsilon| < \frac{\min_{i,j\in\mathbb{N}} |\gamma_i - \gamma_j|}{2\|B\|}, \tag{4.1}$$

then all the eigenvalues of L_ε are simple.

Proof. If (4.1) holds, then $\lambda_i(\varepsilon) \neq \lambda_j(\varepsilon)$ for $i \neq j$ (See [9, Section II-5.1]). The eigenvalues of A are all simple, and therefore the result follows from the previous theorem. □

4.2. Geometric Multiplicity.

THEOREM 4.3. *An eigenvalue $\lambda_k(\varepsilon)$ has geometric multiplicity 1 provided $\lambda_k(\varepsilon) \neq \gamma_j$ for any $j \in \mathbb{N}$.*

Proof. Suppose that for some ε^* and some k, $N(L_{\varepsilon^*} - \lambda_k(\varepsilon^*)) = \text{span}\{u, v\}$ with u and v linearly independent where $\lambda_k(\varepsilon^*) \neq \gamma_j$ for any $j \in \mathbb{N}$. Then, there exist constants a and b with $|a| + |b| \neq 0$ such that $B(au + bv) \equiv 0$. Hence, $au + bv = v_j$ and $\lambda_k(\varepsilon^*) = \gamma_j$ for some $j \in \mathbb{N}$, which is a contradiction and therefore the result is proven. □

5. Nodal Properties of the eigenfunctions of L_ε. As previously stated and as is well-known, the eigenfunction v_k corresponding to the k^{th} eigenvalue of A has exactly $(k-1)$ zeros on $(-1, 1)$, all of which are simple. This result is usually referred to as an *oscillation theorem* for linear Sturm-Liouville boundary value problems. A similar, general oscillation theorem has not yet been successfully derived for non-local boundary value problems of the type discussed here, despite previous efforts in this area, albeit with Neumann boundary conditions (see [3, 4]). In [8], it was observed that the oscillation theorem given in [4] overlooked the possibility of complex eigenvalues. Here we present a further counterexample, which shows that even if L_ε is self-adjoint, i.e. even if the eigenvalues of L_ε remain real $\forall \varepsilon \in \mathbb{R}$, then the oscillation theorem in [4] does not hold.

Note that changing the boundary conditions accompanying the differential equations studied, to Neumann boundary conditions, does not affect the spectral properties of A and L_ε discussed previously.

COUNTEREXAMPLE. Suppose that the domain of L_ε is modified to incorporate homogeneous Neumann boundary conditions. Then let $a(x) \equiv 0$, $c(x) = d(x) = \cos(\pi x)$ in the definition of L_ε. Therefore a, c and d are even functions and thus satisfy the hypotheses of the result in [4]. (In fact, the interval used in [4] is $(0, 1)$ and the conditions are that the corresponding functions are symmetric about $x = 1/2$. This is clearly equivalent to even functions here.) Then,

$$\int_{-1}^{1} c(x)v_k(x)\,\mathrm{d}x = 0 \quad \text{if } k \neq 3, \quad \text{whilst} \quad \int_{-1}^{1} c(x)v_3(x)\,\mathrm{d}x \neq 0.$$

Hence, (by a simple extension of LEMMA 2.1(a) to Neumann boundary conditions), the only eigenvalue which is not fixed is $\lambda_3(\varepsilon)$. Note that L_ε is self-adjoint and hence all eigenvalues remain real for all values of $\varepsilon \in \mathbb{R}$. It also follows that the fixed eigenvalues have eigenfunctions which are fixed. Now consider the only moving eigenvalue. It is straightforward to show that $u(x) = \cos(\pi x)$ is the eigenfunction corresponding to $\lambda_3(\varepsilon)$ for all $\varepsilon \in \mathbb{R}$ and that $\lambda_3(\varepsilon) = -\pi^2 + \varepsilon$. Hence, by choosing ε appropriately, it follows that for any value of $\lambda \in \mathbb{R}$, there exists an eigenfunction corresponding to λ, with exactly 2 interior zeros, both of which are simple. This contradicts the oscillation theorem in [4].

It appears that no progress can made using the method of proof given in [3] irrespective of the boundary conditions imposed. Furthermore, given the simplicity of the above

counterexample it would appear that no general oscillation theorem of the type found in [4] can exist. We now return to Dirichlet boundary conditions, and give some alternative oscillation type results. Note however it is straightforward to show that THEOREM 5.1, COROLLARY 5.2, THEOREM 5.3 and THEOREM 5.5 also hold for homogeneous Neumann boundary conditions.

THEOREM 5.1. *For each eigenvalue $\lambda_k(\varepsilon)$, $\exists$ a constant $\alpha(k)$ dependent on k, such that if $|\varepsilon| < \alpha(k)$, then $u_k(\varepsilon)$ has exactly $(k-1)$ zeros in $(-1,1)$, all of which are simple.*

Proof. For $\varepsilon = 0$ this is the standard Sturm oscillation theorem (see e.g [5, Theorem 2.1 in Chapter 8]). For $|\varepsilon| < \alpha(k)$, the result follows by the continuity of the (1-dimensional) eigenprojections with respect to ε as stated in LEMMA 2.1(b). □

COROLLARY 5.2. *Let $S \subset \Sigma$ be a finite set. Then $\exists$ a constant $\alpha > 0$ such that if $|\varepsilon| < \alpha$ and if $\lambda_k(\varepsilon) \in S$, then $u_k(\varepsilon)$ has exactly $(k-1)$ zeros in $(-1,1)$ all of which are simple.*

THEOREM 5.3. *If a and d are even functions, then for each even $k \in \mathbb{N}$, $\lambda_k(\varepsilon)$ is a fixed eigenvalue, with corresponding fixed eigenfunction $u_k(\varepsilon)$. Consequently, $u_k(\varepsilon) \equiv v_k$ and therefore has exactly $k-1$ zeros on $(-1,1)$, all of which are simple.*

Proof. The function a is even, and therefore v_k is an odd function if k is even. Hence d is even implies that $Bv_k \equiv 0$, $\forall k$ even, and therefore by LEMMA 2.1(a), $\lambda_k(\varepsilon)$ are fixed eigenvalues of L_ε with corresponding fixed eigenfunctions, $u_k(\varepsilon) \equiv v_k$. The nodal properties of v_k are known and the result follows. □

REMARK 5.4. By THEOREM 5.1 and COROLLARY 4.2, there exists a constant $\alpha > 0$ such that if $|\varepsilon| < \alpha$, then the eigenfunction corresponding to $\lambda_1(\varepsilon)$ is unique and may be chosen positive, i.e. $\lambda_1(\varepsilon)$ may be referred to as the *principal eigenvalue* of L_ε. Below, we extend this definition under certain restrictions on the functions a, c and d.

THEOREM 5.5. *Let $c(x)$ be either non-negative or non-positive on $(-1,1)$. Then L_ε has a principal eigenvalue $\lambda_p(\varepsilon) \geq \gamma_1$ (i.e. an eigenvalue corresponding to a positive eigenfunction) either $\forall \varepsilon \geq 0$ or $\forall \varepsilon \leq 0$.*

Proof. First we show that L_ε has an eigenvalue greater than or equal to γ_1, either $\forall \varepsilon \geq 0$ or $\forall \varepsilon \leq 0$. Consider the two possible cases (i) $\lambda_1'(0) \geq 0$ and (ii) $\lambda_1'(0) \leq 0$. Then it can be deduced from LEMMA 2.1 that

$$(i) \quad \lambda_1'(0) \geq 0 \Rightarrow \text{for every } \varepsilon \geq 0,\ \exists k \in \mathbb{N} \text{ such that } \lambda_k(\varepsilon) \geq \gamma_1,$$

$$(ii) \quad \lambda_1'(0) \leq 0 \Rightarrow \text{for every } \varepsilon \leq 0,\ \exists k \in \mathbb{N} \text{ such that } \lambda_k(\varepsilon) \geq \gamma_1.$$

We now show that any eigenvalue, λ of L_ε, satisfying $\lambda \geq \gamma_1$ has a corresponding eigenfunction which is positive on $(-1,1)$. By [1, Theorem 2.4], if $\lambda > \gamma_1$, then $(A - \lambda I)$ satisfies the strong maximum principle. Let $u(x)$ be an eigenfunction corresponding to $\lambda > \gamma_1$. Then as in the proof of [7, Proposition 6.1], we note that $\int_{-1}^{1} d(x)u(x)\,\mathrm{d}x \neq 0$, as there exist no fixed eigenvalues greater than γ_1. Hence, supposing without loss of generality that $-\varepsilon c(x)\int_{-1}^{1} d(x)u(x)\,\mathrm{d}x$ is a non-negative function which is not identical to zero, applying the strong maximum principle yields

$$u(x) = (A - \lambda I)^{-1}\left[-\varepsilon c(x)\int_{-1}^{1} d(x)u(x)\,\mathrm{d}x\right] > 0 \ \ \forall x \in (-1,1).$$

We are left to consider the case $\lambda_k(\varepsilon) = \gamma_1$ for some k. If $\varepsilon = 0$ is the only solution to $\lambda_k(\varepsilon) = \gamma_1$, then as the corresponding eigenfunction, v_1 has no interior zeros, and

the result follows directly. If $\lambda_k(\varepsilon) = \gamma_1$ for some $k \in \mathbb{N}$ and some $\varepsilon \neq 0$, then by LEMMA 2.1(b), $\lambda_1(\varepsilon) \equiv \gamma_1$. A corresponding eigenfunction does not change sign as the following argument shows. The function c is either non-negative or non-positive, therefore $B^*(v_1) \not\equiv 0$, and hence as $\lambda_1(\varepsilon) \equiv \gamma_1$, $B(v_1) \equiv 0$. Therefore v_1 is an eigenfunction corresponding to γ_1, $\forall \varepsilon \in \mathbb{R}$. and the result is proved. □

REMARK 5.6. Non-local perturbations of uniformly elliptic partial differential equations, defined on a domain $\Omega \subset \mathbb{R}^n$, where $n \geq 3$, were considered in [7]. [7, Proposition 6.1] states that if $c(x)$ is strictly of one sign on Ω, and $\lambda < \gamma_1$ is an eigenvalue of L_ε, then any corresponding eigenfunction is strictly of one sign on Ω. THEOREM 5.5 includes the cases where c may have interior zeros, and $\lambda = \gamma_1$.

REMARK 5.7. Note that the principal eigenvalue is not necessarily unique: If $\lambda_1(\varepsilon) \equiv \gamma_1$ and $\lambda_2(\varepsilon)$ is moving such that $\lambda_2(\varepsilon) > \lambda_1(\varepsilon)$ $\forall \varepsilon > \varepsilon^*$ say, then by the arguments above, there exist non-negative eigenfunctions corresponding to both $\lambda_1(\varepsilon)$ and $\lambda_2(\varepsilon)$ $\forall \varepsilon > \varepsilon^*$. The next theorem however, gives conditions for uniqueness of the principal eigenvalue.

THEOREM 5.8. *If L_ε is self-adjoint, if c is strictly of one sign on $[-1, 1]$, and if $\varepsilon > 0$, then the eigenfunction corresponding to $\lambda_1(\varepsilon)$ is the only eigenfunction of L_ε with no interior zeros.*

Proof. Suppose that either $u_1(\varepsilon)$ has an interior zero, or $u_i(\varepsilon)$ has no interior zeros for $i \neq 1$, for some $\varepsilon > 0$. By the continuity of the eigenprojections of L_ε noted in LEMMA 2.1(b), the eigenfunctions of L_ε can be chosen to be continuous functions of ε. Hence $\exists \hat{\varepsilon} > 0$, and $u_i \in C^2[-1, 1]$ such that $[u_i(\hat{\varepsilon})](x) \geq 0$ $\forall x \in [-1, 1]$, whilst $\exists \hat{x} \in [-1, 1]$ such that $[u_i(\hat{\varepsilon})](\hat{x}) = [u_i(\hat{\varepsilon})]'(\hat{x}) = 0$. Then, as $\hat{x}$ is a minimum of u_i it follows that

$$[a(\hat{x})[u_i(\hat{\varepsilon})]'(\hat{x})]' = a'(\hat{x})[u_i(\hat{\varepsilon})]'(\hat{x}) + a(\hat{x})[u_i(\hat{\varepsilon})]''(\hat{x}) = a(\hat{x})[u_i(\hat{\varepsilon})]''(\hat{x}) \geq 0,$$

and also $(b(\hat{x}) - \lambda)u_i(\hat{x}) = 0$, for any $\lambda \in \mathbb{R}$. Hence,

$$\varepsilon c(\hat{x}) \int_{-1}^{1} c(x)u_i(x)\,\mathrm{d}x \leq 0,$$

which contradicts our assumptions on the sign of ε. Hence the result is proven. □

Finally, we present our most complete result, which is obtained under stricter conditions on a, c and d.

THEOREM 5.9. *Suppose that $a(x)$, $b(x)$ and $c(x)$ are equal to constants, and let $d(x)$ be even. Let $\lambda(\varepsilon)$ be an eigenvalue of L_ε. Then*

(a) *If $\lambda(\varepsilon) = \gamma_k$, for any $k \in \mathbb{N}$, then v_k is an eigenfunction corresponding to $\lambda(\varepsilon)$, and has exactly $k - 1$ interior zeros, all of which are simple.*
(b) *Suppose k is even. If $\gamma_{k+2} < \lambda(\varepsilon) < \gamma_k$ and $\lambda(\varepsilon) \neq \gamma_{k+1}$, then the corresponding eigenfunction has exactly k interior zeros, all of which are simple.*
If $\gamma_2 < \lambda(\varepsilon)$, then the corresponding eigenfunction has no interior zeros.

Proof. (a) If $\lambda_k(\varepsilon)$ is fixed, then as L_ε is self-adjoint, $u_k(\varepsilon)$ is also fixed, and is equivalent to v_k. If $\lambda_k(\varepsilon)$ is not fixed, then by THEOREM 3.1, $\lambda(\varepsilon) = \gamma_k$ only when $\varepsilon = 0$, and v_k is a corresponding eigenfunction.

(b) Assume without loss of generality that $a(x) \equiv 1$. For $a(x) \equiv 1$, $b(x) \equiv \alpha$, $\alpha \in \mathbb{R}$, it is straightforward to show that the eigenfunction corresponding to a moving eigenvalue,

λ, is given by

$$u(x) = \begin{cases} \cos(\sqrt{\alpha - \lambda}\, x) - \cos(\sqrt{\alpha - \lambda}) & \text{if } \lambda < \alpha, \\ x^2 - 1 & \text{if } \lambda = \alpha, \\ \cosh(\sqrt{\lambda - \alpha}\, x) - \cosh(\sqrt{\lambda - \alpha}) & \text{if } \lambda > \alpha, \end{cases}$$

whilst $\gamma_k = \alpha - \frac{k^2\pi^2}{4}$. The result then follows by direct calculation. □

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