Proceedings of Equadiff-11
2005, pp. 97–104
ISBN 978-80-227-2624-5

# FINITE TIME SINGULARITIES IN TRANSPORT EQUATIONS WITH NONLOCAL VELOCITIES AND FLUXES

ANTONIO CÓRDOBA*, DIEGO CÓRDOBA†, AND MARCO A. FONTELOS‡

**Abstract.** Navier-Stokes and Euler equations, when written in terms of vorticity, contain nonlinear convective terms involving singular integral (nonlocal) operators of the vorticity itself. This fact suggests the analysis of the role played by nonlocal velocities and fluxes in the formation of singularities. We consider the following one-dimensional analogs of Euler equations, namely:

1) $\theta_t + ((H\theta)\theta)_x = 0$
2) $\theta_t - (H\theta)\theta_x = 0$

with H being the Hilbert transform of, and their viscous versions obtained by adding a dissipative term at the right hand side of the equations. We prove that the inviscid equations do develop singularities in finite time while the solutions of the viscous versions do exist for all time. We also discuss connections of these problems with finite time singularities in Birkhoff-Rott equation.

**Key words.** Nonlocal transport equations, singular integral operators, Hilbert transform, fluid dynamics.

**AMS subject classifications.** 35Q30, 35R35, 74H35, 35Q35

**1. Introduction.** We present some results on partial differential equations of transport type for a scalar $\theta$ with nonlocal velocities or fluxes. We concentrate on the case of one space dimension and the nonlocal operator will be given by the Hilbert transform of $\theta$ defined as

$$H\theta(x) = \frac{1}{\pi} P \cdot V \cdot \int_{-\infty}^{\infty} \frac{\theta(y)}{x-y}\, \mathrm{d}y, \tag{1.1}$$

or

$$H\theta(x) = \frac{1}{2\pi} P \cdot V \cdot \int_{-\pi}^{\pi} \frac{\theta(x-y)}{\tan\frac{y}{2}}\, \mathrm{d}y \tag{1.2}$$

in the periodic case. Specifically, we shall consider the following problems:

$$\theta_t + (H(\theta)\theta)_x = 0, \tag{1.3}$$
$$\theta(x,0) = \theta_0(x). \tag{1.4}$$

and

$$\theta_t - H(\theta)\theta_x = 0, \tag{1.5}$$
$$\theta(x,0) = \theta_0(x). \tag{1.6}$$

*Departamento de Matemáticas, Universitad Autónoma de Madrid, 28049 Madrid, Spain
†Instituto de Matemáticas y Física Fundamental, Consejo Superior de Investigaciones Científicas, C/Serrano 123, Madrid 28006, Spain
‡Departamento de Matemáticas, Universitad Autónoma de Madrid, 28049 Madrid, Spain (marco.fontelos@uam.es)

Equation (1.3) appears as a formal 1D analog to the 2D quasi-geostrophic equation (QG), which models the dynamics of the mixture of cold and hot air and the fronts between them, and reads

$$\theta_t + (u \cdot \nabla)\theta = 0, \tag{1.7}$$

$$u = \nabla^{\perp}\psi, \quad \theta = -(-\Delta)^{\frac{1}{2}}\psi, \tag{1.8}$$

$$\theta(x,0) = \theta_0(x), \tag{1.9}$$

where $\nabla^{\perp} = (-\partial_2, \partial_1)$. Here $\theta(x,t)$ represents the temperature of the air. Besides its direct physical significance ([8, 13]), the quasi-geostrophic equation has very interesting features of resemblance to the 3D Euler equation. Also, the finite time blow-up for (QG) is an outstanding open problem. With respect to that question there are pioneering studies due to Constantin, Majda and Tabak [4]. There are many studies on the equations following that work ([5, 11, 14, 16]). The analogy with (1.3) comes from the fact that

$$u = -\nabla^{\perp}(-\Delta)^{-\frac{1}{2}}\theta = -R^{\perp}\theta, \tag{1.10}$$

and hence

$$\theta_t + \operatorname{div}[(R^{\perp}\theta)\theta] = 0, \tag{1.11}$$

where we have used the notation $R^{\perp}\theta = (-R_2\theta, R_1\theta)$ with $R_j$, $j = 1, 2$, for the two dimensional Riesz transform defined by (see e.g. [15])

$$R_j(\theta)(x,t) = \frac{1}{2\pi} PV \int_{\mathbb{R}^2} \frac{(x_j - y_j)\theta(y,t)}{|x-y|^3}\,\mathrm{d}y. \tag{1.12}$$

The equivalent (in terms of homogeneity) singular integral operator to the Riesz transform in 1D is the Hilbert transform. Therefore, (1.3) is just (1.11) with $R^{\perp}(\cdot)$ replaced by $H(\cdot)$ and $\operatorname{div}(\cdot)$ replaced by $\partial_x$.

Equation (1.5) represents the simplest case of a transport equation with a nonlocal velocity. It is well known that the equivalent equation with a local velocity $v = \theta$, known as Burger's equation, may develop shock-type singularities in finite time. Therefore a natural question to pose is whether the solutions to (1.5) become singular in finite time or not. In fact this question has been previously considered in the literature motivated by the strong analogy with the Birkhoff-Rott equation modelling the evolution of a vortex sheet, where a crucial mathematical difficulty lies in the nonlocality of the velocity.

The analogy of (1.5) with Birkhoff-Rott equations was first established in [1] and [10]. These are integrodifferential equations modelling the evolution of vortex sheets with surface tension. The system can be written in the form

$$\frac{\partial}{\partial t} z^*(\alpha,t) = \frac{1}{2\pi i} PV \int \frac{\tilde{\gamma}(\alpha')\,\mathrm{d}\alpha'}{z(\alpha,t) - z(\alpha',t)} \tag{1.13}$$

$$\frac{\partial \tilde{\gamma}}{\partial t} = \sigma \kappa_{\alpha} \tag{1.14}$$

where $z(\alpha,t) = x(\alpha,t) + \mathrm{i}\, y(\alpha,t)$ represents the two dimensional vortex sheet parametrized with $\alpha$, and where $\kappa$ denotes the mean curvature. Following [1] we substitute, in order to build up the model, the equation (1.13) by its 1D analog

$$\frac{\mathrm{d}x(\alpha,t)}{\mathrm{d}t} = -H(\theta) \tag{1.15}$$

where we have identified $\gamma(\alpha,t)$ with $\theta$. In the limit of $\sigma=0$ in (1.14) we conclude that $\gamma$ is constant along trajectories and this fact leads, in the 1D model, to the equation

$$\theta_t-(H\theta)\,\theta_x=0. \tag{1.16}$$

There is now overwhelming evidence that vortex sheets form curvature singularities in finite time. This evidence comes from the classical paper by Moore [9] where he studied the Fourier spectrum of $z(\alpha,t)$ and, in particular, its asymptotic behavior when the wavenumber $k$ goes to infinity. His numerical results showed that, up to very high values of $k$, this asymptotic behavior is compatible with the formation of a curvature singularity in finite time. Although there has been a very intense activity in order to provide a definitive proof of the formation of such a singularity (see discussions and references in [9], [2] and [1]) all the existing results are mostly supported by numerics or formal asymptotics and do not constitute a full mathematical proof. The same kind of arguments were used in [1] in order to argue the existence of singularities for the 1D analog (1.16).

Problems of the type (1.3), (1.5) were already studied in [1] and [10]. In [10], the following equation was considered

$$\theta_t+\delta(H(\theta)\theta)_x+(1-\delta)H(\theta)\theta_x=0 \qquad \text{with } 0\le\delta\le1. \tag{1.17}$$

In Theorem 1.1 below, we proved existence of singularities for the full range of $0<\delta\le1$. The proof of existence of singularities in the case $\delta=0$ is solved in [6] using a different technique.

We have proved the following theorems:

Theorem 1.1. *Given a periodic non-constant initial data $\theta_0\in C^1([-\pi,\pi])$ such that $\int_{-\pi}^{\pi}\theta_0(x)\,\mathrm{d}x=0$, there is no $C^1([-\pi,\pi]\times[0,\infty))$ solution to (1.17) with $\delta>0$.*

In fact, the result can be improved and extends to any compactly supported initial data in $C^1(\mathbb{R})$ such that $\int_{\mathbb{R}}\theta_0(x)\,\mathrm{d}x=0$ since it can be proved that the support is not expanding as time progresses.

Theorem 1.2. *If $\theta_0$ has compact support and is strictly positive then the solutions of (1.5) will always be such that $\|\theta_x\|_{L^\infty}$ blows up in finite time.*

In Section 2 we show that solutions to (1.3) may develop singularities in finite time by constructing explicit examples. The proof of formation of singularities in (1.5) is the subject of Section 3. Finally, in Section 4 we consider the viscous versions of (1.3), (1.5).

**2. Sketch of proof of Theorem 1.1 and construction of singular solutions of equation (1.3).** In order to prove Theorem 1.1 (see [3] for further details) we define $x(t)$, $\overline{x}(t)$ such that

$$M(t)=\theta(x(t),t),$$
$$m(t)=\theta(\overline{x}(t),t),$$

here $M(t)=\max\theta(\cdot,t)$ and $m(t)=\min\theta(\cdot,t)$ for every $t\ge0$ and notice that

$$\Lambda\theta(x)=H\theta_x(x)=\frac{1}{2\pi}P\cdot V\cdot\int_{-\pi}^{\pi}\frac{\theta(x)-\theta(y)}{\sin^2\frac{x-y}{2}}\,\mathrm{d}y\ . \tag{2.1}$$

By following the point at which the minimum of $\theta$ is achieved one can show from (1.17)

$$m'(t) = -\frac{\delta}{2\pi} m(t) \int_{-\pi}^{\pi} \frac{\theta(\overline{x},t) - \theta(y,t)}{\sin^2 \frac{\overline{x}-y}{2}} \, dy \leq 0 \tag{2.2}$$

at almost every t and hence $m(t) \leq m(0) < 0$. Analogously we can show $M(t) \leq M(0)$ and therefore the set

$$\left\{ y : \theta(y,t) \geq \frac{\theta(\overline{x},t)}{2} \right\} \tag{2.3}$$

has strictly positive measure. Hence there exists a universal positive constant C so that:

$$\frac{\delta}{2\pi} \int_{-\pi}^{\pi} \frac{\theta(y,t) - \theta(\overline{x},t)}{\sin^2 \frac{\overline{x}-y}{2}} \, dy \geq C|\theta(\overline{x},t)|, \tag{2.4}$$

so that

$$|m|'(t) \geq C|m(t)|^2 \tag{2.5}$$

implying blowup for $|m|$ in finite time. This completes the proof.

In order to construct explicit singular solutions to (1.17) with $\delta = 1$, we can transform (1.3) into an equation for complex valued functions. We start by recalling the following well known formulas for the Hilbert transform (see e.g. [12]):

$$H(Hf) = -f, \tag{2.6}$$

$$H(fHg + gHf) = (Hf)(Hg) - fg, \tag{2.7}$$

$$(Hf)_x = H(f_x). \tag{2.8}$$

If we apply the Hilbert transform at both sides of (1.3) and use the relations above we obtain the equation

$$(H\theta)_t + \frac{1}{2}((H\theta)^2 - (\theta)^2)_x = 0. \tag{2.9}$$

Equations (1.3) and (2.9) can be combined into a single equation for the complex function

$$z(x,t) = H\theta(x,t) + \mathrm{i}\,\theta(x,t), \quad z_0(x) = H\theta_0(x) + \mathrm{i}\,\theta_0(x). \tag{2.10}$$

Namely, (1.3) and (2.9) are the real and imaginary parts of the following complex Burger's equation

$$z_t + zz_x = 0, \tag{2.11}$$

$$z(x,0) = z_0(x). \tag{2.12}$$

Another way of writing the system is

$$u_t + uu_x - \theta\theta_x = 0 \ , \tag{2.13}$$

$$\theta_t + u\theta_x + \theta u_x = 0 \ , \tag{2.14}$$

where $u = H\theta$. It is a well known fact that systems of two first order PDEs in two variables can be linearized by using the hodograph transformation which consists in considering $x(u,\theta)$ and $t(u,\theta)$ and writing equations for them through the relations

$$\begin{aligned} u_x &= J t_\theta \ , \\ \theta_x &= -J t_u \ , \\ u_t &= -J x_\theta \ , \\ \theta_t &= J x_u \ , \end{aligned}$$

where $J = (x_u t_\theta - x_\theta t_u)^{-1}$. This leads to the system

$$-x_\theta + u t_\theta + \theta t_u = 0 \ , \tag{2.15}$$

$$x_u - u t_u + \theta t_\theta = 0 \ . \tag{2.16}$$

valid as long as $J^{-1} \neq 0$. By introducing $\eta(u,\theta) \equiv -(x(u,\theta) - t(u,\theta)u)$ and $\xi(u,\theta) \equiv -t(u,\theta)\theta$ one can write (2.15), (2.16) in the form of the following Cauchy-Riemann system

$$\begin{aligned} \xi_u &= \eta_\theta \ , \\ \xi_\theta &= -\eta_u \ . \end{aligned}$$

Hence we can construct solutions from holomorphic functions $f(z) = \xi(u,\theta) + \mathrm{i}\,\eta(u,\theta)$ where $z = u + \mathrm{i}\,\theta$. From the initial data one gets $u(x,0) + \mathrm{i}\,\theta(x,0)$ which represents a curve $\gamma$ in the complex plane parameterized by $x$. On the other hand, at $t = 0$ one has $\eta(u,\theta) = x(u,\theta)$ and $\xi(u,\theta) = 0$ defining the values of $\eta$ and $\xi$ along $\gamma$. Therefore, to solve the initial value problem is equivalent to extend analytically a complex variable function with values given along a certain curve $\gamma$. We do not know how to do this in general, but in [3] we were able to construct solutions developing singularities in finite time by using this technique. Let us consider the example:

$$f(z) = \ln z \ . \tag{2.17}$$

By writing $z = re^{\mathrm{i}\,\varphi}$ we have

$$f(z) = \ln r + \mathrm{i}\,\varphi = \ln\sqrt{u^2+\theta^2} + \mathrm{i}\arctan\frac{\theta}{u} \ . \tag{2.18}$$

The real part of $f(z)$ is zero along the circumference of radius 1: $\gamma = \{(u,\theta) : u^2+\theta^2 = 1\}$. Parameterizing $\gamma$ in the form $(u,\theta) = (\cos\varphi, \sin\varphi)$ one gets $\eta = \mathrm{Im} f(z) = \varphi$. Since along $\gamma$ one has $\eta(u,\theta) = -x(u,\theta)$ it follows that $\varphi = -x$ which yields the following initial data for $z$ :

$$z(x,0) = \cos x - \mathrm{i}\sin x \ . \tag{2.19}$$

This initial data is compatible with (2.9), since $H(\sin x) = -\cos x$. From (2.18) and the definition of $\eta$ and $\xi$ it follows

$$-t\theta = \ln\sqrt{u^2+\theta^2}, \tag{2.20}$$

$$-(x - tu) = \arctan\frac{\theta}{u}, \tag{2.21}$$

which define implicitly the real and imaginary parts $(u(x,t), \theta(x,t))$ of the solution at any given $(x,t)$. From (2.21) one can get

$$\theta = -u\tan(x - tu) \tag{2.22}$$

which inserted in (2.20) yields

$$tu\tan(x-tu)=\ln\left|\frac{u}{\cos(x-tu)}\right| \ . \tag{2.23}$$

Expression (2.23) defines $u(x,t)$ implicitly. Notice that $u(x,0)=\cos x$ satisfies (2.23). If we fix our attention to points in a neighborhood of $x=\frac{\pi}{2}$ it is simple to show from (2.23) that $u(x,t)$ develops cusps at finite time. Writing

$$u\left(\frac{\pi}{2}+\delta x,t\right)\simeq A(t)\delta x \ , \tag{2.24}$$

inserting this into (2.23) and letting $\delta x\to 0$ it follows

$$-tA(t)\frac{1}{1-tA(t)}=\ln\left|\frac{A(t)}{1-tA(t)}\right| \ . \tag{2.25}$$

It is easy to show that $A(t)$, defined implicitly by (2.25) in such a way that $A(0)=-1$ (notice that $u_x(\frac{\pi}{2},0)=-\sin\frac{\pi}{2}=-1$ ), decreases for $t>0$ and blows-up to $-\infty$ at $t=e^{-1}\simeq 0.36788$. Hence, our conclusion is that $u_x(\frac{\pi}{2},t)$ blows-up at finite time.

**3. Idea of the proof of Theorem 1.2.** The result obtained in [6] for symmetric initial data $\theta_0(x)$ and extended recently to general compactly supported positive initial data shows that the solutions to (1.5) develop singularities in finite time. The main idea of the proof is the use of the following inequality:

$$-\int_{-\infty}^{\infty}\frac{f_x(x)\left[(Hf)(x)-(Hf)(0)\right]}{|x|^{1+\delta}}\,\mathrm{d}x\geq C_\delta\int_{-\infty}^{\infty}\frac{(f(x)-f(0))^2}{|x|^{2+\delta}}\,\mathrm{d}x \ , \ 0<\delta<1. \tag{3.1}$$

The proof of (3.1) holds for any function $f(x)$ which is either nonnegative or nonpositive. It is based in a decomposition of $f(x)$ into a sum of a symmetric and an antisymmetric function. Each of them can be estimated by means of Mellin transforms. In the symmetric case, the proof was developed in [6]. The recent work [7] extends the result to general $f's$.

For the sake of simplicity, we will focus on symmetric and positive $\theta_0(x)$ with its support included in $[-L,L]$. In this case, since

$$(H\theta_0)(L)=\frac{1}{\pi}\int_{-L}^{L}\frac{\theta_0(y)}{L-y}\,\mathrm{d}y=\frac{2L}{\pi}\int_0^L\frac{\theta_0(y)}{L^2-y^2}\,\mathrm{d}y\geq 0 \tag{3.2}$$

and analogously $(H\theta_0)(-L)\leq 0$, we conclude that the support of $\theta(x,t)$ shrinks and is, in fact, always included in $[-L,L]$.

Next, from (3.1) it follows

$$(1-\theta)_t=-H(1-\theta)(1-\theta)_x \ . \tag{3.3}$$

Dividing this last expression by $|x|^{1+\delta}$ with $0<\delta<1$ and integrating in $[0,L]$ we obtain:

$$\frac{d}{\mathrm{d}t}\left(\int_0^L\frac{(1-\theta)}{x^{1+\delta}}\,\mathrm{d}x\right)=-\int_0^L\frac{(1-\theta)_xH(1-\theta)}{x^{1+\delta}}\,\mathrm{d}x. \tag{3.4}$$

The fact that $\theta$ cancels outside $[-L,L]$ allows us to write

$$-\int_0^L\frac{(1-\theta)_xH(1-\theta)}{x^{1+\delta}}\,\mathrm{d}x=-\int_0^\infty\frac{(1-\theta)_xH(1-\theta)}{x^{1+\delta}}\,\mathrm{d}x \ . \tag{3.5}$$

Using (3.1) we can estimate,

$$-\int_0^\infty \frac{(1-\theta)_x H(1-\theta)}{x^{1+\delta}}\,\mathrm{d}x \geq C_\delta \int_0^\infty \frac{(1-\theta(x,t))^2}{x^{2+\delta}}\,\mathrm{d}x \tag{3.6}$$

and, consequently,

$$\frac{d}{\mathrm{d}t}\int_0^L \frac{(1-\theta)}{x^{1+\delta}}\,\mathrm{d}x \geq C_\delta \int_0^\infty \frac{(1-\theta)^2}{x^{2+\delta}}\,\mathrm{d}x \geq C_{L,\delta}\left(\int_0^L \frac{(1-\theta)}{x^{1+\delta}}\,\mathrm{d}x\right)^2 \tag{3.7}$$

which implies blow up of $\int_0^L \frac{(1-\theta)}{x^{1+\delta}}\,\mathrm{d}x$ in finite time. This implies, in particular, blow up of $\theta_x$ in finite time since

$$\int_0^L \frac{(1-\theta)}{x^{1+\delta}}\,\mathrm{d}x \leq \sup_x \frac{1-\theta}{x}\int_0^L \frac{\mathrm{d}x}{x^\delta} \leq \frac{L^{1-\delta}}{1-\delta}\sup_x|\theta_x|\,. \tag{3.8}$$

**4. Global regularity for the viscous versions of (1.3) and** (1.5)**.** The addition of higher order (viscous) terms in the equations might prevent the appearance of singularities in the solutions. This might be the case, for instance, of Navier-Stokes equations with respect to Euler equations. It is then natural to study the global regularity, in the context of our nonlocal transport equations, of the solutions to the following equations

$$\begin{aligned} \theta_t + (\theta H(\theta))_x &= -\nu\Lambda^\alpha\theta\ , \\ \theta_0(x) &= \theta(x,0), \end{aligned} \tag{4.1}$$

$$\begin{aligned} \theta_t - (H\theta)\,\theta_x &= -\nu\Lambda^\alpha\theta, \\ \theta_0(x) &= \theta(x,0), \end{aligned} \tag{4.2}$$

where $\Lambda^\alpha\theta = (-\Delta)^{\frac{\alpha}{2}}\theta$.

In [3] the global existence for solutions of equation (4.1) was considered and we obtained the following partial result:

Theorem 4.1. *If $\alpha = 1$ and the initial data $\theta_0$ verifies $\int_{-\pi}^{\pi}\theta_0(x)\,\mathrm{d}x = 0$, $\|\theta_0\|_{L^\infty} < \nu$ and $\|\Lambda^{\frac{3}{2}}\theta_0\|_{L^2} < \infty$, then there is a classical solution of equation* (4.1) *that satisfies $\theta \in C^1([0,\infty); W^{\frac{3}{2}}([-\pi,\pi]))$ and $\|\theta(\cdot,t)\|_{L^\infty} < \nu$ for every $t \geq 0$.*

With respect to global existence for solutions of equation (4.2) the following theorem was proved in [6]:

Theorem 4.2. *Let $0 \leq \theta_0 \in H^2(R)$, $\nu > 0$ and $\alpha > 1$. Then there exists a constant $C$, depending only on $\theta_0$ and $\nu$, such that for $t \geq 0$:*

1) $\|\Lambda^{\frac{1}{2}}\theta\|_{L^2}(t) \leq C,$

2) $\|\Lambda\theta\|_{L^2}(t) \leq C(1+t),$

3) $\|\Delta\theta\|_{L^2}(t) \leq C\,\mathrm{e}^{Ct^3}\,.$

Whether or not solutions of (4.1) remain smooth for all times when $\alpha > 1$ remains an open problem.

## REFERENCES

[1] G. R. Baker, X. Li, and A. C. *Morlet, Analytic structure of 1D-transport equations with nonlocal fluxes*, Physica D **91** (1996), 349–375.

[2] A. L. Bertozzi and A. J. Majda. *Vorticity and the Mathematical Theory of Incompresible Fluid Flow*, Cambridge University Press, Cambridge, 2002.

[3] D. Chae, A. Córdoba, D. Córdoba and M. A. Fontelos, *Finite time singularities in a 1D model of the quasi-geostrophic equation*, Advances in Mathematics **194**(1) (2005), 203–223.

[4] P. Constantin, A. J. Majda, E. Tabak, *Formation of strong fronts in the 2-D quasigeostrophic thermal active scalar*, Nonlinearity, **7** (1994), 1495–1533.

[5] D. Córdoba, *Nonexistence of simple hyperbolic blow-up for the quasi-geostrophic equation*, Ann. of Math. **148** (1998), 1135–1152.

[6] A. Córdoba, D. Córdoba, M. A. Fontelos, *Formation of singularities for a transport equation with nonlocal velocity*, Ann. of Math. **182**, (2005), 1375–1387.

[7] A. Córdoba, D. Córdoba, M. A. Fontelos, *Integral inequalities for the Hilbert transform applied to a non-local transport equation*, J. Math. Pures Appl., **86**(6) (2006), 529-540.

[8] I. Held, R. Pierrehumbert, S. Garner and K. Swanson, *Surface quasi-geostrophic dynamics*, J. Fluid Mech. **282** (1995), 1–20.

[9] D. W. Moore. *The spontaneous appearance of a singularity in the shape of an evolving vortex sheet*, Proc. Roy. Soc. London Ser. A, **365** (1979), 105-119.

[10] A. Morlet. *Further properties of a continuum of model equations with globally defined flux*, Journal of Mathematical Analysis and Applications **221** (1998), 132–160.

[11] K.Ohkitani and M. Yamada. *Inviscid and inviscid-limit behavior of a surface quasi-geostrophic flow*, Phys. Fluids **9** (1997), 876–882.

[12] J. N. Pandey, *The Hilbert transform of Schwartz distributions and applications*, John Wiley & Sons (1996).

[13] J. Pedlosky, *Geophysical Fluid dynamics*, Springer-Verlag, New York, 1987.

[14] S. Resnick, *Dynamical problems in nonlinear advective partial differential equations*. Ph.D. thesis University of Chicago, Chicago (1995).

[15] E. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton, 1970.

[16] J. Wu, *Inviscid limits and regularity estimates for the solutions of the 2-D dissipative Quasi-geostrophic equations*, Indiana Univ. Math. J. **46**(4) (1997), 1113–1124.