Proceedings of Equadiff-11
2005, pp. 105–114
ISBN 978-80-227-2624-5

HYPERBOLIC EQUATIONS AND SYSTEMS WITH DISCONTINUOUS COEFFICIENTS OR SOURCE TERMS

T. GALLOUËT*

Abstract. This paper is devoted to the study of some nonlinear hyperbolic equations or systems with discontinuous coefficients or with source terms. The common feature of the considered problems is the fact that the jacobian matrix of an associated autonomous system is not diagonalizable in $\mathbb{R}$ for many values of the unknown (leading to linear ill posed problems). However, the nonlinear problems appear to be well posed (at least numerically, in the case of systems) in usual functional spaces, even for discontinuous solutions.

Key words. hyperbolic, system, resonance, discontinuous coefficient, source term

AMS subject classifications. 35K65

1. Introduction. In this paper, we present different problems whose common feature is that they correspond to hyperbolic resonant systems, which one defines below.

Let $p \in \mathbb{N}^\star$ and A be a real $p \times p$ matrix. One considers the following Cauchy problem, where the unknown is the function W from $\mathbb{R} \times \mathbb{R}_+$ to $\mathbb{R}^p$:

$$\begin{aligned} &W_t(x,t) + AW_x(x,t) = 0, && (x,t) \in \mathbb{R} \times \mathbb{R}_+, \\ &W(x,0) = W_0(x), && x \in \mathbb{R}. \end{aligned} \tag{1.1}$$

The notation $(\cdot)_t$ stands for the derivative with respect to t and $(\cdot)_x$ stands for the derivative with respect to x.

If A is diagonalizable in $\mathbb{R}$, the first equation of Problem (1.1) is a linear hyperbolic system and Problem (1.1) has a unique weak solution, whatever is W_0 in $L^\infty(\mathbb{R}, \mathbb{R}^p)$ (the Lebesgue space of essentially bounded functions on $\mathbb{R}$,with values in $\mathbb{R}^p$).

If the matrix A has only real eigenvalues but is not diagonalizable, the first equation of Problem (1.1) is said to be a "linear hyperbolic resonant system". In this case, Problem (1.2) is ill posed in the sense that if $W_0 \in L^\infty(\mathbb{R}, \mathbb{R}^p)$ it has, in general, no weak solution $W \in L^\infty(\mathbb{R} \times \mathbb{R}_+, \mathbb{R}^p)$ (however, Problem (1.1) is well posed in C^∞ since it has a unique solution in $C^\infty(\mathbb{R} \times \mathbb{R}_+, \mathbb{R}^p)$ if the initial datum W_0 belongs to $C^\infty(\mathbb{R}, \mathbb{R}^p)$). This ill posedness is due to the fact that there is a lack of regularity between $W(\cdot, t)$ (for $t > 0$) and W_0. For instance, the Riemann problem, that is Problem (1.1) with $W_0(x) = W_l$ for $x < 0$ and $W_0(x) = W_r$ for $x > 0$ (and W_l, $W_r \in \mathbb{R}^p$), does not have a weak solution in $L^\infty(\mathbb{R} \times \mathbb{R}_+, \mathbb{R}^p)$ (except for very particular choices of W_0), but it has a solution in a greater space. In the case $p = 2$, it has a (unique) solution in a space allowing $W(\cdot, t)$ to be, for $t > 0$, a measure on the bounded sets of $\mathbb{R}$, see Section 2 below (in the case $p \geq 3$, the solution $W(\cdot, t)$ may even be less regular).

One considers now that the matrix A in (1.1) is depending on W, leading to the following nonlinear system:

$$\begin{aligned} &W_t(x,t) + A(W(x,t))W_x(x,t) = 0, && (x,t) \in \mathbb{R} \times \mathbb{R}_+, \\ &W(x,0) = W_0(x), && x \in \mathbb{R}. \end{aligned} \tag{1.2}$$

*Universite de Marseille, 39 rue Joliot Curie, 13453 Marseille Cedex 13, France (Thierry.Gallouet@cmi.univ-mrs.fr)

The unknown W is supposed to take values in an admissible set $D \subset \mathbb{R}^p$.

If the matrix $A(w)$ is diagonalizable in $\mathbb{R}$ for all $w \in D$, the first equation of Problem (1.2) is a nonlinear hyperbolic system and Problem (1.2) is expected to be well posed, in a convenient sense (including, for instance, entropy conditions). This well posedness could be suggested by the fact that the linear problem (1.1) with $A = A(w)$ is well posed for any $w \in D$.

Assume now that there exists $R \subset D$, $R \neq \emptyset$, such that the matrix $A(w)$ is diagonalizable in $\mathbb{R}$ for all $w \in D \setminus R$ and has only real eigenvalues but is not diagonalizable if $w \in R$. Then, the first equation of Problem (1.2) is said to be a "nonlinear hyperbolic resonant system". The linear problem (1.1) is ill posed (in L^∞) if $A = A(w)$, for any $w \in R$ (since it corresponds to a linear hyperbolic resonant system). One presents below two examples where Problem (1.2) has a unique solution in L^∞ (in a convenient sense) for a large class of initial data in L^∞, including cases where, for instance, $W_0(x) \in R$ for all $x \in \mathbb{R}$. Indeed, in the first example (two phase flow in an heterogeneous porous medium) one has existence and uniqueness of solution in a convenient sense (weak entropy solution). For the second example (Saint Venant Equations with topography), one only shows the good behavior of some numerical schemes (even if these numerical schemes use, for the computation of numerical fluxes, the resolution of the Riemann problem for some linear hyperbolic resonant systems).

One considers in this paper that the space variable x belongs to $\mathbb{R}$, but extensions to $x \in \mathbb{R}^d$, $d = 2$ or 3, are possibles.

In Section 2, the solution of the Riemann problem of a linear resonant system is given (for $p = 2$), along with a first example of an academic nonlinear resonant system. In Sections 3–5, examples of nonlinear resonant systems (coming from some models in fluid mechanic) are given. Then, in Section 6 are presented some numerical schemes and, in Section 7, some numerical results.

2. Linear resonant systems. Let $p = 2$ and A be a real 2×2 matrix which has only real eigenvalues but is not diagonalizable. Then, using a change of unknown, the Riemann problem for the linear problem (1.1) can be put under the following form, with some $\lambda \in \mathbb{R}$ (which is the unique eigenvalue of A):

$$\begin{bmatrix} u \\ v \end{bmatrix}_t + \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}_x = 0,$$

$$\begin{bmatrix} u(x,0) \\ v(x,0) \end{bmatrix} = \begin{bmatrix} u_l \\ v_l \end{bmatrix}, \text{ if } x < 0, \text{ and } \begin{bmatrix} u_r \\ v_r \end{bmatrix}, \text{ if } x > 0,$$

with u_l, u_r, v_l, $v_r \in \mathbb{R}$. The second equation of the system and the second initial condition are decoupled from the first ones. Then, the unique weak solution for v (uniqueness holds even in the larger possible space of distributions) is $v(\cdot,t) = v(\cdot - \lambda t, 0)$ for all $t > 0$. It is now possible to give the solution for u (which is also unique in in the larger possible space of distributions), it is, for all $t > 0$:

$$u(\cdot,t) = u_l 1_{\{x \in \mathbb{R},\, x < \lambda t\}} + u_r 1_{\{x \in \mathbb{R},\, x > \lambda t\}} + t(v_l - v_r)\delta_{\lambda t},$$

where 1_B is the characteristic function of B, for $B \subset \mathbb{R}$, and δ_a is the Dirac mass at point a, for $a \in \mathbb{R}$. In this example, the problem has no solution in $L^\infty(\mathbb{R} \times \mathbb{R}_+, \mathbb{R}^2)$ but it has a unique weak solution in a space allowing $u(\cdot,t)$ to be a measure on the bounded sets of $\mathbb{R}$.

If $p > 2$, the (unique) solution of the Riemann problem for a linear hyperbolic resonant system may be even less regular. Indeed, the regularity of the solution depends on the difference between the algebraic and the geometric multiplicity of the eigenvalues.

The Riemann problem for a nonlinear hyperbolic resonant system is also often ill posed, as in the following academic simple example:

$$\begin{aligned} &u_t + (au)_x = 0, \\ &a_t = 0, \end{aligned}$$

$$\begin{bmatrix} u(x,0) \\ a(x,0) \end{bmatrix} = \begin{bmatrix} u_l \\ a_l \end{bmatrix}, \text{ if } x < 0, \text{ and } \begin{bmatrix} u_r \\ a_r \end{bmatrix}, \text{ if } x > 0, \tag{2.1}$$

Problem (2.1) has no weak solution in $L^\infty(\mathbb{R} \times \mathbb{R}_+, \mathbb{R}^2)$ if $a_l > 0$, $a_r < 0$ and $a_l u_l \neq a_r u_r$ and has infinitely many solution if $a_l < 0$ and $a_r > 0$. See [4] for the study of such problems.

Problem (2.1) correspond to a nonlinear hyperbolic resonant system since the system is equivalent (for regular solution) to:

$$\begin{bmatrix} u \\ a \end{bmatrix}_t + \begin{bmatrix} a & u \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ a \end{bmatrix}_x = 0.$$

Then, Resonance occurs, for this system, when $a = 0$ and $u \neq 0$ and, as it is said before, the Riemann problem for this nonlinear system is ill posed in $L^\infty(\mathbb{R} \times \mathbb{R}_+, \mathbb{R}^2)$ provided that 0 is between a_r and a_l (except for some particular data).

3. Hyperbolic equation with a discontinuous coefficient. The first example of a nonlinear resonant system which leads to a well posed problem in L^∞, described in this section, is given by a two phase flow in an heterogeneous porous medium, considering only gravity effect (without capillarity and with a total flux equal to zero). The unknown is the saturation, which is a function $u : \mathbb{R}\times\mathbb{R}_+ \to [0,1] \subset \mathbb{R}$. The equation is (forgetting the variable (x,t)):

$$u_t + (kg(u))_x = 0, \text{ in } \mathbb{R} \times \mathbb{R}_+, \tag{3.1}$$

where $k(x) = k_l$, for $x < 0$, and $k(x) = k_r$, for $x > 0$, $k_l, k_r > 0$, $k_l \neq k_r$, the function $g : [0,1] \to \mathbb{R}$ is Lipschitz continuous, nonnegative and such that $g(0) = g(1) = 0$. A typical example, studied in [11], is $g(u) = u(1-u)$.

This hyperbolic equation with a discontinuous coefficient can be viewed has a conservative 2×2 system, adding k has an unknown and the equation $k_t = 0$:

$$\begin{aligned} &u_t + (kg(u))_x = 0, \\ &k_t = 0. \end{aligned}$$

Then, with $W = \begin{bmatrix} u \\ k \end{bmatrix}$ and $F(W) = \begin{bmatrix} kg(u) \\ 0 \end{bmatrix}$, this system is:

$$W_t + (F(W))_x = 0,$$

or equivalently (for regular solutions), with $A(W) = DF(W)$:

$$W_t + A(W)W_x = 0.$$

This leads to problem (1.2) with $p=2$, $W=\begin{bmatrix} u \\ k \end{bmatrix}$, $A(W)=\begin{bmatrix} kg'(u) & g(u) \\ 0 & 0 \end{bmatrix}$.

The admissibility domain is $D=\{(u,k)^t,\ u\in[0,1],\ k>0\}$. Assuming that $g\in C^1$, let $R=\{(u,k)^t\in D,\ g'(u)=0,\ g(u)\neq 0\}$. The matrix $A(w)$ is diagonalizable in $\mathbb{R}$ for $w=(u,k)^t\in D\setminus R$ and has only 0 as eigenvalue but is not diagonalizable if $w\in R$. In the case $g(u)=u(1-u)$, $R=\{1/2\}\times\mathbb{R}_+^\star$. But the domain R corresponding to resonance may be larger. In the case corresponding to Figure 7, R contains $(1/4,3/4)\times\mathbb{R}_+^\star$.

Despite this resonance phenomenon, it is possible to prove existence and uniqueness of an "entropy weak solution" of (3.1) with an initial condition u_0, provided that $u_0\in L^\infty(\mathbb{R})$ takes its values in $[0,1]$. Actually, it is proven in [3] (previous partial results were, for instance, in [11], [8] and [1]) that there exists a unique solution of the following weak entropic formulation of (3.1) with the initial condition u_0:

$$
\begin{aligned}
&u\in L^\infty(\mathbb{R}_+\times\mathbb{R}),\ 0\le u\le 1 \text{ a.e.},\\
&\int_{\mathbb{R}_+}\int_{\mathbb{R}}[|u(x,t)-\kappa|\varphi_t(x,t)+k(x)\phi(u(x,t),\kappa)\varphi_x(x,t)]\,\mathrm{d}x\,\mathrm{d}t\\
&+\int_{\mathbb{R}}|u_0(x)-\kappa|\varphi(x,0)\,\mathrm{d}x+|k_r-k_l|\int_{\mathbb{R}_+}g(\kappa)\varphi(0,t)\,\mathrm{d}t\ge 0,\\
&\hspace{6cm}\forall\kappa\in[0,1],\ \forall\varphi\in C_c^\infty(\mathbb{R}\times\mathbb{R}_+,\mathbb{R}_+),
\end{aligned}
$$

where $\phi(s,\kappa)=\mathrm{sign}(s-k)(g(s)-g(\kappa))$ for $s\in[0,1]$. This definition of entropy weak solution was previously given in [13]. A crucial property, in the proof, is that the constant functions 0 and 1 are solutions of (3.1). Using this same property, a similar result of existence and uniqueness was proven recently in [2] when $k(x)g(u)$ is replaced by $g(x,u)$ provided that $g(\cdot,0)$ and $g(\cdot,1)$ are some constants functions.

4. Hyperbolic system with a source term. The second example of a nonlinear hyperbolic resonant system come from the modelization of shallow water. Considering a non-flat bottom, a classical model is obtained with the nonlinear hyperbolic system of Saint Venant Equations with a source term coming form the topography, which is given by a known function z, Lipschitz continuous, from $\mathbb{R}$ to $\mathbb{R}$ (actually, this model is probably not correct for a discontinuous topography, that is a discontinuous function z). The unknowns, for this model, are: $h,u\ :\ \mathbb{R}\times\mathbb{R}_+\to\mathbb{R}$ (with $h>0$) and the model reads:

$$
\begin{aligned}
h_t+(hu)_x&=0,\\
(hu)_t+(hu^2+\frac{1}{2}gh^2)_x&=-ghz_x,
\end{aligned}
\tag{4.1}
$$

where g is the gravity constant.

This 2×2 conservative hyperbolic system with a source term can be viewed has a nonconservative 3×3 hyperbolic resonant system, adding z has an unknown and the equation $z_t=0$, namely:

$$
W_t+(F(W))_x+B(W)W_x=0,
\tag{4.2}
$$

where $W=\begin{bmatrix} h \\ hu \\ z \end{bmatrix}$, $F(W)=\begin{bmatrix} hu \\ \frac{(hu)^2}{h}+\frac{1}{2}gh^2 \\ 0 \end{bmatrix}$ and $B(W)=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & gh \\ 0 & 0 & 0 \end{bmatrix}$.

Setting $A(W)=DF(W)+B(W)=\begin{bmatrix} 0 & 1 & 0 \\ -u^2+gh & 2u & gh \\ 0 & 0 & 0 \end{bmatrix}$, this system is equivalent (for regular solutions) to:

$$W_t + A(W)W_x = 0, \tag{4.3}$$

which is a hyperbolic resonant system since the eigenvalues of $A(W)$ are $u \pm c$ and 0, with $c = \sqrt{gh}$, and $A(W)$ is not diagonalizable if $u - c = 0$ or $u + c = 0$ (and $h > 0$). Then, for this example, the admissible domain is $D = \{(h, q, z)^t,\ h > 0\}$ and the resonant domain is $R = \{(h, q, z)^t,\ u + c = 0 \text{ or } u - c = 0\}$, with $u = q/h$.

Following [5], the Riemann problem associated to System (4.2) (or (4.3)) has, in general, a unique solution composed of constant states and waves, satisfying a classical entropy condition and assuming continuity of (the Riemann invariants) hu and ψ at the contact discontinuity (at $x = 0$) with $\psi = \frac{1}{2}u^2 + g(h + z)$ (which is the natural condition at the contact discontinuity). Indeed, the Riemann problem has for some very particular choice of the initial condition, three solutions. We probably need some more entropy criterion in order to choose between these 3 solutions. Note however that these 3 solutions are in L^∞, there is not the lack of regularity described for linear resonant systems. For a study of nonlinear hyperbolic resonant systems, one refers to [7].

Concerning the computation of the solution of the Cauchy problem associated to (4.2) (equivalent to (4.1)) with a Lipschitz continuous topography (which is the case of interest), using the resolution of the Riemann problem associated to (4.2), it is then possible to design a numerical scheme (which is the Godunov scheme for the conservative part of this system). It is also possible to design schemes using a linearized version of the Riemann problem (see Section 6). In both cases, the numerical solution seems to converge, as the discretization steps go to 0, toward a unique solution of the Cauchy problem, whatever is the initial condition for h and u, with $h > 0$. (In particular, this solution does not seem to depend of the choice of the solution of the Riemann problem for the exceptional cases where this solution is not unique.)

5. Euler Equations with a particular EOS. One quotes here an ongoing work of E. Godlewki and N. Seguin. It concerns the isentropic Euler equations with an EOS taking into account a simple model of "phase transition", that is:

$$\rho_t + (\rho u)_x = 0,$$
$$(\rho u)_t + (\rho u^2 + p)_x = 0$$

and the EOS is given by:

$$\begin{aligned} p &= a_1\rho, && \text{if } 0 < \rho < \rho_1,\\ p &= a_1\rho_1, && \text{if } \rho_1 \le \rho \le \rho_2,\\ p &= a_2\rho, && \text{if } \rho_2 < \rho, \end{aligned}$$

where ρ_1, ρ_2, a_1, a_2 are given constants, $0 < \rho_1 < \rho_2$, $0 < a_1$, $a_1\rho_1 = a_2\rho_2$.

For $\rho_1 \le \rho \le \rho_2$ and any u, the jacobian matrix of this system has u as unique eigenvalue and is not diagonalizable (and the 2 genuinely nonlinear fields lead to a linearly degenerate field). Then, $D = \{(\rho, q)^t,\ \rho > 0\}$ and the resonant domain is $R = \{(\rho, q)^t,\ \rho_1 \le \rho \le \rho_2\}$. For this system, the Riemann problem has a unique solution composed of constant states and waves, assuming convenient entropy conditions, see [10].

6. Discretization by Finite Volume Schemes. In this section, one considers the discretization of the Cauchy problems described in Sections 3–4, the general form of which is:

$$\begin{aligned} W_t + (F(W))_x + B(W)W_x = 0, \\ W(\cdot,0) = W_0. \end{aligned} \tag{6.1}$$

The time and space steps are denoted by δt and δx. For simplicity, they are assumed to be constant. Let $t_n = n\delta t$ and $x_{i+1/2} = (i+1/2)h$ for $n \in \mathbb{N}$ and $i \in \mathbb{Z}$. The approximate solution is defined by the family $\{W_i^n,\, i \in \mathbb{Z},\, n \in \mathbb{N}\} \subset \mathbb{R}$, where W_n^i is the value of the approximate solution at time for $t \in (t_n, t_{n+1})$ and in the control volume $M_i = (x_{i-1/2}, x_{i+1/2})$.

The initial condition is used to compute $\{W_i^0,\, i \in \mathbb{Z}\}$:

$$W_i^0 = \frac{1}{\delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} W_0(x)\,\mathrm{d}x, \text{ for } i \in \mathbb{Z}. \tag{6.2}$$

One describes now two possibilities for the computation of $\{W_i^{n+1},\, i \in \mathbb{Z}\}$ using $\{W_i^n,$ $i \in \mathbb{Z}\}$. The first one uses the resolution of the Riemann problem associated to (6.1), it is a generalization of the Godunov scheme for a nonconservative system, the second one uses a linearized Riemann problem.

6.1. Godunov scheme for a nonconservative system. Let W_l, $W_r \in \mathbb{R}^p$. The solution of the Riemann problem for W_l and W_r (that is the solution of (6.1) with $W(x,0) = W_l$ if $x < 0$ and $W(x,0) = W_r$ if $x > 0$) is a self similar function. It is denoted by $W(x,t) = R(x/t, W_l, W_r)$ and one sets $W^{\star,\pm}(W_l, W_r) = R(0^\pm, W_l, W_r)$. The values $W^{\star,\pm}(W_l, W_r)$ are always well defined, even if $\{x = 0\}$ is a line of discontinuity for W. Note that in the examples given in Sections 3-4, it is possible to compute this solution (actually, in the case of Section 4, as we said before, this solution is sometimes not unique).

Then, the Godunov scheme for a nonconservative system is defined by:

$$\frac{W_i^{n+1} - W_i^n}{k} + F_{i+\frac{1}{2}}^{n,-} - F_{i-\frac{1}{2}}^{n,+} + B(W_i^n)(W_{i+\frac{1}{2}}^{n,-} - W_{i-\frac{1}{2}}^{n,+}) = 0,\ i \in \mathbb{Z},\ n \in \mathbb{N},$$

with $F_{i+\frac{1}{2}}^{n,\pm} = F(W_{i+\frac{1}{2}}^{n,\pm})$ and $W_{i+\frac{1}{2}}^{n,\pm} = W^{\star,\pm}(W_i^n, W_{i+1}^n)$.

This scheme is very efficient. It uses, as usual for an explicit scheme, a CFL condition which reads $\delta t \le C\delta x$ where C is computed with the eigenvalues of $A(W) = DF(W) + B(W)$. It is sometimes too expansive and it is the reason of the introduction, in the following section, of a modified scheme, using a linearized Riemann problem.

In the case of a conservative system, namely $B(W) = 0$ for all W, the scheme is the classical Godunov scheme and one has $F_{i+1/2}^{n,+} = F_{i+1/2}^{n,-}$, even if $W_{i+1/2}^{n,+} \neq W_{i+1/2}^{n,-}$, thanks to the Rankine-Hugoniot condition for the solution of the Riemann problem.

In the case of Saint Venant Equations with topography, described in Section 4, one has $z_{i+1/2}^{n,-} = z_{i-1/2}^{n,+} = z_i^n$ and then $B(W_i^n)(W_{i+1/2}^{n,-} - W_{i-1/2}^{n,+}) = 0$. The nonconservativity of the system (that is the source term in (4.1)) appears only in the fact that, generally, $F_{i+1/2}^{n,+} \neq F_{i+1/2}^{n,-}$.

6.2. VFRoe-ncv scheme. The objective is still the discretization of Problem (6.1). Recall that the unknown W is a function from $\mathbb{R} \times \mathbb{R}_+$ to $D \subset \mathbb{R}^p$, where D is the so-called admissible domain (The domain D is, for instance, $\mathbb{R}_+^\star \times \mathbb{R} \times \mathbb{R}$ for the system studied in Section 4). One sets $A(w) = DF(w) + B(w)$ for $w \in D$, where $DF(w)$ is the jacobian matrix of F at point $w \in D$. The scheme presented in this section uses the resolution of linear Riemann problems.

Let ϕ be a regular function of $D \subset \mathbb{R}^p$ to $\mathbb{R}^p$. It is not necessary to assume that ϕ is one-to-one from D to $Ra(\phi) = \{\phi(w),\ w \in D\}$. It is sufficient to assume the two following properties for ϕ, where $M_p(\mathbb{R})$ is the set of $p \times p$ matrix with real values and $D\phi(w)$ is the jacobian matrix of ϕ at point $w \in D$:

1. There exists $C \in C(D, M_p(\mathbb{R}))$ such that $D\phi(w)A(w) = C(w)D\phi(w)$, for all $w \in D$.
2. There exist $\tilde{F} \in C(Ra(\phi), \mathbb{R}^p)$ and $\tilde{B} \in C(D, M_p(\mathbb{R}))$ such that $F(w) = \tilde{F}(\phi(w))$ and $B(w)=\tilde{B}(w)D\phi(w)$ for all $w \in D$.

Indeed, the first property on ϕ (existence of the matrix $C(w)$) seems necessary. It allows the definition of the linear Riemann problem (6.4). The second property is perhaps not necessary, but without this hypothesis the numerical flux of the scheme may be not uniquely defined and this may lead to some troubles.

Let $W\ :\ \mathbb{R} \times \mathbb{R}_+ \to D$ be a regular solution of $W_t + A(W)W_x = 0$. Then, $Y = \phi(W)$ satisfy $Y_t + D\phi(W)A(W)W_x = 0$ and, thanks to the first hypothesis on ϕ, the function Y satisfies:

$$Y_t + C(W)Y_x = 0. \tag{6.3}$$

It is now possible to describes the VFRoe-ncv scheme associated to ϕ. For $w_l,\ w_r \in \mathbb{R}^p$, one sets $w_{l,r} = (w_r + w_l)/2$ (it is possible to take another mean value between w_l and w_r) and considers the following linear Riemann problem:

$$\begin{aligned} &Y_t + C(w_{l,r})Y_x = 0, \\ &Y(x,0) = \begin{cases} y_l = \phi(w_l) & \text{if } x < 0, \\ y_r = \phi(w_r) & \text{if } x > 0. \end{cases} \end{aligned} \tag{6.4}$$

If $C(w_{l,r})$ is diagonalizable in $\mathbb{R}$, Problem (6.4) has a unique solution. It is a self similar function: $Y(x,t) = LR(\frac{x}{t}, y_l, y_r)$. Then one sets:

$$y^{\star,\pm}(w_l, w_r) = LR(0^{\pm}, y_l, y_r).$$

If $C(w_{l,r})$ has only real eigenvalues but is not diagonalizable in $\mathbb{R}$, the first equation of (6.4) is a linear hyperbolic resonant system. In this case, Problem (6.4) has also a unique solution but it is not, in general, a function (see Section 2). However, $LR(0^{\pm}, y_l, y_r)$ is always well defined and it is also possible to set $y^{\star,\pm}(w_l, w_r) = LR(0^{\pm}, y_l, y_r)$.

The VFRoe-ncv scheme associated to ϕ is (6.2) and:

$$\frac{W_i^{n+1} - W_i^n}{k} + F^{n,-}_{i+\frac{1}{2}} - F^{n,+}_{i-\frac{1}{2}} + \tilde{B}(W_i^n)D\phi(W_i^n)(Y^{n,-}_{i+\frac{1}{2}} - Y^{n,+}_{i-\frac{1}{2}}) = 0,\ i \in \mathbb{Z},\ n \in \mathbb{N},$$

with $F^{n,\pm}_{i+\frac{1}{2}} = \tilde{F}(Y^{n,\pm}_{i+\frac{1}{2}})$ (assuming that $Y^{n,\pm}_{i+\frac{1}{2}} \in Ra(\phi)$), $Y^{n,\pm}_{i+\frac{1}{2}} = y^{\star,\pm}(W_i^n, W_{i+1}^n)$.

In the case of a conservative system, namely $B(W) = 0$ for all W, one expects to have a conservative scheme, that is $F^{n,+}_{i+1/2} = F^{n,-}_{i+1/2}$. Unfortunately, this is not necessarily the case when $Y^{n,+}_{i+1/2} \neq Y^{n,-}_{i+1/2}$ (which is, in general, the case when 0 is an eigenvalue of $C(w)$ for $w = (W_i^n + W_{i+1}^n)/2$) since in this case it is possible to have $\tilde{F}(Y^{n,+}_{i+1/2}) \neq \tilde{F}(Y^{n,-}_{i+1/2})$. Then, the scheme has to be slightly modified . A possible modification is to take $F^{n,\pm}_{i+1/2} = (\tilde{F}(Y^{n,+}_{i+1/2}) + \tilde{F}(Y^{n,-}_{i+1/2}))/2)$. Therefore, a possible drawback of the method seems to be the fact that the numerical flux of the scheme is not a continuous function of its arguments when an eigenvalue change sign (namely, $F^{n,\pm}_{i+1/2}$ does not depends continuously of W^n_i and W^n_{i+1}). In practice, this drawback does not seem to be so important. In the case of a nonconservative system, a similar modification is sometimes necessary for some components of the numerical flux, namely those corresponding to a conservation law (without source term). This is case, for instance, for the first component of the system of Section 4 when using $\phi(h,q,z) = (2\sqrt{gh}, q/h, z)$ described below.

As for the preceding scheme, the scheme uses a CFL condition which reads $\delta t \le C\delta x$ where C is computed with the eigenvalues of $A(W)$.

One describes now some possible choices of the function ϕ in the cases presented in Sections 3–4.

In the case studied in Section 3 (two phase flow in a porous medium), for $w = (u,k)^t \in D = [0,1] \times \mathbb{R}^\star_+$, one has $F(w) = (kg(u), 0)^t$ and $B(w) = 0$ (the system is a conservative one). A simple choice of ϕ is $\phi(w) = (kg(u), k)^t$ for $w = (u,k)^t$. With this choice of ϕ, the matrix $C(w)$ of the linearized system (6.3) is, for $w = (u,k)^t$:

$$C(w) = \begin{bmatrix} kg'(u) & 0 \\ 0 & 0 \end{bmatrix}.$$

Then, for any $w \in D$, System (6.3) is not a resonant system, it is a linear hyperbolic system since the eigenvalues are $\lambda_1 = kg'(u)$ and $\lambda_2 = 0$ and two independent eigenvectors are $e_1 = (1,0)^t$ and $e_2 = (0,1)^t$.

In the case studied in Section 4 (Saint Venant with topography), the unknown takes values in $D = \{w = (h,q,z)^t \in \mathbb{R}^3,\ h > 0\}$ and the system is a nonconservative one. Two possible choices of ϕ are of particular interest, the first one is, for $w = (h,q,z)^t$, setting $u = q/h$ and $\psi = u^2/2 + g(h+z)$, $\phi(w) = (q,\psi,z)^t$. The second choice is, setting $c = \sqrt{gh}$, $\phi(w) = (2c,u,z)^t$.

With the first choice, $\phi(w) = (q,\psi,z)^t$, ϕ is not one-to-one. The matrix $C(w)$ of the linearized system (6.3) is, for $w = (h,q,z)^t$ (and the same definitions of u, ψ, c):

$$C(w) = \begin{bmatrix} u & h & 0 \\ g & u & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then, here also, for any $w \in D$, System (6.3) is not a resonant system, it is a linear hyperbolic system since the eigenvalues are $\lambda_1 = 0$, $\lambda_2 = u + c$ and $\lambda_3 = u - c$ and a basis of $\mathbb{R}^3$ is obtained with the three following eigenvectors:

$e_1 = (0,0,1)^t$, $e_2 = (h,c,0)^t$ and $e_3 = (-h,c,0)^t$.

With the second choice, $\phi(w) = (2c,u,z)^t$, ϕ is one-to-one. The matrix $C(w)$ of the linearized system (6.3) is, for $w = (h,q,z)^t$ (and the same definitions of u, ψ, c):

$$C(w) = \begin{bmatrix} u & c & 0 \\ c & u & g \\ 0 & 0 & 0 \end{bmatrix}.$$

In this case, System (6.3) is a linear hyperbolic system for $w \in D$ except if $u + c = 0$ or $u - c = 0$. When $u + c = 0$ or $u - c = 0$, System (6.3) is a linear hyperbolic resonant system. Indeed, for any $w \in D$, the eigenvalues are $\lambda_1 = 0$, $\lambda_2 = u + c$ and $\lambda_3 = u - c$. For the eigenvectors, three cases are possible:

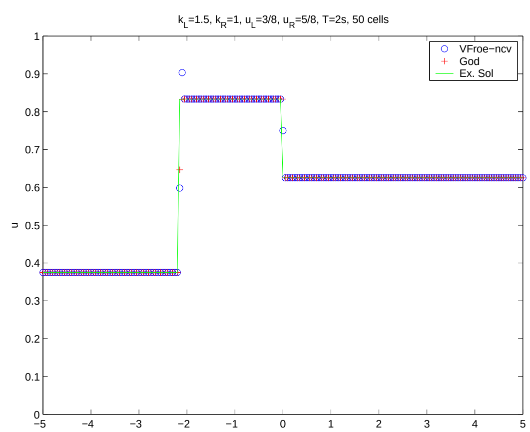
- If $u \pm c \neq 0$, a basis of $\mathbb{R}^3$ is obtained with the 3 eigenvectors:
  $e_1 = (cg, -ug, u^2 - c^2)^t$, $e_2 = (1,1,0)^t$ and $e_3 = (1,-1,0)^t$.

- If $u = c$. One obtains only 2 independent eigenvectors $e_1 = (1,-1,0)^t$ and $e_2 = (1,1,0)^t$.
- If $u = -c$, One obtains also only 2 independent eigenvectors $e_1 = (1,1,0)^t$ and $e_3 = (1,-1,0)^t$.

7. Numerical results. Numerical results are very good with the schemes given in Section 6, for the problems described in Sections 3–4.

For the problem of Saint Venant Equations with topography (Section 4), numerical results with the so-called “Godunov scheme for nonconservative system” described in Section 6 are presented in [5]. Numerical results with the VFRoe-ncv scheme with $\phi(w) = (2c,u,z)^t$, described in Section 6, are presented in [6]. In [6] are also presented the way to take into account boundary conditions and the way to perform a high order scheme. Other efficient schemes are given in [12] and [9].

An interesting question is the preservation, by the numerical scheme, of the steady state solutions of Problem (4.1). Actually, an admissible steady state solution of Problem (4.1) is given by two functions h, u from $\mathbb{R}\times\mathbb{R}_+$ to $\mathbb{R}$ with $h > 0$ and such that $q = hu$ and $\psi = u^2/2 + g(h+z)$ are constant (in space and time). For the Godunov scheme (for nonconservative system) and for the VFRoe-ncv scheme with $\phi(w) = (q,\psi,z)^t$, one has preservation of all the admissible steady state solutions (and not only those with $u = 0$ which are called ”lake at rest”), provided that the discretization of the initial condition is performed in order to have q_i^0 and ψ_i^0 independent of $i \in \mathbb{Z}$. For the VFRoe-ncv scheme with $\phi(w) = (2c,u,z)^t$ (and for other choices of ϕ) one has only preservation of the steady state solutions with $u = 0$. However, the choice of $\phi(w) = (2c,u,z)^t$ is the better choice for the case of “dry bed areas” (corresponding to a vanishing h), see [6].



For the problem of two phase flow in porous media (Section 3), numerical results are presented, for instance, in [11] and [2]. An example is given below for the following case:

$$
\begin{aligned}
&k_l = 1,5,\ k_k = 1, \\
&g(s) = 4s, && \text{if } s \in [0, \tfrac{1}{4}], \\
&g(s) = 1, && \text{if } s \in (\tfrac{1}{4}, \tfrac{3}{4}), \\
&g(s) = 4 - 4s, && \text{if } s \in [\tfrac{3}{4}, 1],
\end{aligned}
$$

For this choice of g, Resonance occurs for all $(k,u)^t \in D = \{(k,u)^t, k > 0\}$ with $u \in (\frac{1}{4}, \frac{3}{4})$. The initial condition is $u_0(x) = 3/8$ for $x < 0$ and $u_0(x) = 5/8$ for $x > 0$, so that $u_0(x) \in R$ for a.e. x.

REFERENCES

[1] F. Bachmann, *Analysis of a scalar conservation law with a flux function with discontinuous coefficients.* Advances in Differential Equations **11–12** (2004), 1317–1338.

[2] F. Bachmann, PhD thesis, université Aix-Marseille 1, and submitted paper.

[3] F. Bachmann and J. Vovelle, *Existence and uniqueness of entropy solution of scalar conservation laws with a flux function involving discontinuous coefficients.* Accepted for publication in CPDE.

[4] F. Bouchut and F. James, *One-dimensional transport equations with discontinuous coefficients.* Nonlinear Analysis, TMA, **32** (1998), 891–933.

[5] A. Chinnayya, A. Y. LeRoux and N. Seguin, *A well-balanced numerical scheme for the approximation of the shallow-water equations with topography: the resonance phenomenon.* International Journal on Finite Volume (electronic), 2004.

[6] T. Gallouët, J. M. Hérard and N. Seguin, *Some approximate Godunov schemes to compute shallow-water equations with topography.* Comput. Fluids, **32**(4) (2003), 479–513.

[7] P. Goatin and P. G. LeFloch, *The Riemann problem for a class of resonant nonlinear systems of balance laws.* Ann. Inst. H. Poincaré – Analyse Non-linéaire **21** (2004), 881–902.

[8] K. H. Karlsen, N. H. Risebro and J. D. Towers, L^1 *stability for entropy solutions of nonlinear degenerate parabolic convection-diffusion equations with discontinuous coefficients.* Sbr. K. Nor. Vidensk. Selsk. **3** (2003), 1–49.

[9] A. Kurganov and D. Levy, *Central-Upwind Schemes for the Saint-Venant System.* Mathematical Modelling and Numerical Analysis, **36** (2002), 397–425.

[10] E. Godlewski and N. Seguin, *The Riemann problem for a simple model of phase transition.* Submitted.

[11] N. Seguin and J. Vovelle, *Analysis and approximation of a scalar conservation law with a flux function with discontinuous coefficients.* Math. Models Methods Appl. Sci., **13**(2) (2003), 221–257.

[12] E. Audusse, F. Bouchut, M.-O. Bristeau, R. Klein and B. Perthame, *A fast and stable well-balanced scheme with hydrostatic reconstruction for shallow water flows.* SIAM J. Sci. Comp., **25** (2004), 2050–2065.

[13] J. D. Towers, *Convergence of a difference scheme for conservation laws with a discontinuous flux.* SIAM J. Numer. Anal., **38**(2) (2000), 681–698.