Proceedings of Equadiff-11
2005, pp. 123–133
ISBN 978-80-227-2624-5

ON EXISTENCE AND SCHAUDER REGULARITY OF SOLUTIONS TO A CLASS OF GENERALIZED STATIONARY STOKES PROBLEMS*

NGUYEN DUC HUY† AND JANA STARÁ‡

Abstract. We investigate existence and Schauder type estimates of weak solutions to a class of generalized Stokes problem. The generalization we consider here consists in two points: Laplace operator is replaced by a general second order linear elliptic operator in divergence form and “pressure” gradient ∇p is replaced by a linear operator of first order.

Key words. Generalized Stokes problem, weak solutions, Hilbert regularity up to the boundary, Schauder type estimates

AMS subject classifications. 76D03, 76D07, 35Q30, 35J55

1. Introduction. Let $\Omega \subset \mathbb{R}^d$ $(d \geq 2)$ be a bounded domain with boundary $\partial\Omega$. We study a generalization of linear Stokes problem : For given $f = (f_1, \cdots, f_d) : \Omega \longrightarrow \mathbb{R}^d$ and $g : \Omega \longrightarrow \mathbb{R}$, $A = \left(A_{ij}^{\alpha\beta}\right)_{i,j,\alpha,\beta=1}^{d} : \Omega \longrightarrow \mathbb{R}^{d^2\times d^2}$ and $B = (B_{ij})_{i,j=1}^{d}$ a $d \times d$ constant matrix we look for $u = (u_1, \cdots, u_d) : \Omega \longrightarrow \mathbb{R}^d$ and $p : \Omega \longrightarrow \mathbb{R}$ solving

$$\begin{aligned} -\text{div}\,(A\nabla u) + B\nabla p &= f \qquad \text{in } \Omega, \\ \text{div}\, u &= g \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega. \end{aligned} \tag{1.1}$$

The generalization of classical Stokes problem consists in two points: instead of Laplace operator we consider a general second order elliptic operator in divergence form and instead of gradient of p we consider a class of general first order linear operators. The new feature of system (1.1) compared with classical Stokes system lies in the fact that operators div u and $B\nabla p$ (for B different from the identity matrix E) do not act as adjoint operators in suitable Banach spaces. While existence of weak solutions and their properties with $-\triangle u$ instead of $-\text{div}\,(A\nabla u)$ and $B = E$ has been studied for a long time (see for instance [6], [16]) both existence and smoothness properties of solutions to system (1.1) – as far as we know – have not been studied yet in full extent.

Our motivation to investigate system (1.1) began with study of smoothness of flows of incompressible fluids with viscosities that depend on shear and pressure (see [17], [5], [11], [13]).

The arrangement of the paper is as follows. In Section 2 we introduce notations and definitions and recall some results used later. In the next section we present the existence and Hilbert regularity results obtained in [18]. In Section 4 we show the interior Schauder estimates of solutions based on a suitable form of Caccioppoli's inequality.

*This work was supported by Grants No.: MSM 0021620839, GACR 201/05/2465 and GACR 201/06/0352.

†Department of Mathematical Analysis, Charles University, Sokolovská 83, 186 75 Prague 8, Czech Republic (ndhuydhsp@yahoo.com)

‡Department of Mathematical Analysis, Charles University, Sokolovská 83, 186 75 Prague 8, Czech Republic (stara@karlin.mff.cuni.cz)

2. Preliminaries. Let Ω be a domain with Lipschitz boundary in $\mathbb{R}^d (d \geq 2)$. For $1 \leq q \leq \infty$, $k \in N$; $L^q(\Omega)$, $W^{k,q}(\Omega)$ and $W_0^{k,q}(\Omega)$ denote the usual Lebesgue and Sobolev spaces and Sobolev spaces with zero trace on the boundary. The norm of $u \in L^q(\Omega)$ is denoted by

$$\|u\|_q = \|u\|_{q,\Omega} := \left(\int_\Omega |\ u\ |^q \ dx \right)^{1/q} \ \mathrm{d}x.$$

The norm of $u \in W^{k,q}(\Omega)$ is defined as

$$\|u\|_{k,q} = \|u\|_{k,q;\Omega} := \left(\int_\Omega \sum_{|\alpha| \leq k} |\ D^\alpha u\ |^q \right)^{1/q} \ \mathrm{d}x.$$

Further, $W_{0,\mathrm{div}}^{1,2}(\Omega) = \left\{ u \in W_0^{1,2}(\Omega)^d;\ \mathrm{div\ u} = 0 \right\}$ where the equation div $u = 0$ is satisfied in distribution sense. We denote by $W^{-1,q'}(\Omega)$ the dual space to $W_0^{1,q}(\Omega)$ for $\frac{1}{q'} + \frac{1}{q} = 1$. If $f \in W^{-1,q'}(\Omega)$, $v \in W_0^{1,q}(\Omega)$ we use the notation $[f, v]$ for the value of the functional f at v. Norms in spaces of scalar functions and of vector valued functions are denoted by the same symbols. For $\mu > 0$, $0 < \alpha \leq 1$; $L^{2,\mu}(\Omega)$, $\mathcal{L}^{2,\mu}(\Omega)$ and $C^{0,\alpha}(\overline{\Omega})$ denote Morrey, Campanato and Hölder spaces with norms $\|u\|_{L^{2,\mu}(\Omega)}, \|u\|_{\mathcal{L}^{2,\mu}(\Omega)} := \|u\|_2 + [u]_{2,\mu;\Omega}$ and $\|u\|_{C^{0,\alpha}(\overline{\Omega})}$, respectively.

For $f : \mathbb{R}^d \to \mathbb{R}^d$ we will use notations

$$D_j f := \frac{\partial f}{\partial x_j}, \quad D_j^2 f := \frac{\partial^2 f}{\partial x_j^2}, \quad \nabla f := (D_j f)_{j=1}^d, \quad \nabla^2 f := (D_j D_l f)_{j,l=1}^d.$$

Next, we review some well-known facts needed later. We start with the following lemma showing the property of the orthogonal projection of $W_0^{1,2}(B_R(x_0))^d$ on $W_{0,\mathrm{div}}^{1,2}(B_R(x_0))$.

Lemma 2.1. *Denote by*

$$P_{B_R(x_0)}\ :\ W_0^{1,2}(B_R(x_0))^d \longrightarrow W_{0,\mathrm{div}}^{1,2}(B_R(x_0))$$

the orthogonal projection of $W_0^{1,2}(B_R(x_0))^d$ *onto* $W_{0,\mathrm{div}}^{1,2}(B_R(x_0))$. *Then there is a constant* $C > 0$ *such that for all* $\Phi \in W_0^{1,2}(B_R(x_0))^d$, *we have*

$$\|\nabla(P_{B_R(x_0)}\Phi)\|_{2,B_R(x_0)} \leq C\|\mathrm{div}\ \Phi\|_{2,\mathrm{B_R(x_0)}}.$$

The constant C *does not depend on* x_0 *and* R.

See in [10, Corollary 0.4].

Lemma 2.2. *Let* $f \in L^2(B_R(x_0))^d$ *with* $\int_{B_R(x_0)} f \ \ \mathrm{d}x = 0$. *Then there exists a unique solution* $u \in W_0^{1,2}(B_R(x_0))^d \cap (W_{0,\mathrm{div}}^{1,2}(B_R(x_0))^d)^\perp$ *to the equation* div u=f, *moreover*

$$\|\nabla u\|_{2,B_R(x_0)} \leq C\|f\|_{2,B_R(x_0)}$$

with the positive constant C *which does not depend on* x_0 *and* R.

See [10, Theorem 0.3 and Corollary 0.4].

Finally, we recall a well-know facts verified easily by iterations.

Lemma 2.3. *Let $f(t)$ be a nonnegative bounded function defined in $[\tau_0, \tau_1]$ where $\tau_0 \le 0$. Suppose that for $\tau_0 \le t < s \le \tau_1$ we have*

$$f(t) \le [A(s-t)^{-\alpha} + B] + \theta f(s),$$

where A, B, α, θ are nonnegative constants with $0 \le \theta < 1$. Then for all $\tau_0 \le t < s \le \tau_1$ we have

$$f(t) \le C[A(s-t)^{-\alpha} + B],$$

where C is a constant depending on α and θ.

See [8, Lemma 3.1, page 161].

Lemma 2.4. *Let $\Phi(\rho)$ be a nonnegative and nondecreasing function on $(0, R_0]$. Suppose that there are nonnegative constants A, B, α, β with $\alpha > \beta$ so that*

$$\Phi(\rho) \le A\left[\left(\frac{\rho}{R}\right)^{\alpha} + \varepsilon\right]\Phi(R) + BR^{\beta} \qquad \text{for all } 0 < \rho < R \le R_0. \tag{2.1}$$

Then there exists positive $\varepsilon_0 = \varepsilon_0(\alpha, \beta, A)$ such that the following holds:
If (2.1) is true for some $\varepsilon < \varepsilon_0$ then

$$\Phi(\rho) \le C(\alpha, \beta, A)\left[\left(\frac{\rho}{R}\right)^{\alpha}\Phi(R) + B\rho^{\beta}\right].$$

See [9, Section III. 2, page 51].

3. Existence and Hilbert regularity of solutions to (1.1)**.** Let $\Omega \subset \mathbb{R}^d$ $(d \ge 2)$ be a bounded Lipschitz domain with boundary $\partial\Omega$. The existence and uniqueness of solutions for the generalized linear Stokes system which we quote further was proved in [18].

We consider system

$$\begin{aligned} -\text{div}\,(A\nabla u) + B\nabla p &= f \qquad && \text{in } \Omega \\ \text{div}\, u &= g && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{3.1}$$

Let $f \in W^{-1,2}(\Omega)^d$. Recall that by a weak solution to system (3.1) we understand a pair $(u, p) \in W_0^{1,2}(\Omega)^d \times L^2(\Omega)$ if div u = g a.e. on Ω and

$$-\,\text{div}\,(A\nabla u) + B\nabla p = f \tag{3.2}$$

holds in the sense of distributions, i.e.

$$\sum_{\alpha,\beta,i,j=1}^{d} \int_\Omega A_{ij}^{\alpha\beta} D_\beta u_j D_\alpha v_i \;\mathrm{d}x - \sum_{i,j=1}^{d} \int_\Omega p D_j(B_{ij} v_i) \;\mathrm{d}x = [f, v] \tag{3.3}$$

holds for all $v \in W_0^{1,2}(\Omega)^d$.

For the existence of solutions, we assume that A, B satisfy the following conditions

(3a) B is a constant regular matrix,

(3b) $A_{ij}^{\alpha\beta}$ belong to $L^\infty(\Omega)$ and there is a positive Λ_A such that

$$\|A_{ij}^{\alpha\beta}\|_\infty \le \Lambda_A \quad \text{for all } \; i,j,\alpha,\beta = 1,\cdots,d.$$

(3c) $B^{-1}A$ generates elliptic (generally nonsymetric) bilinear form on $W_0^{1,2}(\Omega)^{d^2}$, i.e. there exists a $\lambda > 0$ such that

$$a(u,v) = \int_\Omega (B^{-1}A\nabla u) : \nabla v \; \mathrm{d}x = \int_\Omega \sum_{\alpha,\beta,i,j=1}^{d} \sum_{k=1}^{d} (B^{-1})_{ik} A_{kj}^{\alpha\beta} \; D^\alpha v_i \; D^\beta v_j \; \mathrm{d}x$$
$$\ge \lambda ||\nabla v||_2^2 \; \text{ for all } v \in W_0^{1,2}(\Omega)^{d^2}.$$

Theorem 3.1. *Let the assumptions* (3a), (3b), (3c) *be in force and* Ω *be a bounded Lipschitz domain, let* Ω_0 *be a nonempty subdomain of* Ω *and* $f \in W^{-1,2}(\Omega)^d$. *Then there exists unique pair* $(u,p) \in W_0^{1,2}(\Omega)^d \times L^2(\Omega)$ *satisfying* $\int_{\Omega_0} p \; \mathrm{d}x = 0$ *and solving system* (3.1).

Moreover, the inequality

$$\|u\|_{1,2} + \|p\|_2 \le C\|f\|_{-1,2} \tag{3.4}$$

holds with a constant $C = C(A, B, \Omega, \Omega_0) > 0$.

Examples. To illustrate the type of systems we have in mind we show some examples that satisfy conditions (3a), (3b), (3c).

Proposition 3.2 (A elliptic, B near to identity). *Suppose that*

- $A_{ij}^{\alpha\beta}$ *belong to* $L^\infty(\Omega)$ *and there is a positive* Λ_A *such that*

$$\|A_{ij}^{\alpha\beta}\|_\infty \le \Lambda_A \qquad \textit{for all}\, i,j,\alpha,\beta = 1,\cdots,d,$$

- A *generates an elliptic bilinear form* a *on* $W_0^{1,2}(\Omega)^d$) *i.e. there is a positive constant* λ_A *such that*

$$a(v,v) = \int_\Omega \sum_{\alpha,\beta,i,j=1}^{d} A_{ij}^{\alpha\beta} \; D^\alpha v_i \; D^\beta v_j \; \mathrm{d}x \ge \lambda_A \|\nabla v\|_2^2 \quad \textit{for all}\, v \in W_0^{1,2}(\Omega)^d,$$

- B *is a constant* $d \times d$ *matrix such that*

$$\nu = |B - E| < \frac{\lambda_A}{\lambda_A + d^4 \Lambda_A} < 1, \tag{3.5}$$

  where E *is the identity* $d \times d$ *matrix.*

Then conditions (3a), (3b), (3c) *hold.*

Proposition 3.3 (A Laplace operator on the diagonal, B positive definite). *Suppose that* div (A∇v) *is Laplace operator on* v_j *in* j*-th equation,* $j = 1,\cdots,d$ *i.e.*

$$A_{ij}^{\alpha\beta} = \delta_{\alpha\beta}\delta_{ij} \; \textit{for all } i,j,\alpha,\beta = 1,\cdots,d$$

and B *is constant, self adjoint and positive definite matrix.*
Then conditions (3a), (3b), (3c) *are satisfied.*

Next, we show the regularity of solutions to (3.1) in $W^{k,2}(\Omega)$. For Hilbert regularity we assume that A,B satisfy the following conditions

(3d) B is regular,

(3e) $B^{-1}A$ satisfies uniformly strong ellipticity condition, i.e. there exists a positive λ such that

$$\sum_{\alpha,\beta,i,j=1}^{d}\sum_{k=1}^{d}(B^{-1})_{ik}A_{kj}^{\alpha\beta}\xi_i^\alpha\xi_j^\beta \ \geq \lambda|\xi|^2 \quad \text{for all } \xi \in R^{d\times d}.$$

In [18] we proved that under natural conditions on f, g, A, B, Ω any solution pair $u \in W^{1,2}(\Omega), p \in L^2(\Omega)$ satisfies $u \in W^{k+2}(\Omega)$ and $p \in W^{k+1}(\Omega)$, $(k \in \mathbb{N})$.

Theorem 3.4. *Let assumptions* (3d), (3e) *be in force and* $k \in \mathbb{N}$, Ω *be a bounded* C^{k+2}*-domain in* $\mathbb{R}^d, (d \geq 2)$*. Suppose that* $f \in W^{k,2}(\Omega)^d$, $g \in W^{k+1,2}(\Omega)$ $\int_\Omega g \ \mathrm{d}x = 0$ $A \in C^{k+1}(\overline{\Omega})^{d^4}$, $B \in C^{k+1}(\overline{\Omega})^{d^2}$, $(u,p) \in W_0^{1,2}(\Omega)^d \times L^2(\Omega)$ *be a weak solution of system* (3.1)*. Then we have*

$$u \in W^{k+2,2}(\Omega)^d, \qquad p \in W^{k+1,2}(\Omega), \tag{3.6}$$

and inequality

$$\|u\|_{k+2,2} + \|p\|_{k+1,2} \leq C\left(\|f\|_{k,2} + \|g\|_{k+1,2} + \|u\|_{1,2}\right) \tag{3.7}$$

holds with a constant $C = C(A, B, \Omega) > 0$.

As a consequence we get the following result on the interior regularity

Corollary 3.5. *Let assumptions* (3d), (3e) *be in force. Suppose that* $f \in C^\infty(\overline{\Omega})^d$, $g \in C^\infty(\overline{\Omega})$*;* $A \in C^\infty(\overline{\Omega})^{d^4}$, $B \in C^\infty(\overline{\Omega})^{d^2}$*. If* $(u,p) \in W_{loc}^{1,2}(\Omega)^d \times L_{loc}^2(\Omega)$ *is a local weak solution of system* (3.1)*, then* $u \in C^\infty(\overline{\Omega})^d$, $p \in C^\infty(\overline{\Omega})$.

4. Schauder estimates in Hölder spaces. In this section, we study Schauder type estimates for systems

$$\begin{aligned} -\text{div }(A\nabla u) + B\nabla p &= \text{div } F && \text{in } \Omega,\\ \text{div } u &= g && \text{in } \Omega,\\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{4.1}$$

where A and B are matrices of sufficiently smooth functions, $F \ : \ \Omega \longrightarrow \mathbb{R}^{d^2}$ and $g \ : \ \Omega \longrightarrow \mathbb{R}^d$. We assume throughout this section that A, B satisfy the following conditions

(4a) $A_{ij}^{\alpha,\beta} \in L^\infty(\Omega)$ for all $i, j, \alpha, \beta = 1, \cdots, d$, B is regular, $B_{ij} \in C^{0,1}(\overline{\Omega})$ for all $i, j = 1, \cdots, d$,

(4b) $B^{-1}A$ satisfies strong Legendre-Hadamard ellipticity condition i.e. there exists a positive λ so that

$$\sum_{\alpha,\beta,i,j=1}^{d}\sum_{k=1}^{d}(B^{-1})_{ik}A_{kj}^{\alpha\beta}\xi_\alpha\xi_\beta\eta^i\eta^j \ \geq \lambda|\xi|^2|\eta|^2 \qquad \text{in } \Omega \text{ for all } \xi, \eta \in \mathbb{R}^d.$$

Under the assumption (4a), system (4.1) can be transformed to

$$\begin{aligned} -\text{div }(\bar{A}\nabla u) + H\nabla u + \nabla p &= B^{-1}\text{div } F && \text{in } \Omega,\\ \text{div } u &= g && \text{in } \Omega,\\ u &= 0 && \text{on } \partial\Omega \end{aligned} \tag{4.2}$$

where $\bar{A} := B^{-1}A$, $\quad H = \left(H_{ij}^{\beta}\right)_{\beta,i,j=1}^{d} := \left(\sum_{\alpha,k=1}^{d} D_{\alpha}(B^{-1})_{ik}A_{kj}^{\alpha\beta})\right)_{\beta,i,j=1}^{d}$,
$H\nabla u = \left(\sum_{\beta,j=1}^{d} H_{ij}^{\beta} D_{\beta} u_j\right)_{i=1}^{d}$.

First we show a local energy estimate – Caccioppoli's inequality.

Theorem 4.1 (Caccioppoli's inequality). *Suppose* (4a), (3c) *are satisfied. Let* $(u,p) \in W_{loc}^{1,2}(\Omega)^d \times L_{loc}^2(\Omega)$ *be a weak solution of system* (4.1). *Then there is a positive constant* C *such that for all* $x_0 \in \Omega$ *and all* ρ, R *with* $0 < \rho < R < \mathrm{dist}(x_0, \partial\Omega)$, *we have*

$$\|\nabla u\|_{2,B_\rho(x_0)}^2 \le C\left[\frac{1}{(R-\rho)^2}\|(u-\nu)\|_{2,B_R(x_0)}^2 + \|F\|_{2,B_R(x_0)}^2 + \|g\|_{2,B_R(x_0)}^2\right] \tag{4.3}$$

$$\|(p-p_{x_0,R})\|_{2,B_R(x_0)}^2 \le C\left[\|\nabla u\|_{2,B_R(x_0)}^2 + \|F\|_{2,B_R(x_0)}^2\right] \tag{4.4}$$

where $\nu \in \mathbb{R}^d$ *is an arbitrary constant vector.*

Proof. Let u, p be a weak solution of system (4.1) and $\nu \in \mathbb{R}^d$. Choose a test function $\varphi_1 = \eta^2(u-\nu)$, where $\eta \in C_0^\infty(R^d)$ is a cut-off function :

$$\mathrm{supp}\eta \subset B_R(x_0), \quad 0 \le \eta \le 1 \quad \text{on } \mathrm{B_R(x_0)},$$
$$\eta \equiv 1 \quad \text{on } \mathrm{B_\rho(x_0)}; \quad \eta, \quad |\nabla\eta| \le \frac{\mathrm{C}}{\mathrm{R}-\rho}.$$

Then we have

$$\begin{aligned}\int_\Omega \eta^2 \bar{A}\nabla u : \nabla u \; dx = & -\int_\Omega \Big\{2\bar{A}_{ij}^{\alpha\beta} D_\beta u_j (u_i - \nu_i)\eta D_\alpha \eta + \eta^2 (H\nabla u)\cdot(u-\nu) \\ & - (p-p_{x_0,R})2\eta\nabla\eta\cdot(u-\nu) - (p-p_{x_0,R})\eta^2 g \\ & -F\cdot\nabla(\eta^2 B^{-1}(u-\nu))\Big\}\, dx.\end{aligned}$$

Therefore

$$\begin{aligned}\lambda\|\eta\nabla u\|_{2,B_R(x_0)}^2 \le\ & 2\|\bar{A}\|_\infty \|\eta\nabla u\|_{2,B_R(x_0)}\|\nabla\eta(u-\nu)\|_{2,B_R(x_0)} \\ & + \|H\|_\infty\|\eta\nabla u\|_{2,B_R(x_0)}\|\eta(u-\nu)\|_{2,B_R(x_0)} \\ & + 2\|\eta(p-p_{x_0,R})\|_{2,B_R(x_0)}\|\nabla\eta(u-\nu)\|_{2,B_R(x_0)} \\ & + \|\eta(p-p_{x_0,R})\|_{2,B_R(x_0)}\|g\|_{2,B_R(x_0)} \\ & + C(\|\eta\nabla u\|_{2,B_R(x_0)}^2 + \|\nabla\eta(u-\nu)\|_{2,B_R(x_0)})\|F\|_{2,B_R(x_0)}\end{aligned}$$

and then by Young's inequality we have estimate

$$\begin{aligned}\|\eta\nabla u\|_{2,B_R(x_0)}^2 \le\ & \varepsilon\|\eta(p-p_{x_0,R})\|_{2,B_R(x_0)}^2 \\ & + C(\varepsilon)\left[\frac{1}{(R-\rho)^2}\|u-\nu\|_{2,B_R(x_0)}^2 + \|F\|_{2,B_R(x_0)}^2 + \|g\|_{2,B_R(x_0)}^2\right].\end{aligned} \tag{4.5}$$

Next, because of Lemma 2.2, we choose a test function $\varphi_2 \in W_0^{1,2}(B_R(x_0))^d \cap (W_{0,\mathrm{div}}^{1,2}(B_R(x_0)))^\perp$ which is a unique solution of equation

$$\mathrm{div}\, \varphi_2 = \eta^2(p-p_{x_0,R}) - (\eta^2(p-p_{x_0,R}))_{x_0,R} \qquad \text{in } B_R(x_0),$$

Recall that $\|\nabla\varphi_2\|_{L^2} \le C\|\eta(p-p_{x_0,R})\|_{L^2}$. We have

$$\int_\Omega \eta^2 \bar{A}\nabla u : \nabla\varphi_2 \ + (H\nabla u)\cdot\varphi_2 + \eta^2(p-p_{x_0,R})^2 \ \mathrm{d}x = \int_\Omega F\cdot\nabla(B^{-1}\cdot\varphi_2) \ \mathrm{d}x.$$

Applying Lemma 2.2, we have

$$\begin{aligned} \|\eta(p-p_{x_0,R})\|^2_{2,B_R(x_0)} \le C & \left[\|\nabla u\|_{2,B_R(x_0)}\|\eta(p-p_{x_0,R})\|_{2,B_R(x_0)}\right. \\ & + \ R^2\|\nabla u\|_{2,B_R(x_0)}\|\eta(p-p_{x_0,R})\|_{2,B_R(x_0)}] \\ & + \ \|F\|_{2,B_R(x_0)}\|\eta(p-p_{x_0,R})\|_{2,B_R(x_0)}] \\ \le & \ \frac{1}{2}\|\eta(p-p_{x_0,R})\|^2_{2,B_R(x_0)} + C\left[\|\nabla u\|^2_{2,B_R(x_0)} + \|F\|^2_{2,B_R(x_0)}\right]. \end{aligned}$$

Thus

$$\|\eta(p-p_{x_0,R})\|^2_{2,B_R(x_0)} \le C(\|\nabla u\|^2_{2,B_R(x_0)} + \|F\|^2_{2,B_R(x_0)}) \tag{4.6}$$

It is easy to check that as the inequality (4.6) holds for all $\eta \in C_0^\infty(B_R(x_0))$ with $0 \le \eta \le 1$, it implies the inequality (4.4).
Substituting (4.6) into (4.5), we conclude that

$$\begin{aligned} \|\nabla u\|^2_{2,B_\rho(x_0)} \le & \ \varepsilon\|\nabla u\|^2_{2,B_R(x_0)} \\ & + C\left[\frac{1}{(R-\rho)^2}\|u-\nu\|^2_{2,B_R(x_0)} + \|F\|^2_{2,B_R(x_0)} + \|g\|^2_{2,B_R(x_0)}\right] \end{aligned} \tag{4.7}$$

By application of Lemma 2.3, we obtain the inequality (4.3) and Theorem 4.1 is proved. □

Now, we consider systems with constant coefficients

$$\begin{aligned} -\operatorname{div}(A_0\nabla u) + \nabla p = 0 \ \text{ in } \Omega, \\ \operatorname{div} u = 0 \ \text{ in } \Omega, \end{aligned} \tag{4.8}$$

where A_0 is a $d^2 \times d^2$ constant matrix.
In a standard way (see for instance first [9, Theorem in III.2]), we have first estimates for the Hölder continuity by a following proposition

Proposition 4.2 (Campanato's inequality). *Suppose that A_0 is a constant matrix satisfying strong Legendre-Hadamard condition (see (4b)). Then there is a positive constant C such that for any weak solution $(u,p) \in W^{1,2}_{loc}(\Omega)^d \times L^2_{loc}(\Omega)$ of system (4.1), for all $x_0 \in \Omega$ and all ρ, R with $0 < \rho \le R < \operatorname{dist}(x_0, \partial\Omega)$, the following two estimates are valid:*

$$\|u\|^2_{2,B_\rho(x_0)} \le C\left(\frac{\rho}{R}\right)^d \|u\|^2_{2,B_R(x_0)} \tag{4.9}$$

$$\|u-u_{x_0,\rho}\|^2_{2,B_\rho(x_0)} \le C\left(\frac{\rho}{R}\right)^{d+2} \|u-u_{x_0,\rho}\|^2_{2,B_R(x_0)}. \tag{4.10}$$

The constant C depends on the dimension d and on A_0.

Next, we state first regularity theorem in Morrey spaces.

Theorem 4.3. *Suppose assumptions (4a), (4b) be satisfied, $A \in C^0(\Omega)^{d^4}$, $F \in L^{2,\mu}(\Omega)^{d^2}$, $g \in L^{2,\mu}(\Omega)$ with $0 < \mu < d$ and $(u,p) \in W_0^{1,2}(\Omega)^d \times L^2(\Omega)$ be a weak solution of system (4.1). Then $Du \in L^{2,\mu}_{loc}(\Omega)^{d^2}$, $p \in L^{2,\mu}_{loc}(\Omega)$, and for all $\tilde\Omega \subset\subset \Omega$ we have the estimates*

$$\|\nabla u\|_{L^{2,\mu}(\tilde\Omega)^{d^2}} + \|p\|_{L^{2,\mu}(\tilde\Omega)} \le C\left(\|\nabla u\|_2 + \|F\|_{L^{2,\mu}(\Omega)^{d^2}} + \|g\|_{L^{2,\mu}(\Omega)}\right) \tag{4.11}$$

with a constant $C = C(A, B, \operatorname{dist}(\tilde\Omega, \Omega)) > 0$.

Proof. Fix $x_0 \in \Omega$ and $R < \mathrm{dist}(x_0, \partial\Omega)$. Let v be the weak solution to system

$$\int_{B_R(x_0)} \bar{A}(x_0)\nabla v : \nabla\varphi \;\mathrm{d}x = 0 \qquad \text{for all } \varphi \in W^{1,2}_{0,div}(B_{R(x_0)})$$
$$\mathrm{div}\; v = 0, \qquad u - v \in W^{1,2}_0(B_{R(x_0)})^d$$

The existence of such a solution is guaranteed by Lax-Milgram theorem. By Proposition 4.2 we get that if $0 < \rho < R$

$$\|\nabla v\|^2_{2,B_\rho(x_0)} \le C\left(\frac{\rho}{R}\right)^d \|\nabla v\|^2_{2,B_R(x_0)}$$

Set $w = u - v$, then

$$\|\nabla u\|^2_{2,B_\rho(x_0)} \le C\left(\left(\frac{\rho}{R}\right)^d \|\nabla u\|^2_{2,B_R(x_0)} + \|\nabla w\|^2_{2,B_R(x_0)}\right) \tag{4.12}$$

For each $\varphi \in W^{1,2}_0(B_{R(x_0)})^d$ we test (4.1) by $\varphi - P_{B_R(x_0)}\varphi$ with $P_{B_R(x_0)}\varphi$ given in Lemma 2.2 and use definition of w. It is easily seen that $w \in W^{1,2}_0(B_{R(x_0)})^d$ satisfies

$$\begin{aligned}\int_{B_R(x_0)} \bar{A}(x_0)\nabla w : \nabla[\varphi - P_{B_R(x_0)}\varphi] \;\mathrm{d}x = \int_{B_R(x_0)} & [\bar{A}(x_0) - \bar{A}(x)]\nabla u : \nabla[\varphi - P_{B_R(x_0)}\varphi] \\ & - (H\nabla u)\cdot[\varphi - P_{B_R(x_0)}\varphi] \\ & + F\cdot\nabla(B^{-1}[\varphi - P_{B_R(x_0)}\varphi]) \;\mathrm{d}x, \\ \mathrm{div} \quad w = g. & \end{aligned}$$

By choosing $\varphi = w$, using the assumptions of Theorem 4.3, Lemma 2.1 and Poincaré's inequality, we deduce

$$\begin{aligned}\|\nabla w\|^2_{2,B_R(x_0)} \le & C\left\{\left[\omega(R)\|\nabla u\|_{2,B_R(x_0)} + \|F\|_{2,B_R(x_0)} + RH_c\|\nabla u\|_{2,B_R(x_0)}\right]\right. \\ & \left.\left[\|\nabla w\|_{2,B_R(x_0)} + \|g\|_{2,B_R(x_0)}\right]\right\},\end{aligned}$$

where

$$\omega(R) = \max_{i,j,\alpha,\beta}[\sup_{B_R(x_0)} |\bar{A}^{\alpha\beta}_{ij}(x) - \bar{A}^{\alpha\beta}_{ij}(x_0)|], \qquad H_c = \max_{i,j,\beta}[\sup_{B_R(x_0)} |H^\beta_{ij}(x)|].$$

Thus using smallness of ω and Young inequality

$$\|\nabla w\|^2_{2,B_R(x_0)} \le C\left[(\omega^2(R) + R^2H_c^2)\|\nabla u\|^2_{2,B_R(x_0)} + \|F\|^2_{2,B_R(x_0)} + \|g\|^2_{2,B_R(x_0)}\right] \tag{4.13}$$

Substituting (4.13) into (4.12), we get

$$\|\nabla u\|^2_{2,B_\rho(x_0)} \le C\left\{\left[\left(\frac{\rho}{R}\right)^d + \omega^2(R) + R^2H_c^2\right]\|\nabla u\|^2_{2,B_R(x_0)} + \|F\|^2_{2,B_R(x_0)} + \|g\|^2_{2,B_R(x_0)}\right\}.$$

Since $\bar{A}, H \in C^0(\overline{\Omega})$ there exists $R_0 < \mathrm{dist}(x_0, \partial\Omega)$ such that for $0 < \rho \le R \le R_0$, we have

$$\|\nabla u\|^2_{2,B_\rho(x_0)} \le C\left[\left(\frac{\rho}{R}\right)^d + \varepsilon\right]\|\nabla u\|^2_{2,B_R(x_0)} + C(\|F\|^2_{L^{2,\mu}} + \|g\|^2_{L^{2,\mu}}).$$

Thus applying the LEMMA 2.4 with $\varepsilon < \varepsilon_0$ we have that for $0 < \rho \le R \le R_0$

$$\|\nabla u\|^2_{2,B_\rho(x_0)} \le C(R^{-\mu}\|\nabla u\|^2_{2,B_R(x_0)} + \|F\|^2_{L^{2,\mu}} + \|g\|^2_{L^{2,\mu}})\rho^\mu. \tag{4.14}$$

It follows that $\nabla u \in L^{2,\mu}_{loc}(\Omega)^{d^2}$, and for all $\tilde\Omega \subset\subset \Omega$ we have the estimates

$$\|\nabla u\|_{L^{2,\mu}(\tilde\Omega)^{d^2}} \le C\ (\|\nabla u\|_2 + \|F\|_{L^{2,\mu}(\Omega)^{d^2}} + \|g\|_{L^{2,\mu}})$$

with a constant $C = C(A, B, \text{dist}\ (\tilde\Omega, \Omega)) > 0$.
Thanks to the inequalities (4.4), (4.14), it follows that $p \in L^{2,\mu}_{loc}(\Omega)$ and we also obtain the inequality (4.11). The proof is complete. ∎

THEOREM 4.4. *Suppose that the assumptions* (4a), (4b) *are satisfied and, moreover,* $A \in C^{0,\delta}(\Omega)^{d^4}$, $A \in L^\infty(\Omega)^{d^4}$, $F \in C^{0,\delta}(\Omega)^{d^2}$, $g \in C^{0,\delta}(\Omega)$ *with of system* (4.1). *Then* $\nabla u \in C^{0,\delta}_{loc}(\Omega)^{d^2}$, $p \in C^{0,\delta}_{loc}(\Omega)$, *and for all* $\tilde\Omega \subset\subset \Omega$ *we have the estimates*

$$\|\nabla u\|_{C^{0,\delta}(\tilde\Omega)^{d^2}} + \|p\|_{C^{0,\delta}(\tilde\Omega)} \le C\left[\|\nabla u\|_{2,\Omega'} + \|F\|_{C^{0,\delta}(\overline{\Omega'})^{d^2}} + \|g\|_{C^{0,\delta}(\overline{\Omega'})}\right] \tag{4.15}$$

holds with a constant $C = C(A, B, \text{dist}(\tilde\Omega, \Omega)$, diam $\Omega) > 0$. *Here*

$$\Omega' = \left\{x \in \Omega; \text{dist}(\text{x}, \partial\Omega) > \frac{1}{2}\,\text{dist}(\tilde\Omega, \partial\Omega)\right\}.$$

Proof. Fix $x_0 \in \Omega, R > 0$ sufficiently small. Let v be the weak solution to

$$\int_{B_R(x_0)} \bar A(x_0)\nabla v : \nabla\varphi\,\text{d}x = 0 \qquad \text{for all } \varphi \in W^{1,2}_{0,\text{div}}(B_R(x_0))$$
$$\text{div}\ v = g_{x_0,R}, \quad u - v \in W^{1,2}_0(B_R(x_0))^d$$

The existence of such a solution is guaranteed by Lax-Milgram Theorem. Clearly, as v solves a system with constant coefficients and zero right hand side, also ∇v is a solution of the same problem. Thus (4.10) is valid for ∇v i.e. if $0 < \rho \le R < \frac{1}{2}\text{dist}(\tilde\Omega, \partial\Omega)$ then

$$\|\nabla v - (\nabla v)_{x_0,\rho}\|^2_{2,B_\rho(x_0)} \le C\left(\frac{\rho}{R}\right)^{d+2}\|\nabla v - (\nabla v)_{x_0,R}\|^2_{2,B_R(x_0)}$$

Set $w = u - v$, then

$$\|\nabla u - (\nabla u)_{x_0,\rho}\|^2_{2,B_\rho(x_0)} \le C\left[\left(\frac{\rho}{R}\right)^d\|\nabla v - (\nabla v)_{x_0,R}\|^2_{2,B_R(x_0)} + \|\nabla w\|^2_{2,B_R(x_0)}\right] \tag{4.16}$$

Moreover, $w \in W^{1,2}_0(B_R(x_0))^d$ solves the system

$$\int_{B_R(x_0)} \bar A(x_0)\nabla w : \nabla[\varphi - P_{B_R(x_0)}\varphi]\ \text{d}x = \int_{B_R(x_0)} [\bar A(x_0) - \bar A(x)]\nabla u : \nabla[\varphi - P_{B_R(x_0)}\varphi]$$
$$-(H\nabla u).[\varphi - P_{B_R(x_0)}\varphi] + F.\nabla(B^{-1}\cdot[\varphi - P_{B_R(x_0)}\varphi])\ \text{d}x,$$
$$\forall\varphi \in W^{1,2}_0(B_{R(x_0)})^d;\ \text{div}\ w = g - g_{x_0,R}.$$

As in the proof of THEOREM 4.3, we choose $\varphi = w$ and get

$$\|\nabla w\|^2_{2,B_\rho(x_0)} \le C\left[(\omega^2(R) + R^2H_c^2)\|\nabla u\|^2_{2,B_R(x_0)}\right.$$
$$\left. + \|F - F_{x_0,R}\|^2_{2,B_R(x_0)} + \|g - g_{x_0,R}\|^2_{2,B_R(x_0)}\right].$$

Since $A \in C^{0,\delta}(\overline{\Omega})$ we have $\omega(R) \le [A]_{0,\delta,\Omega'} R^{\delta}$. Thanks to the assumptions on F, g it implies that

$$\begin{aligned}\|\nabla u - (\nabla u)_{x_0,\rho}\|^2_{2,B_\rho(x_0)} \le C \Bigg[\left(\frac{\rho}{R}\right)^{d+2} & \|\nabla u - (\nabla u)_{x_0,R}\|^2_{2,B_R(x_0)} \\ & + R^{2\delta}\|\nabla u\|^2_{2,B_R(x_0)} \\ & + ([F]^2_{2,d+2\delta;\Omega'} + [g]^2_{2,d+2\delta;\Omega'}) R^{d+2\delta}\Bigg].\end{aligned} \tag{4.17}$$

For any $\varepsilon > 0$, we have $F, g \in C^{0,\delta}(\Omega) \simeq \mathcal{L}^{2,d+2\delta}(\Omega) \subset \mathcal{L}^{2,d-\varepsilon}(\Omega) \subset L^{2,d-\varepsilon}(\Omega)$. Therefore, $f \in L^{2,d-\varepsilon}(\Omega)^{d^2}$. According to Theorem 4.3, we have $\nabla u \in L^{2,d-\varepsilon}(\tilde{\Omega})^{d^2}$ and inequality

$$\|\nabla u\|^2_{2,B_R(x_0)} \le C[\|\nabla u\|^2_{2,\Omega'} + \|F\|^2_{L^{2,d-\varepsilon}(\Omega')^{d^2}} + \|g\|^2_{L^{2,d-\varepsilon}(\Omega')}] R^{d-\varepsilon}. \tag{4.18}$$

It implies

$$\begin{aligned}\|\nabla u - (\nabla u)_{x_0,\rho}\|^2_{2,B_\rho(x_0)} \le C \Bigg[\left(\frac{\rho}{R}\right)^{d+2} & \|\nabla u - (\nabla u)_{x_0,R}\|^2_{2,B_R(x_0)} \\ & + (\|\nabla u\|^2_{2,\Omega'} + \|F\|^2_{L^{2,d-\varepsilon}(\Omega')^{d^2}} \\ & + \|g\|^2_{L^{2,d-\varepsilon}(\Omega')}) R^{d+2\delta-\varepsilon} + ([F]^2_{2,d+2\delta;\Omega'} + [g]^2_{2,d+2\delta;\Omega'}) R^{d+2\delta}\Bigg].\end{aligned}$$

Applying Lemma 2.4, we obtain

$$\|\nabla u - (\nabla u)_{x_0,\rho}\|^2_{2,B_\rho(x_0)} \le C \left[\|\nabla u\|^2_{2,\Omega'} + \|F\|^2_{L^{2,d-\varepsilon}(\Omega')^{d^2}} + \|g\|^2_{L^{2,d-\varepsilon}(\Omega')}\right] \rho^{d+2\delta-\varepsilon}$$

which implies that $\nabla u \in C^{0,\delta-\varepsilon/2}_{loc}(\Omega)^{d^2}$ for all $\varepsilon > 0$, and the inequality

$$\|\nabla u\|^2_{C^{0,\delta-\varepsilon/2}(\tilde{\Omega})^{d^2}} \le C \left[\|\nabla u\|^2_{2,\Omega'} + \|F\|^2_{\mathcal{L}^{2,d+2\delta}(\Omega')^{d^2}} + \|g\|^2_{\mathcal{L}^{2,d+2\delta}(\Omega')}\right].$$

In particular, ∇u is locally bounded and thus

$$\|\nabla u\|^2_{2,B_R(x_0)} \le C \left[\|\nabla u\|^2_{2,\Omega'} + \|F\|^2_{C^{0,\delta}(\overline{\Omega'})^{d^2}} + \|g\|^2_{C^{0,\delta}(\overline{\Omega'})}\right] R^d. \tag{4.19}$$

Substituting this inequality into (4.17), we get

$$\begin{aligned}\|\nabla u - (\nabla u)_{x_0,\rho}\|^2_{2,B_\rho(x_0)} \le C \left(\frac{\rho}{R}\right)^{d+2} & \|\nabla u - (\nabla u)_{x_0,R}\|^2_{2,B_R(x_0)} \\ & + C \left[\|\nabla u\|^2_{2,\Omega'} + \|F\|^2_{C^{0,\delta}(\overline{\Omega'})^{d^2}} + \|g\|^2_{C^{0,\delta}(\overline{\Omega'})}\right] R^{d+2\delta}.\end{aligned}$$

Applying again the Lemma 2.4, we conclude that $\nabla u \in C^{0,\delta}_{loc}(\Omega)^{d^2}$, and the estimate

$$\|\nabla u\|^2_{C^{0,\delta}(\tilde{\Omega})^{d^2}} \le C \left[\|\nabla u\|^2_{2,\Omega'} + \|F\|^2_{C^{0,\delta}(\overline{\Omega'})^{d^2}} + \|g\|^2_{C^{0,\delta}(\overline{\Omega'})}\right] \tag{4.20}$$

holds with a constant $C = C(A, B, \Omega) > 0$.
In a similar way, we conclude that $p \in C^{0,\delta}_{loc}(\Omega)$ and it satisfies the required estimate. ◻

REFERENCES

[1] R. A. Adams, *Sobolev spaces*, Academic press, New York, 1975.

[2] S. Agmon, A. Douglis and L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions.* I. Comm. Pure Appl. Math. **12** (1959) 623–727.

[3] S. Agmon, A. Douglis and L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions.* II. Comm. Pure Appl. Math. **17** (1964) 35–92.

[4] C. Amrouche and V. Girault, *Decomposition of vector spaces and application to the Stokes problem in arbitrary dimension.* Czechoslovak Math. J., **44**(1) (1994), 109–140.

[5] M. Franta, J. Málek and K. R. Rajagopal,*On steady flows of fluids with pressure- and shear-dependent viscosities.* Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. **461**(2055), (2005), 651–670.

[6] M. Fuchs and G. Seregin, *Variational methods for problems from plasticity theory and for generalized Newtonian fluids.* Lecture Notes in Mathematics, 1749. Springer-Verlag, Berlin, 2000.

[7] G. P. Galdi, *An introduction to the mathematical theory of the Navier-Stokes equations.* Vol. I. Linearized steady problems. 38. Springer-Verlag, New York, 1994

[8] M. Giaquinta, *Multiple integrals in the calculus of variations and nonlinear elliptic systems*, Princeton University Press, 1983.

[9] M. Giaquinta, *Introduction to regularity theory for nonlinear elliptic systems*, Birkhäuser Verlag, 1993.

[10] M. Giaquinta; G. Modica, *Nonlinear systems of the type of the stationary Navier-Stokes system*, J.Reine Angew.Math., **330** (1982), 173–214.

[11] 1J. Hron, J. Málek, J.Nečas and K. R. Rajagopal, *Numerical simulations and global existence of solutions of two-dimensional flows of fluids with pressure- and shear-dependent viscosities.* Mathematics and Computer in Simulation **61**(3-6) (2003), 297–315.

[12] P. Kaplický, J. Málek and J. Stará, $C^{1,\alpha}$*-solution to a class of nonlinear fluids in two dimensions-stationary Dirichlet problem.* Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **259** (1999), Kraev. Zadachi Mat. Fiz. i Smezh. Vopr. Teor. Funkts. 30, 89–121, 297; translation in J. Math. Sci. (New York) **109**(5) (2002), 1867–1893

[13] J. Málek, J. Nečas and K. R. Rajagopal, *Global existence of solutions for flows of fluids with pressure and shear dependent viscosities.* Appl. Math. Lett. **15**(8) (2002), 961–967.

[14] J. Nečas, *Les méthodes directes en théorie des équations elliptiques.* Academia, Prague 1967.

[15] H. Sohr, *The Navier-Stokes equations. An elementary functional analytic approach.* Birkhäuser Verlag, Basel, 2001.

[16] V. A. Solonnikov, *Initial-boundary value problem for generalized Stokes equations.* Math. Bohem. **126**(2) (2001), 505–519.

[17] J. Málek, G. Mingione and J. Stará, *Fluids with pressure dependent viscosity: partial regularity of steady flows.* to appear.

[18] N. D. Huy and J. Stará, *On existence and regularity of solutions to a class of generalized stationary Stokes problem.* Comm. Math. Univ. Carolin. **47**(2) (2006), 241–264.