Proceedings of Equadiff-11
2005, pp. 559–566
ISBN 978-80-227-2624-5

ON A GROUND STATE AND TWO SYMMETRIC GROUND STATE SOLUTIONS IN A DOMAIN

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Abstract. Let Ω be a domain in $\mathbb{R}^N$, $N \geq 1$, and $2^* = \infty$ if $N = 1, 2$, $2^* = \frac{2N}{N-2}$ if $N > 2$, $2 < p < 2^*$. Consider the semilinear elliptic equation

$$\begin{aligned} -\Delta u + u &= |u|^{p-2}u \quad \text{in } \Omega; \\ u &\in H_0^1(\Omega). \end{aligned} \tag{*}$$

The existence, the nonexistence, and the multiplicity of positive solutions of equation (*) are affected by the geometry and the topology of the domain Ω. In the article, we first present various analyses and use them to characterize which domain Ω is a ground state domain or a non-ground state domain. Secondly, for a y-symmetric domain Ω, we study their index $\alpha(\Omega)$ and y-symmetric index $\alpha_s(\Omega)$. We determine whether $\alpha(\Omega) = \alpha_s(\Omega)$ or $\alpha(\Omega) < \alpha_s(\Omega)$. In case that $\alpha(\Omega) < \alpha_s(\Omega)$ and that both $\alpha(\Omega)$ and $\alpha_s(\Omega)$ admits ground state solutions, then we obtain that in Ω, the Equation (*) has three positive solutions, of which one is y-symmetric and other two are not y-symmetric.

1. Introduction.

Let Ω be a domain in $\mathbb{R}^N$, $N \geq 1$, and $2^* = \infty$ if $N = 1, 2$, $2^* = \frac{2N}{N-2}$ if $N > 2$, $2 < p < 2^*$. Consider the semilinear elliptic equation

$$\begin{aligned} -\Delta u + u &= |u|^{p-2}u \quad \text{in } \Omega; \\ u &\in H_0^1(\Omega). \end{aligned} \tag{1.1}$$

Let $H_0^1(\Omega)$ be the Sobolev space in Ω. Associated with Equation (1.1), we consider the energy functionals a, b and J for $u \in H_0^1(\Omega)$

$$a(u) = \int_\Omega (|\nabla u|^2 + u^2); \quad b(u) = \int_\Omega |u|^p; \quad J(u) = \frac{1}{2}a(u) - \frac{1}{p}b(u).$$

It is well known that the solutions of Equation (1.1) and the critical points of the energy functional J are the same.

The existence, the nonexistence, and the multiplicity of positive solutions of Equation (1.1) in a domain Ω have been the focus of a great deal of research in recent years. They are affected by the geometry and the topology of the domain Ω. To characterize in what kind of domains we have the existence, the nonexistence, and the multiplicity of positive solutions of Equation (1.1) is an open question. In this article, we try to answer partially the question.

The aim of our study is twofold: first, we characterize what kind of domains Ω are ground state domains or non-ground state domains; secondly, we characterize in what kind of y-symmetric domain Ω, the index $\alpha(\Omega)$ is equal to or less than the $y-$ symmetric index $\alpha_s(\Omega)$, Suppose that Ω is a ground state domain for $H_0^1(\Omega)$ and for $H_s^1(\Omega)$ such that $\alpha(\Omega) < \alpha_s(\Omega)$, then Equation (1.1) on Ω has three positive solutions, of which one is y-symmetric and other two are not y-symmetric.

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2. ***y*-symmetric Domains.** Let $z = (x, y) \in \mathbb{R}^{N-1} \times \mathbb{R} = \mathbb{R}^N$, $N \geq 3$. In the followings, we denote the infinite strip $\mathbf{A}^r = \{(x, y) \in \mathbb{R}^N : |x| < r\}$; a finite strip $\mathbf{A}^r_{s,t} = \{(x, y) \in \mathbf{A}^r : s < y < t\}$; an upper semi-strip $\mathbf{A}^r_s = \{(x, y) \in \mathbf{A}^r : s < y\}$;, the infinite strip with a hole $\mathbf{A}^r \backslash \omega$, a ball $B^N(z_0; s)$, an interior flask domain $\mathbf{F}^r_s = \mathbf{A}^r_0 \cup B^N(0; s)$; the upper half space $\mathbb{R}^N_+$, a horizontal infinite strip $\mathbb{R}^N_{-\rho,\rho} = \{(x, y) \in \mathbb{R}^N : -\rho < y < \rho\}$; a positive paraboloid $\mathbf{P}^+ = \{(x, y) \in \mathbb{R}^N : y > |x|^2\}$; an infinite cone $\mathbf{C} = \{(x, y) \in \mathbb{R}^N : |x| < y\}$; and an epigraph $\Pi = \{(x, y) \in \mathbb{R}^N \mid f(x) < y\}$.

We have the following definitions of y-symmetric domains:

Definition 2.1.
- (i) Suppose that $(x, y) \in \Omega$ if and only if $(x, -y) \in \Omega$, then we call Ω a y-symmetric domain;
- (ii) Let Ω be a y-symmetric domain in R^N. If a function $u : \Omega \to R$ satisfies $u(x, y) = u(x, -y)$ for $(x, y) \in \Omega$, then we call u a y-symmetric function.

Example 2.2. The whole space $\mathbb{R}^N$, a ball $B^N(0; s)$, the infinite strip $\mathbf{A}^r = \{(x, y) \in \mathbb{R}^N : |x| < r\}$, a finite strip $\mathbf{A}^r_{-s,s} = \{(x, y) \in \mathbf{A}^r : -s < y < s\}$, and the infinite strip with a hole $\mathbf{A}^r \backslash B^N(0; r/2)$ are y-symmetric domains in $\mathbb{R}^N$.

Let Ω be a y-symmetric domain in R^N and denote the space $H^1_s(\Omega)$ by the H^1-closure of the space $\{u \in C^\infty_0(\Omega) : u \text{ is } y\text{-symmetric}\}$. Note that $H^1_s(\Omega)$ is a closed linear subspace of $H^1_0(\Omega)$. Let $H^{-1}_s(\Omega)$ be the dual space of $H^1_s(\Omega)$. Throughout this article, let $X(\Omega)$ be either the whole space $H^1_0(\Omega)$ or the y-symmetric space $H^1_s(\Omega)$. Let $X^{-1}(\Omega)$ be the dual space of $X(\Omega)$.

3. Palais-Smale Theory. We define the Palais-Smale (denoted by (PS)) sequences, (PS)-values, and (PS)-conditions in $X(\Omega)$ for J as follows.

Definition 3.1.
- (i) For $\beta \in R$, a sequence $\{u_n\}$ is a $(\mathrm{PS})_\beta$-sequence in $X(\Omega)$ for J if $J(u_n) = \beta + o(1)$ and $J'(u_n) = o(1)$ strongly in $X^{-1}(\Omega)$ as $n \to \infty$;
- (ii) $\beta \in R$ is a (PS)-value in $X(\Omega)$ for J if there is a $(\mathrm{PS})_\beta$-sequence in $X(\Omega)$ for J;
- (iii) J satisfies the $(\mathrm{PS})_\beta$-condition in $X(\Omega)$ if every $(\mathrm{PS})_\beta$-sequence in $X(\Omega)$ for J contains a convergent subsequence;
- (iv) J satisfies the (PS)-condition in $X(\Omega)$ if for every $\beta \in R$, J satisfies the $(\mathrm{PS})_\beta$-condition in $X(\Omega)$.

For any $\beta \in \mathbb{R}$, a $(\mathrm{PS})_\beta$-sequence in $X(\Omega)$ for J is bounded and a (PS)-value β is non-negative.

Lemma 3.2. *Let $\beta \in \mathbb{R}$ and let $\{u_n\}$ be a (PS)$_\beta$-sequence in $X(\Omega)$ for J, then a positive sequence $\{c_n(\beta)\}$ exists such that $\|u_n\|_{H^1} \leq c_n(\beta) \leq c$ for each n and $c_n(\beta) = o(1)$ as $n \to \infty$ and $\beta \to 0$. Furthermore,*

$$a(u_n) = b(u_n) + o(1) = \frac{2p}{p-2}\beta + o(1)$$

and $\beta \geq 0$.

Consider the Nehari minimizing problem

$$\alpha_{\mathbf{X}}(\Omega) = \inf_{v \in \mathbf{M}_X(\Omega)} J(v),$$

where $\mathbf{M}_X(\Omega) = \{u \in X(\Omega) \backslash \{0\} : a(u) = b(u)\}$. Note that $\mathbf{M}_X(\Omega)$ contains every nonzero solution of Equation (1.1) in $X(\Omega)$.

We have the following three useful results:

THEOREM 3.3. *If $\{u_n\}$ is a (PS)$_\beta$-sequence in $X(\Omega)$ for J, then there is a sequence $\{s_n\}$ in $\mathbb{R}^+$ such that $\{s_n u_n\}$ is in $\mathbf{M}_X(\Omega)$ and is a (PS)$_\beta$-sequence in $X(\Omega)$ for J.*

THEOREM 3.4. *Every minimizing sequence $\{u_n\}$ in $\mathbf{M}_X(\Omega)$ of $\alpha_X(\Omega)$ is a (PS)$_{\alpha_X(\Omega)}$--sequence in $X(\Omega)$ for J.*

By THEOREM 3.4, we have that $\alpha_X(\Omega)$ is a positive (PS)$_{\alpha_X(\Omega)}-$value in $X(\Omega)$ for J.

THEOREM 3.5. *Let $u \in \mathbf{M}_X(\Omega)$ be such that $J(u) = \min_{v\in\mathbf{M}_X(\Omega)} J(v)$. Then u is a nonzero solution of Equation (1.1) such that $J(u) = \alpha_X(\Omega)$.*

DEFINITION 3.6. $\alpha_X(\Omega)$ is called the index of J in $X(\Omega)$. *If* u is a nonzero solution of Equation (1.1), then $u \in M_X(\Omega)$. Thus, $J(u) \geq \alpha_X(\Omega)$. We say that a nonzero solution u in $X(\Omega)$ of Equation (1.1) is a ground state solution for $X(\Omega)$ if $J(u) = \alpha_X(\Omega)$, and is a higher energy solution if $J(u) > \alpha_X(\Omega)$.

REMARK 3.7. We denote $\alpha_X(\Omega)$ by the (general) index $\alpha(\Omega)$ of a domain Ω for $X(\Omega) = H_0^1(\Omega)$ and by the $y-$ symmetric index $\alpha_s(\Omega)$ of a y-symmetric domain Ω for $X(\Omega) = H_s^1(\Omega)$. Therefore, a y-symmetric domain Ω has two indexes: the index $\alpha(\Omega)$ and the $y-$ symmetric index $\alpha_s(\Omega)$ such that $\alpha(\Omega) \leq \alpha_s(\Omega)$. The two indexes $\alpha(\Omega)$ and $\alpha_s(\Omega)$ may be equality or inequality.

The Palais-Smale conditions are conditions for compactness. They are useful in asserting the existence, the nonexistence, and the multiplicity of solutions of Equation (1.1).

THEOREM 3.8. *If J satisfies the (PS)$_{\alpha_X(\Omega)}$-condition, then there is a ground state solution for $X(\Omega)$ of Equation* (1.1).

Let $\Omega^1 \subsetneqq \Omega^2$ and $\alpha_X^i = \alpha_X(\Omega^i)$ for $i = 1, 2$, then clearly $\alpha_X^2 \leq \alpha_X^1$. If $\alpha_X^2 = \alpha_X^1$, then we have the following useful results.

THEOREM 3.9. *Let $\Omega^1 \subsetneqq \Omega^2$ and $J : X(\Omega^2) \to \mathbb{R}$ be the energy functional. Suppose that $\alpha_X^2 = \alpha_X^1$. Then*

(i) *α_X^1 does not admit any ground state solution;*
(ii) *J does not satisfy the (PS)$_{\alpha_X^1}$-condition;*
(iii) *J does not satisfy the (PS)$_{\alpha_X^2}$-condition.*

4. Non-ground State Domains and Higher Energy Solutions. We give the following definition.

DEFINITION 4.1. Let Ω be a domain in R^N.

(i) Ω is a compactness domain for $X(\Omega)$ if the embedding $X(\Omega) \hookrightarrow L^p(\Omega)$ is compact;
(ii) Ω is a ground state domain for $X(\Omega)$ if there is a ground state solution u in $X(\Omega)$ of Equation (1.1). Otherwise, we say that Ω is a non-ground state domain for $X(\Omega)$.

REMARK 4.2. By a domain Ω for $X(\Omega)$, if $X(\Omega) = H_s^1(\Omega)$, then Ω should be a y-symmetric domain in R^N.

By THEOREM 3.9, we have
THEOREM 4.3. *Let $\Omega^1 \subsetneqq \Omega^2$ and $\alpha_X^2 = \alpha_X^1$. Then Ω^1 is a non-ground state domain for $X(\Omega)$.*

In this section, we prove that proper large domains, Esteban-Lions domains, and some interior flask domains $\mathbf{F}_s^r = \mathbf{A}_0^r \cup B^N(0;s)$ are non-ground state .

Definition 4.4.

(i) For $\Omega \subset R^N$, we call Ω a large domain in R^N if for any $r > 0$, $z \in \Omega$ exists such that $B^N(z;r) \subset \Omega$;

(ii) For $\Omega \subset A^r$, we call Ω a large domain in A^r if for any positive number m, a, b exist such that $b - a = m$ and $A_{a,b}^r \subset \Omega$.

Example 4.5.

(i) The whole space $\mathbb{R}^N$, the upper half space $\mathbb{R}_+^N$, a positive paraboloid $\mathbf{P}^+$, an infinite cone $\mathbf{C}$, and an epigraph Π are large domains in R^N;

(ii) The infinite strip $\mathbf{A}^r$, an upper semi-strip $\mathbf{A}_s^r$, the infinite strip with a hole $\mathbf{A}^r \backslash \omega$, and the interior flask domain $\mathbf{F}_s^r$ are large domains in $\mathbf{A}^r$.

Definition 4.6. A proper smooth unbounded domain Ω in R^N is an Esteban-Lions domain if $\chi \in R^N$ exists with $\|\chi\| = 1$ such that $n(z) \cdot \chi \geq 0$, and $n(z) \cdot \chi \not\equiv 0$ on $\partial\Omega$, where $n(z)$ is the unit outward normal vector to $\partial\Omega$ at the point z.

Example 4.7. An upper half strip A_s^r, an epigraph Π, an infinite cone C, the upper half space R_+^N, and a positive paraboloid P^+ are Esteban-Lions domains.

We have the following result.

Theorem 4.8.

(i) *A proper large domain Ω in $\mathbb{R}^N$ or in $\mathbf{A}^r$ is a non-ground state domain for $X(\Omega)$;*

(ii) *An Esteban-Lions domain Ω is a non-ground state domain for $H_0^1(\Omega)$;*

(iii) *$s_0 > 0$ exists such that the interior flask domain $\mathbf{F}_s^r$, for each $s < s_0$, is a non-ground state domain for $H_0^1(\mathbf{F}_s^r)$.*

For $h > r$ and $\mathbf{B} = B^N((0,h); r/2)$, let $\Omega_h = (\mathbf{A}_0^r \cup B^N(0;r)) \backslash \overline{\mathbf{B}}$ be the upper half strip with a hole. Ω_h is a proper large domain in $\mathbf{A}^r$. By Theorem 4.8 (i), there is no any ground state solutions of Equation (1.1) in Ω_h. However, we prove that a positive higher energy solution of Equation (1.1) exists in Ω_h for large h.

Theorem 4.9. *Suppose that the positive solution of Equation* (1.1) *in the infinite strip $\mathbf{A}^r$ is unique up to y-translations. $h_0 > 0$ exists such that if $h \geq h_0$, then there is a positive higher energy solution v of Equation* (1.1) *in the upper half strip with a hole Ω_h such that $\alpha(\mathbf{A}^r) < J(v) < 2^{\frac{p-2}{p}} \alpha(\mathbf{A}^r)$.*

5. Ground State Solutions and Ground State Domains. In this section, we prove that comapctness domains for $X(\Omega)$, the union of a finite numbers of ground state domains for $X(\Omega)$, periodic domains, and the interior flask domains F_s^r *for* $s > s_0$ are ground state domains for $X(\Omega)$.

Remark 5.1. It is known that a ground state solution in $X(\Omega)$ is of constant sign. Note that if u is a solution of Equation (1.1), then $-u$ is also a solution of (1.1). By the maximum principle, if u is a nonzero and nonnegative solution of (1.1), then u is positive. By a ground state solution in $X(\Omega)$, we mean a positive solution of Equation (1.1). If Ω is a $C^{1,1}$ domain in R^N, then by the regularity, by a ground state solution in $X(\Omega)$, we mean a C^2−positive solution of Equation (1.1). In this article, the example of domains may not of $C^{1,1}$, but after smoothing it out, then it is of $C^{1,1}$.

Theorem 5.2. *Let Ω be a domain in $\mathbb{R}^N$ of finite measure. Then the embedding $X(\Omega) \hookrightarrow L^p(\Omega)$ is compact.*

THEOREM 5.3. *If the embedding* $X(\Omega) \hookrightarrow L^p(\Omega)$ *is compact, then* J *satisfies the* $(PS)_{\alpha_{X(\Omega)}}$*-condition. In particular, there is a ground state solution for* $X(\Omega)$ *of Equation* (1.1).

THEOREM 5.4. *Let* $A_1, A_2, \ldots, A_n$ *be domains such that there is a ground state solution for* $X(A_i)$ *of Equation* (1.1), *for each* i. *Let* $A = A_1 \cup A_2 \cup \ldots \cup A_n$. *Then there is a ground state solution for* $X(A)$ *of Equation* (1.1).

DEFINITION 5.5. A domain Θ in $\mathbb{R}^N$ is a periodic domain if a partition $\{Q_m\}_{m=0}^{\infty}$ of Θ and points $\{z_m\}_{m=1}^{\infty}$ in $\mathbb{R}^N$ exist, satisfying the following conditions:
(i) $\{z_m\}_{m=1}^{\infty}$ forms a subgroup of $\mathbb{R}^N$;
(ii) Q_0 is bounded;
(iii) $Q_m = z_m + Q_0$ for each m.

Typical examples of periodic domains are the infinite strip $\mathbf{A}^r$ and the whole space $\mathbb{R}^N$.

THEOREM 5.6. *Let* Θ *be a periodic domain. There is a ground state solution for* $H_0^1(\Theta)$ *of Equation* (1.1). *In particular, there is a ground state solution for* $H_0^1(\mathbf{A}^r)$ *and for* $H_0^1(\mathbb{R}^N)$ *of Equation* (1.1).

We have the following results for the whole space $\mathbb{R}^N$.
THEOREM 5.7.
(i) *Every positive ground state solution for* $H^1(\mathbb{R}^N)$ *of Equation* (1.1) *is spherically symmetric about some point* x_0 *in* $\mathbb{R}^N$, $\bar{u}'(r) < 0$ *for* $r = |x - x_0|$, *and*

$$\lim_{r\to\infty} r^{\frac{N-1}{2}} e^r \bar{u}(r) = \gamma > 0,$$
$$\lim_{r\to\infty} r^{\frac{N-1}{2}} e^r \bar{u}'(r) = -\gamma;$$

(ii) *There is a ground state solution for* $H^1(\mathbb{R}^N)$ *of Equation* (1.1);
(iii) *The positive solution of Equation* (1.1) *in* $\mathbb{R}^N$ *is unique.*

In the infinite strip $\mathbf{A}^r = B^{N-1}(0;r) \times \mathbb{R}$, let λ_1 be the first eigenvalue of $-\Delta$ in $B^{N-1}(0;r)$ with the Dirichlet problem, and ϕ_1 the corresponding positive eigenfunction to λ_1. We have the following results for the infinite strip $\mathbf{A}^r$.

THEOREM 5.8.
(i) *Let* $u(x,y)$ *be a* C^2- *solution of Equation* (1.1) *in* $\mathbf{A}^r$. *Then* u *is radially symmetric in* x *and in* y*; that is to say,* $u(x,y) = u(|x|,|y|)$*;*
(ii) *For each positive solution* u *of Equation* (1.1) *in* $\mathbf{A}^r$, *and for every* $0 < \delta < 1+\lambda_1$ $\gamma > 0$ *and* $\beta > 0$ *exist such that*

$$\gamma\phi_1(x)e^{-\sqrt{1+\lambda_1+\delta}\,|y|} \le u(z) \le \beta\phi_1(x)e^{-\sqrt{1+\lambda_1-\delta}\,|y|} \quad \textit{for } z = (x,y) \in \mathbf{A}^r;$$

(iii) *There is a a ground state solution for* $H_0^1(\mathbf{A}^r)$ *of Equation* (1.1).

Next we present ground state domains from the perturbations $\mathbf{F}_s^r$ of a non-ground state domain $\mathbf{A}_0^r$.
THEOREM 5.9. $s_0 > 0$ *exists such that Equation* (1.1) *has a ground state solution for* $H_0^1(\mathbf{F}_s^r)$ *if* $s > s_0$, *but does not have any ground state solution for* $H_0^1(\mathbf{F}_s^r)$ *if* $s < s_0$.

REMARK 5.10. In THEOREM 5.9 we have asserted that the interior flask domains $F_s^r = A_0^r \cup B^N(0;s)$ are ground state domains if $s > s_0$. In fact, if we replace $A_0^r \cup B^N(0;s)$ by $A_0^r \cup \Omega$, where Ω is a bounded domain containing $B^N(0;s)$, the theorem still holds.

We have the following theorem.

THEOREM 5.11.

(i) *A domain in* $\mathbb{R}^N$ *of finite measure is a compactness domain for* $X(\Omega)$;
(ii) *A comapctness domain for* $X(\Omega)$ *is a ground state domain for* $X(\Omega)$;
(iii) *The union of a finite numbers of ground state domains for* $X(\Omega)$ *is a ground state domain for* $X(\Omega)$;
(iv) *The whole space* $\mathbb{R}^N$ *is a ground state domain for* $H_0^1(\mathbb{R}^N)$ *and for* $H_s(\mathbb{R}^N)$;
(v) *The whole space* $\mathbf{A}^r$ *is a ground state domain for* $H_0^1(\mathbf{A}^r)$ *and for* $H_s(\mathbf{A}^r)$, *Moreover,* $\mathbf{A}^r$ *is a comapctness domain for* $H_s(\mathbf{A}^r)$;
(vi) *A periodic domain* Θ *is a ground state domain for* $H_0^1(\Theta)$;
(vii) *An interior flask domains* $\mathbf{F}_s^r$ *for* $s > s_0$ *is a ground state domain for* $H_0^1(\mathbf{F}_{\mathrm{s}}^{\mathrm{r}})$.

6. A y**-symmetric Domain with Two Same Indexes.** In this section, we assert that if Ω is one of a ball $B^N(0;R)$, the whole space $\mathbb{R}^N$, a finite strip $\mathbf{A}^r_{-t,t}$, and the infinite strip $\mathbf{A}^r$, then $\alpha(\Omega) = \alpha_s(\Omega)$.

THEOREM 6.1. *Suppose that each positive solution* $u \in H_0^1(\Omega)$ *of Equation* (1.1) *is in* $H_s^1(\Omega)$ *and there is a ground state solution for* $H_0^1(\Omega)$ *of Equation* (1.1). *Then* $\alpha(\Omega) = \alpha_s(\Omega)$.

By THEOREM 6.1, we have

THEOREM 6.2.

(i) $\alpha(B^N(0;R)) = \alpha_s(B^N(0;R))$;
(ii) $\alpha(\mathbb{R}^N) = \alpha_s(\mathbb{R}^N)$;
(iii) $\alpha(\mathbf{A}^r_{-t,t}) = \alpha_s(\mathbf{A}^r_{-t,t})$;
(iv) $\alpha(\mathbf{A}^r) = \alpha_s(\mathbf{A}^r)$.

7. A y**-symmetric Domain with Two Different Indexes.** We have the following multiplicity result.

THEOREM 7.1. *Suppose that* Ω *is a ground state domain for* $H_0^1(\Omega)$ *and for* $H_s^1(\Omega)$ *such that* $\alpha(\Omega) < \alpha_s(\Omega)$, *then Equation* (1.1) *on* Ω *has three positive solutions, of which one is* y*-symmetric and other two are not* y*-symmetric.*

In this section, we prove that if Ω is one of a y-symmetric large domain separated by a y-symmetric bounded domain, the finite strip with a hole, and a two-bumps domain, then $\alpha(\Omega) < \alpha_s(\Omega)$. Related results see Wang-Wu [13] and [14].

(**Case A**)

DEFINITION 7.2. Let Ω be a y-symmetric domain and Θ a y-symmetric bounded domain in R^N. If two disjoint subdomains Ω_1 and Ω_2 of Ω exist such that $\Omega\backslash\bar{\Theta} = \Omega_1 \cup \Omega_2$, where $(x, y) \in \Omega_2$ if and only if $(x, -y) \in \Omega_1$. Then we say that the y-symmetric domain Ω is separated by a bounded domain Θ;

EXAMPLE 7.3. Let $0 < r_1 < r$, $t > 0$, x_0 in $B^{N-1}(0; r + r_1)$, and

$$\Theta_t = \mathbf{A}^r \setminus \left[\overline{B^N((x_0, t + r_1); r_1)} \cup \overline{B^N((x_0, -(t + r_1)); r_1)}\right].$$

Then the infinite strip with holes Θ_t is a y-symmetric large domain in A^r separated by a bounded domain $A^r_{-t,t}$.

Let Ω be a y-symmetric large domain separated by a y-symmetric bounded domain, then we have the following results:

THEOREM 7.4. *Let E be either $\mathbb{R}^N$ or $\mathbf{A}^r$. Suppose that Ω is a proper y-symmetric large domain in E separated by a y-symmetric bounded domain, then $\alpha(\Omega) < \alpha_s(\Omega)$. Moreover, If $\alpha_s(\Omega) < 2\alpha(\Omega)$, then J satisfies the $(PS)_{\alpha_s(\Omega)}-$condition in $H_s(\Omega)$.*

By THEOREM 7.4, we have
THEOREM 7.5. *There exists $t_0 > 0$ such that $\alpha_s(\Theta_t) < 2\alpha(\Theta_t)$ for all $t \geq t_0$. In particular, there is a ground state solution for $H_s^1(\Theta_t)$ of Equation (1.1) in Θ_t, for all $t \geq t_0$.*

(**Case B**)
Let $0 < r_1 < r$, $t > 0$, and x_0 in $B^{N-1}(0; r + r_1)$, consider the finite strip with a hole Ψ_t, where

$$\Psi_t = \mathbf{A}^r_{-t,t} \backslash \overline{B^N((x,0); r_1)}.$$

Then we have the following assertion.

THEOREM 7.6. *There exists $t_0 > 0$ such that for $t \geq t_0$, we have $\alpha(\Psi_t) < \alpha_s(\Psi_t)$. Moreover, Equation (1.1) on Ψ_t has three positive solutions of Equation (1.1), in which one is y-symmetric and the other two are not y-symmetric.*

(Case C)
We consider the two-bumps domains.

DEFINITION 7.7. Let Θ be a proper ground state domain for $H_s^1(\Theta)$ in $\mathbb{R}^N$ bounded in the $x-$direction. For $R > 0$, let Ω_R^1 and Ω_R^2 be two bounded domains in $\mathbb{R}^N$ such that Ω_R^1 contains a ball of radius R, $\Omega_R^1 \subset \mathbb{R}^N \setminus \overline{\mathbb{R}^N_{-\rho,\rho}}$ for some $\rho > 0$, and $\Omega_R^2 = \{(x,y) : (x,-y) \in \Omega_R^1\}$. The y-symmetric domain D_R is called the two-bumps domain, where

$$D_R = \Omega_R^1 \cup \Theta \cup \Omega_R^2.$$

By THEOREM 5.4, the two-bumps domain D_R is a ground state domain for $H_s^1(D_R)$. Here are some examples of two-bumps domains.

EXAMPLE 7.8.

(i) For $t > R > r > 0$. The bounded dumbbell domain D_R^b is a two-bumps domain, where

$$D_R^b = B^N((0,-t), R) \cup \mathbf{A}^r_{-t,t} \cup B^N((0,t), R);$$

(ii) For $t > R > r > 0$. The unbounded dumbbell domain D_R^u is a two-bumps domain, where

$$D_R^u = B^N((0,-t), R) \cup \mathbf{A}^r \cup B^N((0,t), R);$$

(iii) For $t > R > r > 0$. The curved dumbbell domain $D_R^c = \Omega_R^1 \cup \Theta \cup \Omega_R^2$ is a two bumps domain, where Ω_R^1 and Ω_R^2 be two bounded domains in R^N such that $dist\{0, \Omega_R^1\} > 0$, Ω_R^1 contains a ball of radius R, and $\Omega_R^2 = \{(x,y) : (x,-y) \in \Omega_R^1\}$, and Θ is a curved bounded y-symmetric domain in R^N.

THEOREM 7.9. *Let D_R be a two bumps domain. There is an $R_0 > 0$ such that for $R \geq R_0$, we have $\alpha(D_R) < \alpha_s(D_R)$. Moreover, Equation (1.1) in D_R has three positive solutions, of which one is y-symmetric and other two are not y-symmetric.*

Since finite dumbbell is a two bumps domain, the results of Byeon [1], Chen-Ni-Zhou [2], and Dancer [6] are the consequences of our THEOREM 7.9.

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