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ROBIN TYPE CONDITIONS ARISING FROM CONCENTRATED POTENTIALS*

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Abstract. We analyze the limit of the solutions of an elliptic problem with zero flux boundary condition when the potential functions are concentrated in a neighborhood of the boundary and this neighborhood shrinks to the boundary as a parameter goes to zero.

We prove that this family of solutions converges in the sup norm to the solution of an elliptic problem with Robin type condition.

This Robin type conditions for the limiting problem comes from the concentrated potentials around the boundary of the domain.

Key words. Robin flux conditions, elliptic boundary problems, concentrating integrals

1. Introduction.

As it is usually accepted that the environmental influence on a reacting media is modelled by a boundary condition in a boundary value problem, it is also common sense to state that boundary conditions of flux type, must also be equivalent to some external action localized in a thin neighborhood of the boundary.

In fact elementary application of finite difference schemes to one dimensional boundary value problems, show that this analogy or equivalence must have a sounded mathematical foundation.

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Therefore in this paper we present a rigurous mathematical analysis of this question for some elliptic (i.e. time independent) linear problems. In particular we show how linear reaction and flux terms on the boundary condition can be obtained as a result of a limiting process from terms distibuted and concentrated near the boundary.

2. Concentrating integrals. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary $\partial\Omega = \Gamma$ and outward normal vector $\vec{n}(x)$ for $x \in \Gamma$.

We consider the neighborhood of Γ defined as $\omega_\varepsilon = \{x - s\vec{n}(x), x \in \Gamma, s \in [0, \varepsilon)\} \subset \bar{\Omega}$ for sufficiently small ε. Then, if we consider $\Gamma_\sigma = \{x - \sigma\vec{n}(x), x \in \Gamma\}$, $\Gamma_0 = \Gamma$, $0 < \varepsilon < \varepsilon_0$ the "parallel" interior boundary, we have $\omega_\varepsilon = \cup_{0 \le \sigma < \varepsilon} \Gamma_\sigma$.

Using the standard trace theory and local parameterization of the boundary and partitions of unity we can prove the following results.

LEMMA 2.1. *Assume that* $v \in W^{s,p}(\Omega) \subset L^q(\Gamma_\sigma)$ *with* $s > \frac{1}{p}$ *and* $s - \frac{N}{p} \ge -\frac{(N-1)}{q}$ *i.e.* $q \le \frac{p(N-1)}{N-sp}$. *Then for* ε *small enough:*

i) *The map* $\sigma \mapsto \int_{\Gamma_\sigma} v^q$ *is continuous.*

ii) *There exists* $C > 0$ *independent of* ε *such that*

$$\sup_{\sigma \in [0,\varepsilon)} \|v\|_{L^q(\Gamma_\sigma)} \le C\|v\|_{W^{s,p}(\Omega)} \tag{2.1}$$

iii)

$$\int_{\omega_\varepsilon} |v|^q = \int_0^\varepsilon \int_{\Gamma_\sigma} |v|^q \, dS \, d\sigma. \tag{2.2}$$

iv) *In particular*

$$\frac{1}{\varepsilon} \int_{\omega_\varepsilon} |v|^q \le c\|v\|^q_{W^{s,p}(\Omega)} \tag{2.3}$$

and

$$\lim_{\varepsilon\mapsto 0}\frac{1}{\varepsilon}\int_{\omega_\varepsilon}|v|^q=\int_\Gamma |v|^q$$

Note that these results assert that for smooth test functions, integrals concentrated close to the boundary converge to integrals on the boundary.

With this we can prove then the following result that, in some sense, allows to pass to the limit in concentrating integrals, with nonsmooth integrands.

Lemma 2.2. *We assume that the functions h_ε on ω_ε are such that*

$$\frac{1}{\varepsilon}\int_{\omega_\varepsilon}|h_\varepsilon(x)|^r dx\le C<\infty,\qquad \text{for some } 1\le r<\infty \tag{2.4}$$

for some constant C independent of ε. Then there exists a subsequence, still denoted by ε, and a function $h_0\in L^r(\Gamma)$ (or $h_0\in\mathcal{M}(\Gamma)$ if $r=1$), such that

i) *For any continuous function in ω_{ε_0}, φ, we have*

$$\lim_{\varepsilon\mapsto 0}\frac{1}{\varepsilon}\int_{\omega_\varepsilon}h_\varepsilon(x)\varphi(x)\,\mathrm{d}x=\int_\Gamma h_0(y)\varphi(y)\,\mathrm{d}\sigma. \tag{2.5}$$

ii) *If u_ε converge to u weakly in $W^{s,p}(\Omega)$ with $s>\frac{1}{p}$ and $s-\frac{N}{p}\ge-\frac{N-1}{r'}$, then*

$$\lim_{\varepsilon\mapsto 0}\frac{1}{\varepsilon}\int_{\omega_\varepsilon}h_\varepsilon(x)u_\varepsilon\,\mathrm{d}x=\int_\Gamma h_0(y)u(y)\,\mathrm{d}\sigma. \tag{2.6}$$

iii) *If $u_\varepsilon\mapsto u_0$ in $W^{s,p}(\Omega)$ and $\varphi_\varepsilon\mapsto\varphi_0$ in $W^{\sigma,q}(\Omega)$ with $s>\frac{1}{p}$ and $\sigma>\frac{1}{q}$ and $s+\sigma-\frac{N}{p}-\frac{N}{q}>-\frac{N-1}{r'}$, we have that*

$$\lim_{\varepsilon\mapsto 0}\frac{1}{\varepsilon}\int_{\omega_\varepsilon}h_\varepsilon(x)u_\varepsilon\varphi_\varepsilon(x)\,\mathrm{d}x=\int_\Gamma h_0(y)u_0(y)\varphi(y)\,\mathrm{d}\sigma. \tag{2.7}$$

iv) *In particular if we assume the above the hypothesis, and consider* $(W^{\sigma,q}(\Omega))' = W^{-\sigma,q'}(\Omega)$ *such that* $W^{\sigma,q}(\Omega) \subset L^{r'}(\Gamma)$, *i.e.* $\sigma > \frac{1}{q}$ *and* $\sigma - \frac{N}{q} \geq -\frac{N-1}{r'}$. *Then,*

1. $L_\varepsilon = \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} h_\varepsilon$ *is a bounded family in* $(W^{\sigma,q}(\Omega))' := W^{-\sigma,q'}(\Omega)$.
2. *The operators* B_ε *converge to the operator* B_0 *in* $\mathcal{L}(W^{s,p}(\Omega), W^{-\sigma,q'}(\Omega))$ *with* $s > \frac{1}{p}$ *and* $\sigma > \frac{1}{q}$ *and* $s + \sigma - \frac{N}{p} - \frac{N}{q} > -\frac{N-1}{r'}$, *where*

$$< B_\varepsilon(u), \varphi >= \frac{1}{\varepsilon}\int_{\omega_\varepsilon} h_\varepsilon u\varphi \quad \longmapsto \quad < B_0(u), \varphi >= \int_\Gamma h_0 u\varphi \tag{2.8}$$

for $u \in W^{s,p}(\Omega)$ *and* $\varphi \in W^{\sigma,q}(\Omega)$.

Proof. 1. If $\varphi \in W^{s,p}(\Omega)$ from Lemma 2.1, we have that

$$|L_\varepsilon(\varphi)| \leq \frac{1}{\varepsilon}\int_{\omega_\varepsilon} |h_\varepsilon||\varphi| \leq \left[\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |h_\varepsilon|^r\right]^{\frac{1}{r}} \left[\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |\varphi|^{r'}\right]^{\frac{1}{r'}} \leq C\|\varphi\|_{W^{s,p}(\Omega)}.$$

2. If $\frac{1}{r} + \frac{1}{m} + \frac{1}{n} = 1$

$$\left|\frac{1}{\varepsilon}\int_{\omega_\varepsilon} h_\varepsilon u\varphi\right| \leq \left[\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |h_\varepsilon|^r\right]^{\frac{1}{r}} \left[\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |u|^m\right]^{\frac{1}{m}} \left[\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |\varphi|^n\right]^{\frac{1}{n}}.$$

If r, m, n are such that $s_0 - \frac{N}{p_0} \geq -\frac{N-1}{m}$ with $s_0 > \frac{1}{p_0}$ and $\sigma_0 - \frac{N}{q_0} \geq -\frac{N-1}{n}$ with $\sigma_0 > \frac{1}{q_0}$, then from Lemma 2.2 and (2.4)

$$\left|\frac{1}{\varepsilon}\int_{\omega_\varepsilon} h_\varepsilon u\varphi\right| \leq C\|u\|_{W^{s_0,p_0}(\Omega)}\|\varphi\|_{W^{\sigma_0,q_0}(\Omega)}$$

and B_ε from $W^{s_0,p_0}(\Omega)$ into $W^{-\sigma_0,q_0'}(\Omega))$ is uniformly bounded. Hence, fixed $u \in W^{s_0,p_0}(\Omega)$ we have Lemma 2.1 that

$$< B_\varepsilon(u), \varphi >= \frac{1}{\varepsilon}\int_{\omega_\varepsilon} h_\varepsilon u\varphi \quad \mapsto \quad \int_\Gamma h_0 u\varphi =< B_0(u), \varphi >$$

uniformly for φ in compact sets of $W^{\sigma_0,q_0}(\Omega)$. Hence if $q \geq q_0$ with $\sigma > \frac{1}{q}$ and $\sigma - \frac{N}{q} > \sigma_0 - \frac{N}{q_0}$ then $W^{\sigma,q}(\Omega) \subset W^{\sigma_0,q_0}(\Omega)$ with compact embedding, and then, in particular

$$B_\varepsilon(u) \quad \mapsto \quad B_0(u) \text{ in } W^{-\sigma,q'}(\Omega).$$

Again this implies uniform convergence for u in compact sets of $W^{s_0,p_0}(\Omega)$. Hence if $p \geq p_0$ with $s > \frac{1}{p}$ and $s - \frac{N}{p} \leq s_0 - \frac{N}{p_0}$ then $W^{s,p}(\Omega) \subset W^{s_0,p_0}(\Omega)$ with compact embedding, and then, in particular, we have

$$B_\varepsilon \quad \mapsto \quad B_0 \qquad \text{in } \mathcal{L}(W^{s,p}(\Omega), W^{-\sigma,q'}(\Omega))$$

which gives the result. □

3. An elliptic problem with concentrating non homogeneous terms. With the previous results, in this section we study the behavior, for small ε, of the solutions of the elliptic problem

$$\begin{cases} -\Delta u_\varepsilon + u_\varepsilon = \dfrac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} f_\varepsilon & \text{in } \Omega \\ \dfrac{\partial u_\varepsilon}{\partial n} = 0 & \text{on } \Gamma \end{cases} \tag{3.1}$$

with a given f_ε and where $\mathcal{X}_{\omega_\varepsilon}$ denote the characteristic function of the set ω_ε. Hence the effective reaction is concentrated on this set.

The goal in this section is to prove that, the family of solutions, u_ε, converges when the parameter ε goes to zero, to a limit function, which is given by the solution of the homogeneous elliptic problem with nonhomogeneous flux conditions on the boundary:

$$\begin{cases} -\Delta u + u = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial n} = f_0 & \text{on } \Gamma. \end{cases} \tag{3.2}$$

Moreover, we give conditions on f_ε that guarantee that the convergence is uniform in $\bar\Omega$ and in $\mathcal{C}^\infty_{loc}(\Omega)$. For this we will assume, that f_ε satisfies

$$\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |f_\varepsilon(x)|^r \,\mathrm{d}x \le C < \infty \tag{3.3}$$

for some constant C independent of ε and there exists a function, $f_0 \in L^r(\Gamma)$ such that for any smooth function φ, we have

$$\lim_{\varepsilon\mapsto 0}\frac{1}{\varepsilon}\int_{\omega_\varepsilon} f_\varepsilon(x)\varphi(x)\,\mathrm{d}x = \int_\Gamma f_0(y)\varphi(y)\,\mathrm{d}\sigma. \tag{3.4}$$

with $\varepsilon_0 > 0$ sufficiently small.

With the notations above, we can prove first

Theorem 3.1.

i) *Assume the notations above and the hypotheses (3.3) with $r = 1$ and (3.4). Then the family u_ε converges to u in $W^{1,q}(\Omega)$ for every $q < \frac{N}{N-1}$, in $L^s(\Omega)$ for every $s < \frac{N}{N-2}$ and almost everywhere, where u is the unique solution of (3.2). Moreover the family u_ε also converges in $\mathcal{C}^j(K)$ for every j and $K \subset\subset \Omega$ a compact set.*

ii) *Assume in addition that, for $r > 1$ we have*

$$\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |f_\varepsilon(x)|^r \,\mathrm{d}x \le C < \infty$$

for some constant C independent of ε. Then u_ε is uniformly bounded in $W^{1+\frac{1}{r},r}(\Omega)$ and therefore

$$u_\varepsilon \quad\mapsto\quad u \text{ in } W^{\alpha,q}(\Omega)$$

with $1+\frac{1}{r}-\frac{N}{r} > \alpha - \frac{N}{q}$. In particular, if $r > N-1$ then

$$u_\varepsilon \quad\mapsto\quad u \text{ in } \mathcal{C}^\beta(\bar\Omega)$$

for some $\beta > 0$.

Proof. i) Let $F_\varepsilon = \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} f_\varepsilon$ which is uniformly bounded in $L^1(\Omega)$. Then, there exists λ_0 such that if $\lambda \geq \lambda_0$ then the elliptic operator is invertible. Hence, by elliptic regularity we have a uniform bound of solutions, u_ε, in $W^{2-\delta,1}(\Omega)$ for every $\delta > 0$. Thus, there exists a subsequence which converges weakly to u in $W^{2-\delta,1}(\Omega)$, for every $\delta > 0$.

On the other hand, again from Sobolev's embedding, we have that for any $q < \frac{N}{N-1}$ we can find $\delta > 0$ such that $W^{2-\delta,1}(\Omega) \subset W^{1,q}(\Omega)$ with compact inclusion. From this, we can ensure the subsequence converges strongly in $W^{1,q}(\Omega)$ for every $q < \frac{N}{N-1}$. Moreover, taking into account that, again from Sobolev's embeddings, for every $s < \frac{N}{N-2}$ we can find $q < \frac{N}{N-1}$ such that $W^{1,q}(\Omega) \subset L^s(\Omega)$, then the subsequence converge also strongly in $L^s(\Omega)$ and thus almost everywhere. Also, the traces of u_ε converge to the trace of u in $W^{1-\frac{1}{q},q}(\Gamma)$.

Next, we prove u verifies the homogeneous elliptic problem (3.2). In effect, multiplying the equation from (3.1) by $\varphi \in \mathcal{C}^\infty(\Omega)$ we obtain

$$\int_\Omega (\nabla u_\varepsilon \nabla \varphi + u_\varepsilon)\varphi = \frac{1}{\varepsilon}\int_{\omega_\varepsilon} f_\varepsilon \varphi.$$

Now we assume first $\varphi \in \mathcal{C}_c^\infty(\Omega)$ and taking the limit as ε goes to zero, using (3.4), we get

$$\int_\Omega \nabla u \nabla \varphi + u\varphi = 0, \quad \forall \varphi \in \mathcal{C}_c^\infty(\Omega). \tag{3.5}$$

Therefore the limit function satisfies $-\Delta u + u = 0$ in Ω. In particular $u \in \mathcal{C}_{loc}^\infty(\Omega)$. Now, we consider a test function $\varphi \in \mathcal{C}^\infty(\bar{\Omega})$ and using again (3.4) together with the convergence of the traces, we get

$$\int_\Omega \nabla u \nabla \varphi + \int_\Omega u\varphi = \int_\Gamma f_0 \varphi, \qquad \text{for all } \varphi \in \mathcal{C}^\infty(\bar{\Omega}). \tag{3.6}$$

Using the equation for u in Ω and integrating by parts, we get now

$$\int_\Gamma \frac{\partial u}{\partial n}\varphi = \int_\Gamma f_0\varphi, \quad \forall \varphi \in \mathcal{C}^\infty(\bar{\Omega})$$

where $\frac{\partial u}{\partial n}$ denote the normal derivative defined by

$$\int_\Gamma \frac{\partial u}{\partial n}\varphi = \int_\Omega \nabla u \nabla \varphi - \int_\Omega (-\Delta u)\varphi$$

for every $\varphi \in \mathcal{C}^\infty(\bar{\Omega})$. Therefore u solves (3.2).

Now, since $\lambda > \lambda_0$, from the uniqueness of solutions for the limit problem (3.2), we have that the whole family u_ε converges to u.

Next, to improve the convergence in Ω, given $K \subset\subset \Omega$ we consider the auxiliar function $v_\varepsilon = u_\varepsilon \varphi$ where $\varphi \in \mathcal{C}_c^\infty(\Omega)$ is a regular function, such that is equal to 1 in K, and 0 in the set $\Omega \setminus \Omega'$ where Ω' is a domain such that $K \subset \Omega' \subset \Omega$.

Now observe that considering ε small enough such that $\omega_\varepsilon \subset \Omega \setminus \Omega'$ and thus $F_\varepsilon \varphi = 0$ in Ω. Then, v_ε, satisfies the elliptic problem

$$\begin{cases} -\Delta v_\varepsilon + v_\varepsilon = g_\varepsilon & \text{in } \Omega \\ v_\varepsilon = 0 & \text{on } \Gamma \end{cases} \tag{3.7}$$

with $g_\varepsilon = -2\nabla u_\varepsilon \nabla \varphi - u_\varepsilon \Delta \varphi$, which satisfies $g_\varepsilon \in L^q(\Omega)$. From uniform estimates of the family u_ε in $W^{1,q}(\Omega)$, with $q < \frac{N}{N-1}$, we obtain $\|g_\varepsilon\|_{L^q(\Omega)} \le C_1$ for some constant $C_1 > 0$ independent of ε. Then, from (3.7) there exists $C_2 > 0$ such that $\|v_\varepsilon\|_{W^{2,q}(\Omega)\cap W_0^{1,q}(\Omega)} \le C_2$.

Now, applying a bootstrap argument to v_ε and taking into account that $v_\varepsilon = u_\varepsilon$ on K we have that, for any j there exists $C(j)$ positive constant independent of ε such that

$$\|u_\varepsilon\|_{\mathcal{C}^j(K)} \le C(j).$$

Thus, using the Ascoli-Arzela's theorem, and the uniqueness of u we get the result.

ii) From Lemma 2.2 the right hand side in the equation of u_ε defines a bounded family in the Sobolev's space $(W^{s,p}(\Omega))' = W^{-s,p'}(\Omega)$ for $s > \frac{1}{p}$ and $s - \frac{N}{p} = -\frac{N-1}{r'}$.

Now, from the elliptic regularity, we have the solution of (3.1) satisfies that $u_\varepsilon \in W^{2-s,p'}(\Omega)$ and is uniformly bounded in this space.

Now note that $W^{2-s,p'}(\Omega) \subset W^{\alpha,q}(\Omega)$, with compact embedding, provided

$$2 - s - \frac{N}{p'} = 1 - \frac{N-1}{r} = 1 + \frac{1}{r} - \frac{N}{r} > \alpha - \frac{N}{q}.$$

In particular, if $r > N-1$ we have $1 - \frac{N-1}{r} > 0$ and then $W^{2-s,p'}(\Omega) \subset C^\beta(\bar{\Omega})$ with compact embedding, for some β. □

4. An elliptic problem with concentrating potentials. We assume now that we have concentrating potentials V_ε and nonhomogeneous terms f_ε satisfying

$$\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |V_\varepsilon(x)|^\rho \, \mathrm{d}x \le C < \infty \tag{4.1}$$

$$\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |f_\varepsilon(x)|^r \, \mathrm{d}x \le C < \infty \tag{4.2}$$

for some constant C independent of ε, and there exists functions $f_0 \in L^r(\Gamma)$, and $V_0 \in L^\rho(\Gamma)$ (or $f_0, V_0 \in \mathcal{M}(\Gamma)$ if $r = \rho = 1$), such that for any function φ, continuous in ω_{ε_0}, we have

$$\lim_{\varepsilon \mapsto 0} \frac{1}{\varepsilon}\int_{\omega_\varepsilon} f_\varepsilon(x)\varphi(x)\, \mathrm{d}x = \int_\Gamma f_0(y)\varphi(y)\, \mathrm{d}\sigma. \tag{4.3}$$

$$\lim_{\varepsilon \mapsto 0} \frac{1}{\varepsilon}\int_{\omega_\varepsilon} V_\varepsilon(x)\varphi(x) dx = \int_\Gamma V_0(y)\varphi(y) d\sigma. \tag{4.4}$$

In this section we prove that for such concentrating potentials and nonhomogeneous terms, we get $u_\varepsilon \mapsto u$ in $\mathcal{C}(\bar{\Omega})$ where u_ε satisfies that

$$\begin{cases} -\Delta u_\varepsilon + \lambda u_\varepsilon = \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} V_\varepsilon u_\varepsilon + \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} f_\varepsilon + g & \text{in } \Omega \\ \dfrac{\partial u_\varepsilon}{\partial n} = 0 & \text{on } \Gamma \end{cases} \tag{4.5}$$

and u satisfies

$$\begin{cases} -\Delta u + \lambda u = g & \text{in } \Omega \\ \dfrac{\partial u}{\partial n} = V_0 u + f_0 & \text{on } \Gamma. \end{cases}$$

In fact if s, p, σ are as in Lemma 2.2 iii) and if we define the operator $P_\varepsilon \in \mathcal{L}(W^{s,p}(\Omega), W^{-\sigma,p}(\Omega))$ as

$$< P_\varepsilon(u_\varepsilon), \varphi >= \frac{1}{\varepsilon}\int_{\omega_\varepsilon} V_\varepsilon u_\varepsilon \varphi,$$

then we have

Theorem 4.1. *Assume V_ε satisfies (4.1) with $\rho > N-1$. Then there exists some λ_0 such that for $\lambda \geq \lambda_0$ the the elliptic operator $A_0 + \lambda I - P_\varepsilon$ in (4.5) is invertible and*

$$\|(A_0 + \lambda I - P_\varepsilon)^{-1}\|_{W^{-\sigma,p}(\Omega)} \leq \frac{C}{|\lambda|} \tag{4.6}$$

for some constant C independent of ε.

Proof. Writing (4.5) as a perturbation of a fixed elliptic operator i.e.

$$A_0 u_\varepsilon + \lambda u_\varepsilon = P_\varepsilon u_\varepsilon + h_\varepsilon$$

from Lemma 2.2 we have $P_\varepsilon \mapsto P_0$ in $\mathcal{L}(W^{s,p}(\Omega), W^{-\sigma,p}(\Omega))$ with

$$< P_\varepsilon(u_\varepsilon), \varphi >= \frac{1}{\varepsilon}\int_{\omega_\varepsilon} V_\varepsilon u_\varepsilon \varphi \quad \mapsto \quad \int_\Gamma V_0 u \varphi =< P_0(u), \varphi > \tag{4.7}$$

for every $u \in W^{s,p}(\Omega)$ and $\varphi \in W^{\sigma,p'}(\Omega)$.

Since $\rho > N-1$ we can take $s+\sigma < 2$ and we get that $A_0 + \lambda I - P_\varepsilon$ is well defined from $W^{2-\sigma,p}(\Omega)$ into $W^{-\sigma,p}(\Omega)$. In particular for given $g \in W^{-\sigma,p}(\Omega)$ the equation $A_0 u_\varepsilon + \lambda u_\varepsilon - P_\varepsilon u_\varepsilon = g$ can be written as

$$u_\varepsilon = (A_0 u_\varepsilon + \lambda I)^{-1}(g + P_\varepsilon u_\varepsilon) = T^\varepsilon_\lambda(u_\varepsilon).$$

Now the resolvent estimates for A_0 imply that the Lipschitz constant of T^ε_λ is bounded by $\frac{C}{|\lambda|}$, for some constant C independent of ε, and therefore it is a contraction for large enough λ. □

With this we get

COROLLARY 4.2. *Assume that $\lambda \geq \lambda_0$ and in addition that, for $r > 1$ we have*

$$\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |f_\varepsilon(x)|^r \, \mathrm{d}x \leq C < \infty$$

Then u_ε is uniformly bounded in $W^{1+\frac{1}{r},r}(\Omega)$ and therefore

$$u_\varepsilon \mapsto u \text{ in } W^{\alpha,q}(\Omega) \text{ as } \varepsilon \mapsto 0$$

with $1 + \frac{1}{r} - 1 + \frac{N}{r} > \alpha - \frac{N}{q}$ where u is the unique solution of

$$\begin{cases} -\Delta u + \lambda u = g & \text{in } \Omega \\ \dfrac{\partial u}{\partial n} = V_0 u + f_0 & \text{on } \Gamma \end{cases} \tag{4.8}$$

In particular, if $r > N-1$ then

$$u_\varepsilon \mapsto u \text{ in } C^\beta(\bar{\Omega}) \text{ as } \varepsilon \mapsto 0$$

for some $\beta > 0$.

The proofs above can be extended along the same lines above for more general elliptic problems with smooth coefficients.

THEOREM 4.3. *Under the above notations and the hypotheses for $V_\varepsilon, f_\varepsilon$ and g, we consider the family u_ε given by the solutions of the following problems*

$$\begin{cases} -\mathrm{div}(a(x)\nabla u_\varepsilon) + c(x)u_\varepsilon + \lambda u_\varepsilon = \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} V_\varepsilon u_\varepsilon + \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} f_\varepsilon + g & \text{in } \Omega \\ u_\varepsilon = h, & \text{on } \Gamma_0, \\ a(x)\dfrac{\partial u_\varepsilon}{\partial n} + b(x)u_\varepsilon = j & \text{on } \Gamma_1 \end{cases} \tag{4.9}$$

for sufficiently large λ, and smooth coefficients $a, c \in \mathcal{C}^1(\bar{\Omega})$, and $b \in \mathcal{C}^1(\bar{\Gamma}_1)$ with $a(x) \geq a_0 > 0$.

Then, given $h \in \mathcal{C}(\bar{\Gamma}_0)$ and $j \in L^r(\Gamma_1)$, we have that $u_\varepsilon \mapsto u$ in $\mathcal{C}(\bar{\Omega})$ where u is the solution of

$$\begin{cases} -\mathrm{div}(a(x)\nabla u + c(x)u + \lambda u = g \\ u = h, & \text{on } \Gamma_0, \\ a(x)\dfrac{\partial u}{\partial n} + b(x)u = j + V_0 u + f_0 & \text{on } \Gamma_1 \end{cases} \tag{4.10}$$

The key argument for the proof, relies in considering the scale of extrapolation spaces associated to these operators, as constructed in [1]. In fact it can be shown that this scale does not depend on ε and that the concentrating potentials and nonhomogeneous terms remain uniformly bounded in the corresponding dual norms.

5. Singular coefficients. The analysis above fails if the smoothness of the coefficients in (4.9) is removed, since then the scales of spaces in [1] is no longer available.

In this case however, by using a different approach we are still able to prove analogous convergence results as above; in particular, we are still able to prove the convergence in $\mathcal{C}(\bar{\Omega})$. See [2] for details.

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