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ASYMPTOTIC PROPERTIES OF A TWO-DIMENSIONAL DIFFERENTIAL SYSTEM WITH DELAY*

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Abstract. A useful method for the investigation of the asymptotic behaviour of the solutions of two-dimensional systems of ordinary differential equations is the method of complexification. This method was used e. g. in papers [6] and [5], where asymptotic properties of a real two-dimensional system $x' = \mathsf{A}(t)x + h(t,x)$ were studied. In the present contribution we shall examine the asymptotic nature of the solutions of a real two-dimensional system of retarded differential equations $x'(t) = \mathsf{A}(t)x(t) + \mathsf{B}(t)x(t-r) + h(t,x(t),x(t-r))$, where $r > 0$ is a constant delay, A, B and h being matrix functions and a vector function, respectively. The method of complexification transforms this system to one equation with complex-valued coefficients. Stability and the asymptotic properties of this equation are studied by means of a suitable Lyapunov-Krasovskii functional and by virtue of the Ważewski topological principle. The contribution has a character of an overview article, nevertheless several results are given in a somewhat modified and more general form than those given in [4] and [3].

Key words. delayed differential equations, asymptotic behaviour, stability, boundedness of solutions, two-dimensional systems, Lyapunov method, Ważewski topological principle

AMS subject classifications. 34K15

1. Introduction. Consider the real two-dimensional system

$$x'(t) = \mathsf{A}(t)x(t) + \mathsf{B}(t)x(t-r) + h(t,x(t),x(t-r)), \tag{1.1}$$

where $\mathsf{A}(t) = (a_{jk}(t))$, $\mathsf{B}(t) = (b_{jk}(t))$ $(j,k = 1,2)$ are real square matrices and $h(t,x,y) = (h_1(t,x,y), h_2(t,x,y))$ is a real vector function, $x = (x_1,x_2)$, $y = (y_1,y_2)$. It is supposed that the functions a_{jk} are locally absolutely continuous on $[t_0,\infty)$, b_{jk} are locally Lebesgue integrable on $[t_0,\infty)$ and the function h satisfies Carathéodory conditions on $[t_0,\infty)\times\mathbb{R}^4$.

There is a lot of papers dealing with the stability and asymptotic behaviour of n-dimensional real vector equations with delay, for references see e. g. [1] or [2]. Since the plane has special topological properties different from those of n-dimensional space, where $n \geq 3$ or $n = 1$, it is interesting to study asymptotic behaviour of two-dimensional systems by using tools which are typical and effective for two-dimensional systems. The method of complexification allows to simplify some considerations and estimations and, combined with the technique of Lyapunov-Krasovskii functional and Razumikhin-type version of Ważewski topological method, it leads to new, effective and easy applicable results in the two-dimensional case. We shall give results both for the stable and instable case of the equation (1.1). More details and further results can be found in [4] and in [3]. For a similar results dealing with ordinary differential equations without delay, the reader is referred to [6] and [5]. Notice that the Razumikhin-type versions of Ważewski principle for retarded functional differential equations were formulated in papers of K. P. Rybakowski [7, 8].

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2. Preliminaries. Introducing complex variables $z = x_1 + \mathrm{i}\, x_2$, $w = y_1 + \mathrm{i}\, y_2$, we can rewrite the system (1.1) into an equivalent equation with complex-valued coefficients

$$z'(t) = a(t)z(t) + b(t)\bar{z}(t) + A(t)z(t-r) + B(t)\bar{z}(t-r) + g(t, z(t), z(t-r)), \quad (2.1)$$

where

$$\begin{aligned}
a(t) &= \frac{1}{2}(a_{11}(t) + a_{22}(t)) + \frac{\mathrm{i}}{2}(a_{21}(t) - a_{12}(t)),\\
b(t) &= \frac{1}{2}(a_{11}(t) - a_{22}(t)) + \frac{\mathrm{i}}{2}(a_{21}(t) + a_{12}(t)),\\
A(t) &= \frac{1}{2}(b_{11}(t) + b_{22}(t)) + \frac{\mathrm{i}}{2}(b_{21}(t) - b_{12}(t)),\\
B(t) &= \frac{1}{2}(b_{11}(t) - b_{22}(t)) + \frac{\mathrm{i}}{2}(b_{21}(t) + b_{12}(t)),\\
g(t,z,w) &= h_1\left(t, \frac{1}{2}(z+\bar{z}), \frac{1}{2\,\mathrm{i}}(z-\bar{z}), \frac{1}{2}(w+\bar{w}), \frac{1}{2\,\mathrm{i}}(w-\bar{w})\right)\\
&\qquad + \mathrm{i}\, h_2\left(t, \frac{1}{2}(z+\bar{z}), \frac{1}{2\,\mathrm{i}}(z-\bar{z}), \frac{1}{2}(w+\bar{w}), \frac{1}{2\,\mathrm{i}}(w-\bar{w})\right).
\end{aligned}$$

Conversely, putting $a_{11}(t) = \mathrm{Re}[a(t) + b(t)]$, $a_{12}(t) = \mathrm{Im}[b(t) - a(t)]$, $a_{21}(t) = \mathrm{Im}[a(t) + b(t)]$, $a_{22}(t) = \mathrm{Re}[a(t) - b(t)]$, $b_{11}(t) = \mathrm{Re}[A(t) + B(t)]$, $b_{12}(t) = \mathrm{Im}[B(t) - A(t)]$, $b_{21}(t) = \mathrm{Im}[A(t) + B(t)]$, $b_{22}(t) = \mathrm{Re}[A(t) - B(t)]$, $h_1(t,x,y) = \mathrm{Re}\, g(t, x_1 + \mathrm{i}\, x_2, y_1 + \mathrm{i}\, y_2)$, $h_2(t,x,y) = \mathrm{Im}\, g(t, x_1 + \mathrm{i}\, x_2, y_1 + \mathrm{i}\, y_2)$, $\mathsf{A}(t) = (a_{ij}(t))$, $\mathsf{B}(t) = (b_{ij}(t))$, the equation (2.1) can be written in the real form (1.1).

We shall use the following notation:

$\mathbb{R}$	set of all real numbers,
$\mathbb{R}_+$	set of all positive real numbers,
$\mathbb{R}_+^0$	set of all non-negative real numbers,
$\mathbb{R}_-$	set of all negative real numbers,
$\mathbb{R}_-^0$	set of all non-positive real numbers,
$\mathbb{C}$	set of all complex numbers,
$AC_{\mathrm{loc}}(I, M)$	class of all locally absolutely continuous functions $I \to M$,
$L_{\mathrm{loc}}(I, M)$	class of all locally Lebesgue integrable functions $I \to M$,
$K(I \times \Omega, M)$	class of all functions $I \times \Omega \to M$ satisfying Carathéodory conditions on $I \times \Omega$,
$\mathrm{Re}\, z$	real part of z,
$\mathrm{Im}\, z$	imaginary part of z,
$\bar{z}$	complex conjugate of z.

3. Assumptions. Consider the equation

$$z'(t) = a(t)z(t) + b(t)\bar{z}(t) + A(t)z(t-r) + B(t)\bar{z}(t-r) + g(t, z(t), z(t-r)), \quad (3.1)$$

where $r > 0$ is a constant, $a, b \in AC_{\mathrm{loc}}(J, \mathbb{C})$, $A, B \in L_{\mathrm{loc}}(J, \mathbb{C})$, $g \in K(J \times \mathbb{C}^2, \mathbb{C})$, $J = [t_0, \infty)$. Throughout the paper we shall suppose that (3.1) satisfies the uniqueness property of solutions. We shall consider a case

$$\liminf_{t\to\infty}(|a(t)| - |b(t)|) > 0.$$

Clearly, the last inequality is equivalent to the existence of $T \geq t_0 + r$ and $\mu > 0$ such that

$$|a(t)| > |b(t)| + \mu \quad \text{for } t \geq T - r. \quad (3.2)$$

Define functions γ, c by

$$\gamma(t) = |a(t)| + \sqrt{|a(t)|^2 - |b(t)|^2}, \qquad c(t) = \frac{\bar{a}(t)b(t)}{|a(t)|}. \tag{3.3}$$

As $\gamma(t) > |a(t)|$ and $|c(t)| = |b(t)|$, the inequality

$$\gamma(t) > |c(t)| + \mu \tag{3.4}$$

holds for $t \geq T - r$. It can be easily verified that $\gamma, c \in AC_{\text{loc}}([T-r,\infty),\mathbb{C})$.

The equation (3.1) will be studied subject to suitable subsets of the following assumptions:

(i) The numbers $T \geq t_0 + r$ and $\mu > 0$ are such that (3.2) holds.

(ii) There exist functions $\varkappa, \kappa, \varrho : [T,\infty) \to \mathbb{R}$ such that

$$|\gamma(t)g(t,z,w) + c(t)\bar{g}(t,z,w)| \leq \varkappa(t)|\gamma(t)z + c(t)\bar{z}| + \kappa(t)|\gamma(t-r)w + c(t-r)\bar{w}| + \varrho(t)$$

for $t \geq T$, $z, w \in \mathbb{C}$, where ϱ is continuous on $[T,\infty)$.

(ii_n) There exist numbers $\tau_n \geq T$, $R_n \geq 0$ and functions $\varkappa_n, \kappa_n : [T,\infty) \to \mathbb{R}$ such that

$$|\gamma(t)g(t,z,w) + c(t)\bar{g}(t,z,w)| \leq \varkappa_n(t)|\gamma(t)z + c(t)\bar{z}| + \kappa_n(t)|\gamma(t-r)w + c(t-r)\bar{w}|$$

for $t \geq \tau_n$, $|z| > R_n$, $|w| > R_n$.

(iii) $\beta \in AC_{\text{loc}}([T,\infty),\mathbb{R}^0_+)$ is a function satisfying

$$\beta(t) \geq \lambda(t) \quad \text{a. e. on } [T,\infty), \tag{3.5}$$

where λ is defined by

$$\lambda(t) = \kappa(t) + (|A(t)| + |B(t)|)\frac{\gamma(t) + |c(t)|}{\gamma(t-r) - |c(t-r)|} \tag{3.6}$$

for $t \geq T$.

(iv) $\beta_- \in AC_{\text{loc}}([T,\infty),\mathbb{R}^0_-)$ is a function satisfying

$$\beta_-(t) \leq -\lambda(t) \quad \text{a. e. on } [T,\infty), \tag{3.7}$$

where λ is defined by (3.6) for $t \geq T$.

(iv_n) $\beta_n \in AC_{\text{loc}}([T,\infty),\mathbb{R}^0_-)$ is a function satisfying

$$\beta_n(t) \leq -\lambda_n(t) \quad \text{a. e. on } [T,\infty), \tag{3.8}$$

where λ_n is defined by

$$\lambda_n(t) = \kappa_n(t) + (|A(t)| + |B(t)|)\frac{\gamma(t) + |c(t)|}{\gamma(t-r) - |c(t-r)|} \tag{3.9}$$

for $t \geq T$.

(v) $\Lambda : [T,\infty) \to \mathbb{R}$ is a locally Lebesgue integrable function satisfying the inequalities $\beta'(t) \leq \Lambda(t)\beta(t)$, $\theta(t) \leq \Lambda(t)$ for $t \in [T,\infty)$, where θ is defined by (3.10).

($\mathrm{v_n}$) $\Lambda_n : [T,\infty) \to \mathbb{R}$ is a locally Lebesgue integrable function satisfying the inequalities $\beta_n'(t) \ge \Lambda_n(t)\beta_n(t)$, $\Theta_n(t) \ge \Lambda_n(t)$ for almost all $t \in [\tau_n,\infty)$, where Θ_n is defined by (3.10).

Obviously, if A, B, κ are locally absolutely continuous on $[T,\infty)$ and $\lambda(t) \ge 0$, the choice $\beta(t) = \lambda(t)$ or $\beta_-(t) = -\lambda(t)$ is admissible in (iii) or (iv), respectively. Similarly, if A, B, κ_n are locally absolutely continuous on $[T,\infty)$ and $\lambda_n(t) \ge 0$, the choice $\beta_n(t) = -\lambda_n(t)$ is admissible in ($\mathrm{iv_n}$).

Throughout the paper we denote

$$\begin{aligned}
\alpha(t) &= 1 + \left|\frac{b(t)}{a(t)}\right| \operatorname{sgn} \operatorname{Re} a(t),\\
\alpha_-(t) &= 1 - \left|\frac{b(t)}{a(t)}\right| \operatorname{sgn} \operatorname{Re} a(t),\\
\vartheta(t) &= \frac{\operatorname{Re}(\gamma(t)\gamma'(t) - \bar{c}(t)c'(t)) + |\gamma(t)c'(t) - \gamma'(t)c(t)|}{\gamma^2(t) - |c(t)|^2},\\
\vartheta_-(t) &= \frac{\operatorname{Re}(\gamma(t)\gamma'(t) - \bar{c}(t)c'(t)) - |\gamma(t)c'(t) - \gamma'(t)c(t)|}{\gamma^2(t) - |c(t)|^2},\\
\theta(t) &= \alpha(t) \operatorname{Re} a(t) + \vartheta(t) + \varkappa(t) + \beta(t),\\
\Theta(t) &= \alpha_-(t) \operatorname{Re} a(t) + \vartheta_-(t) - \varkappa(t),\\
\Theta_n(t) &= \alpha_-(t) \operatorname{Re} a(t) + \vartheta_-(t) - \varkappa_n(t) + \beta_n(t).
\end{aligned} \tag{3.10}$$

It can be easily verified that the functions ϑ, ϑ_- are locally Lebesgue integrable on $[T,\infty)$ under the assumption (i). If $\beta \in AC_{\mathrm{loc}}([T,\infty),\mathbb{R}_+)$, $\varkappa \in L_{\mathrm{loc}}([T,\infty),\mathbb{R})$ and $\beta'(t)/\beta(t) \le \theta(t)$ for almost all $t \ge T$ together with the conditions (i), (ii) are fulfilled, then we can choose $\Lambda(t) = \theta(t)$ for $t \in [T,\infty)$ in (v). If relations $\beta_n \in AC_{\mathrm{loc}}([T,\infty),\mathbb{R}_-)$, $\varkappa_n \in L_{\mathrm{loc}}([T,\infty),\mathbb{R})$ and $\beta_n'(t)/\beta_n(t) \le \Theta_n(t)$ for almost all $t \ge \tau_n$ together with the conditions (i), ($\mathrm{ii_n}$) are satisfied, then we can choose $\Lambda_n(t) = \Theta_n(t)$ for $t \in [T,\infty)$ in ($\mathrm{v_n}$).

4. Results.

4.1. Stable case. The results on asymptotic and stability properties of the solutions of (3.1) were intensively studied in Kalas and Baráková [4]. Our first theorem is a simple modification of [4, Theorem 1].

THEOREM 4.1. *Let the assumptions* (i), (ii), (iii) *and* (v) *be fulfilled with* $\varrho(t) \equiv 0$. *If*

$$\limsup_{t\to\infty} \int^t \Lambda(s)\ \mathrm{d}s < \infty, \tag{4.1}$$

then the trivial solution of (3.1) *is stable on* $[T,\infty)$*; if*

$$\lim_{t\to\infty} \int^t \Lambda(s)\ \mathrm{d}s = -\infty, \tag{4.2}$$

it is asymptotically stable on $[T,\infty)$.

Proof. The proof, based on the technique of Lyapunov-Krasovskii functional, is a small modification of that of [4, Theorem 1]. ∎

REMARK 1. If we consider an ordinary differential equation

$$z' = a(t)z + b(t)\bar{z} + g(t,z), \tag{4.3}$$

instead of the equation (3.1), then we have $A(t) \equiv 0$, $B(t) \equiv 0$. Thus we can take $\kappa(t) \equiv 0$, $\lambda(t) \equiv 0$, $\beta(t) \equiv 0$ and $\Lambda(t) = \theta(t) = \alpha(t) \operatorname{Re} a(t) + \vartheta(t) + \varkappa(t)$ in (iii) and (v), respectively, and we obtain [6, Theorem 1] as a consequence of Theorem 4.1.

The following remark allows to simplify the function ϑ in (3.10).

Remark 2. Since

$$\vartheta = \frac{\operatorname{Re}(\gamma\gamma' - \bar{c}c') + |\gamma c' - \gamma' c|}{\gamma^2 - |c|^2} \le \frac{(|\gamma'| + |c'|)(|\gamma| + |c|)}{\gamma^2 - |c|^2} = \frac{|\gamma'| + |c'|}{\gamma - |c|},$$

it follows from (3.4), that the function ϑ in (3.10) may be replaced by $\frac{1}{\mu}(|\gamma'| + |c'|)$.

From Theorem 4.1 we easily obtain (in the same way as in [4]) the following two corollaries

Corollary 4.2. *Let $a(t) \equiv a \in \mathbb{C}$, $b(t) \equiv b \in \mathbb{C}$, $|a| > |b|$. Assume that $\varrho_0, \varrho_1 : [T, \infty) \to \mathbb{R}$ are such that*

$$|g(t, z, w)| \le \varrho_0(t)|z| + \varrho_1(t)|w| \tag{4.4}$$

for $t \ge T$, $|z| < R$, $|w| < R$ and ϱ_0 is locally Lebesgue integrable on $[T, \infty)$. Let $\beta \in AC_{loc}([T, \infty), \mathbb{R}_+)$ be such that

$$\beta(t) \ge \left(\frac{|a| + |b|}{|a| - |b|}\right)^{\frac{1}{2}} \big(\varrho_1(t) + |A(t)| + |B(t)|\big) \quad \textit{a. e. on } [T, \infty).$$

If

$$\limsup_{t\to\infty} \int^t \max\left(\frac{|a| - |b|}{|a|} \operatorname{Re} a + \left(\frac{|a| + |b|}{|a| - |b|}\right)^{\frac{1}{2}} \varrho_0(s) + \beta(s), \frac{\beta'(s)}{\beta(s)}\right) \mathrm{d}s < \infty, \tag{4.5}$$

then the trivial solution of the equation (3.1) *is stable; if*

$$\lim_{t\to\infty} \int^t \max\left(\frac{|a| - |b|}{|a|} \operatorname{Re} a + \left(\frac{|a| + |b|}{|a| - |b|}\right)^{\frac{1}{2}} \varrho_0(s) + \beta(s), \frac{\beta'(s)}{\beta(s)}\right) \mathrm{d}s = -\infty, \tag{4.6}$$

then the trivial solution of (3.1) *is asymptotically stable.*

In the next corollary, we denote

$$H_1(t) = \sqrt{\frac{(|a| - |b|)^3}{|a| + |b|}} \frac{\operatorname{Re} a}{|a|} + |A| + |B| + \varrho_0(t) + \varrho_1(t),$$

$$H_2(t) = \sqrt{\frac{|a| - |b|}{|a| + |b|}} \frac{\varrho_1'(t)}{\varrho_1(t) + |A| + |B|}.$$

Corollary 4.3. *Let $a(t) \equiv a \in \mathbb{C}$, $b(t) \equiv b \in \mathbb{C}$, $|a| > |b|$ and $A(t) \equiv A \in \mathbb{C}$, $B(t) \equiv B \in \mathbb{C}$. Let there exist $\varrho_0, \varrho_1 : [T, \infty) \to \mathbb{R}$, ϱ_0 being locally Lebesgue integrable and ϱ_1 locally absolutely continuous, such that* (4.4) *holds for $t \ge T$, $|z| < R$, $|w| < R$. Suppose $\varrho_1(t) + |A| + |B| > 0$ on $[T, \infty)$. If*

$$\limsup_{t\to\infty} \int^t \max(H_1(s), H_2(s)) \,\mathrm{d}s < \infty,$$

then the trivial solution of the equation (3.1) *is stable; if*

$$\lim_{t\to\infty}\int^t \max(H_1(s),H_2(s))\,\mathrm{d}s=-\infty,$$

then the trivial solution of (3.1) *is asymptotically stable.*

The next theorem gives a useful estimation for the solution $z(t)$ of the equation (3.1).

Theorem 4.4. *Let the assumptions* (i), (ii), (iii) *and* (v) *be fulfilled, and*

$$V(t)=|\gamma(t)z(t)+c(t)\bar z(t)|+\beta(t)\int_{t-r}^{t}|\gamma(s)z(s)+c(s)\bar z(s)|\,\mathrm{d}s, \tag{4.7}$$

where $z(t)$ *is any solution of* (3.1) *defined on* $[t_1,\infty)$, *where* $t_1\ge T$. *Then*

$$\mu|z(t)|\le V(s)\exp\biggl(\int_s^t \Lambda(\tau)\,\mathrm{d}\tau\biggr)+\int_s^t\varrho(\tau)\exp\biggl(\int_\tau^t\Lambda(\sigma)d\sigma\biggr)\,\mathrm{d}\tau \tag{4.8}$$

for $t\ge s\ge t_1$.

Proof. The proof of the theorem can be found in [4, Theorem 2]. □

Remark 3. If we consider the ordinary differential equation (4.3) instead of the equation (3.1), we can take $A(t)\equiv 0$, $B(t)\equiv 0$, $\kappa(t)\equiv 0$, $\lambda(t)\equiv 0$, $\beta(t)\equiv 0$, $\Lambda(t)=\theta(t)=\alpha(t)\operatorname{Re}a(t)+\vartheta(t)+\varkappa(t)$ and we obtain [6, Theorem 2] as a consequence of Theorem 4.4.

Theorem 4.4 has several further consequences. Their proofs follow from those of corresponding corollaries in [4] and therefore the most of them are omitted.

Corollary 4.5. *Let the assumptions* (i), (ii), (iii) *and* (v) *be fulfilled. Let*

$$\limsup_{t\to\infty}\int_s^t\varrho(\tau)\exp\biggl(-\int_s^\tau\Lambda(\sigma)d\sigma\biggr)\,\mathrm{d}\tau<\infty.$$

If $z(t)$ *is any solution of* (3.1) *defined for* $t\to\infty$, *then*

$$z(t)=O\left[\exp\biggl(\int_s^t\Lambda(\tau)\,\mathrm{d}\tau\biggr)\right].$$

Corollary 4.6. *Let the assumptions* (i), (ii), (iii), (v) *be fulfilled and let*

$$\limsup_{t\to\infty}\Lambda(t)<\infty \quad \textit{and} \quad \varrho(t)=O(\mathrm{e}^{\eta t}),\quad \textit{where}\quad \eta>\limsup_{t\to\infty}\Lambda(t). \tag{4.9}$$

If $z(t)$ *is any solution of* (3.1) *defined for* $t\to\infty$, *then* $z(t)=O(\mathrm{e}^{\eta t})$.

Proof. In view of (4.9) , there exist $L>0$, $\eta^*<\eta$ and $s>T$ such that $\eta^*>\Lambda(t)$ and $\varrho(t)\,\mathrm{e}^{-\eta t}\le L$ for $t\ge s$. From (4.8), we get

$$\begin{aligned}\mu|z(t)|&\le V(s)\,\mathrm{e}^{\eta^*(t-s)}+L\int_s^t\mathrm{e}^{\eta\tau}\,\mathrm{e}^{\eta^*(t-\tau)}\,\mathrm{d}\tau\\&\le V(s)\,\mathrm{e}^{\eta^*(t-s)}+L\,\mathrm{e}^{\eta^*t}(\eta-\eta^*)^{-1}[\mathrm{e}^{(\eta-\eta^*)t}-e^{(\eta-\eta^*)s}]\\&\le V(s)\,\mathrm{e}^{\eta^*(t-s)}+L(\eta-\eta^*)^{-1}\,\mathrm{e}^{\eta t}=O(\mathrm{e}^{\eta t}).\end{aligned} \tag{4.10}$$

□

REMARK 4. If $\varrho(t) \equiv 0$, we can obtain the following statement: there exists an $\eta_0 < \eta$ such that $z(t) = o\,(e^{\eta_0 t})$ holds for the solution $z(t)$.

Consider now a special case of the equation (3.1) with $g(t,z,w) \equiv h(t)$:

$$z'(t) = a(t)z(t) + b(t)\bar{z}(t) + A(t)z(t-r) + B(t)\bar{z}(t-r) + h(t), \tag{4.11}$$

where $h : [t_0, \infty) \to \mathbb{C}$ is a locally Lebesgue integrable function.

COROLLARY 4.7. *Suppose that the assumption* (i) *is satisfied and*

$$\limsup_{t\to\infty}(\gamma(t) + |c(t)|) < \infty. \tag{4.12}$$

Let $\tilde{\beta} \in AC_{loc}([T,\infty), \mathbb{R}_+)$ *be such that*

$$\tilde{\beta}(t) \geq (|A(t)| + |B(t)|)\frac{\gamma(t) + |c(t)|}{\gamma(t-r) - |c(t-r)|} \quad a.\ e.\ on\ [T,\infty). \tag{4.13}$$

Assume that h *is a bounded function. If*

$$\limsup_{t\to\infty}[\alpha(t)\operatorname{Re} a(t) + \vartheta(t) + \tilde{\beta}(t)] < 0 \tag{4.14}$$

and

$$\limsup_{t\to\infty}\frac{\tilde{\beta}'(t)}{\tilde{\beta}(t)} < 0, \tag{4.15}$$

then any solution of (4.11) *is bounded. If* $h(t) = O(\mathrm{e}^{\eta t})$ *for any* $\eta > 0$,

$$\limsup_{t\to\infty}[\alpha(t)\operatorname{Re} a(t) + \vartheta(t) + \tilde{\beta}(t)] \leq 0 \qquad and \qquad \limsup_{t\to\infty}\frac{\tilde{\beta}'(t)}{\tilde{\beta}(t)} \leq 0,$$

then any solution $z(t)$ *of* (4.11) *satisfies* $z(t) = o(\mathrm{e}^{\eta t})$ *for any* $\eta > 0$.

REMARK 5. If $h(t) \equiv 0$ in COROLLARY 4.7, then, with respect to COROLLARY 4.6 and REMARK 4, we get the following statement:

Suppose that assumptions (i) and (4.12) are satisfied and for $\tilde{\beta}$ from COROLLARY 4.7 the inequality (4.13) is true. If (4.14) and (4.15) hold, then there exists $\eta_0 < 0$ such that $z(t) = o(e^{\eta_0 t})$ for any solution $z(t)$ of

$$z'(t) = a(t)z(t) + b(t)\bar{z}(t) + A(t)z(t-r) + B(t)\bar{z}(t-r).$$

In THEOREM 4.8 we give a sufficient condition ensuring that $\lim_{t\to\infty} z(t) = 0$ for any solution $z(t)$ of (3.1) defined for $t \to \infty$ (for the proof see [4]).

THEOREM 4.8. *Let the assumptions* (i), (ii), (iii) *and* (v) *be fulfilled. Let* $\Lambda(t)$ *satisfy* $\Lambda(t) \leq 0$ *a. e. on* $[T^*, \infty)$,

$$\lim_{t\to\infty}\int^t \Lambda(s)\,\mathrm{d}s = -\infty \quad and \quad \varrho(t) = o(\Lambda(t)), \tag{4.16}$$

where $T^* \in [T,\infty)$. *Then any solution* $z(t)$ *of* (3.1) *defined for* $t \to \infty$ *satisfies*

$$\lim_{t\to\infty} z(t) = 0.$$

REMARK 6. If we consider the ordinary differential equation (4.3) instead of the equation (3.1), we can take $A(t) \equiv 0$, $B(t) \equiv 0$, $\kappa(t) \equiv 0$, $\lambda(t) \equiv 0$, $\beta(t) \equiv 0$, $\Lambda(t) = \theta(t) = \alpha(t)\operatorname{Re} a(t) + \vartheta(t) + \varkappa(t)$ and we obtain [6, Theorem 3] as a consequence of THEOREM 4.8.

THEOREM 4.8 yields also the consequence:

COROLLARY 4.9. *Let the assumptions* (i) *and* (4.12) *be satisfied and* $\tilde\beta : [T,\infty) \to \mathbb{R}_+$ *be a locally absolutely continuous function satisfying* (4.13). *If the relations* (4.14) *and* (4.15) *are fulfilled and* $h : [t_0,\infty) \to \mathbb{C}$ *is a locally Lebesgue integrable function satisfying*

$$\lim_{t\to\infty} h(t) = 0,$$

then

$$\lim_{t\to\infty} z(t) = 0$$

for any solution $z(t)$ *of* (4.11) *defined for* $t \to \infty$.

4.2. Unstable case. In this case we study not only the solutions of (3.1) with the unstable behaviour, but mainly the the existence of solutions which are bounded or tending to the origin as $t \to \infty$.

THEOREM 4.10. *Let the assumptions* (i), (ii$_0$), (iv$_0$), (v$_0$) *be fulfilled for some* $\tau_0 \geq T$. *Suppose there exist* $t_1 \geq \tau_0$ *and* $\nu \in (-\infty,\infty)$ *such that*

$$\inf_{t\geq t_1}\left[\int_{t_1}^{t} \Lambda_0(s)\,\mathrm{d}s - \ln(\gamma(t) + |c(t)|)\right] \geq \nu. \tag{4.17}$$

If $z(t)$ *is any solution of* (3.1) *satisfying*

$$\min_{s\in[t_1-r,t_1]} |z(s)| > R_0, \qquad \Delta(t_1) > R_0\,\mathrm{e}^{-\nu}, \tag{4.18}$$

where $\Delta(t) = (\gamma(t) - |c(t)|)|z(t)| + \beta_0(t)\max_{s\in[t-r,t]}|z(s)|\int_{t_1-r}^{t_1}(\gamma(s)+|c(s)|)\,\mathrm{d}s$, *then*

$$|z(t)| \geq \frac{\Delta(t_1)}{\gamma(t)+|c(t)|}\exp\left[\int_{t_1}^{t}\Lambda_0(s)\,\mathrm{d}s\right] \tag{4.19}$$

for all $t \geq t_1$ *for which* $z(t)$ *is defined.*

Proof. The proof can be accomplished by the help of Lyapunov-Krasovskii functional, the reader is referred to the proof of [3, Theorem 2] for details. ∎

Similarly as in [3] we can obtain the following two corollaries.

COROLLARY 4.11. *Let the assumptions of* THEOREM 4.10 *be fulfilled with* $R_0 > 0$. *If*

$$\liminf_{t\to\infty}\left[\int_{t_1}^{t}\Lambda_0(s)\,\mathrm{d}s - \ln(\gamma(t)+|c(t)|)\right] = \varsigma > \nu, \tag{4.20}$$

then to any ε, $0 < \varepsilon < R_0\,\mathrm{e}^{\varsigma-\nu}$, *there is a* $t_2 \geq t_1$ *such that*

$$|z(t)| > \varepsilon \tag{4.21}$$

for all $t \geq t_2$ *for which* $z(t)$ *is defined.*

Corollary 4.12. *Let the assumptions of* Theorem 4.10 *be fulfilled with* $R_0 > 0$. *If*

$$\lim_{t\to\infty}\left[\int_{t_1}^{t} \Lambda_0(s)\ \mathrm{d}s - \ln(\gamma(t) + |c(t)|)\right] = \infty,$$

then for any $\varepsilon > 0$ *there exists a* $t_2 \geq t_1$ *such that* (4.21) *holds for all* $t \geq t_2$ *for which* $z(t)$ *is defined.*

The following theorem is a little generalization of [3, Theorem 5]. The proof of this theorem is based on the results of K. P. Rybakowski [8] on a Ważewski topological principle for retarded functional differential equations of Carathéodory type. The details of the proof are similar to those of the proof of [3, Theorem 5].

Theorem 4.13. *Let the conditions* (i), (ii), (iv) *be fulfilled and* $\tilde{\Lambda}$ *be a continuous function satisfying the inequality* $\tilde{\Lambda}(t) \leq \Theta(t)$ *a. e. on* $[T, \infty)$, *where* Θ *is defined by* (3.10). *If* $\xi : [T - r, \infty) \to \mathbb{R}$ *is a continuous function such that*

$$\tilde{\Lambda}(t) + \beta_-(t)\exp\left[-\int_{t-r}^{t} \xi(s)\ \mathrm{d}s\right] - \xi(t) > \varrho(t)C^{-1}\exp\left(-\int_{T}^{t} \xi(s)\ \mathrm{d}s\right) \quad (4.22)$$

for $t \in [T, \infty]$ *and some constant* $C > 0$, *then there exists a* $t_2 > T$ *and a solution* $z_0(t)$ *of* (3.1) *satisfying*

$$|z_0(t)| \leq \frac{C}{\gamma(t) - |c(t)|}\exp\left[\int_{T}^{t} \xi(s)\ \mathrm{d}s\right] \quad (4.23)$$

for $t \geq t_2$.

Remark 7. If $\eta_1(t)\tilde{\Lambda}(t) > |\beta_-(t)| + C^{-1}\varrho(t) > 0$, where $0 < \eta_1(t) \leq 1$, the functions η_1, $\tilde{\Lambda}$ are continuous on $[T, \infty)$ and $\tilde{\Lambda}(t) \leq \Theta(t)$ a. e. on $[T, \infty)$, then the choice $\xi(t) = \eta_1(t)\tilde{\Lambda}(t) + \beta_-(t) - C^{-1}\varrho(t)$ is possible in (4.22). Moreover, the condition $|\beta_-(t)| + C^{-1}\varrho(t) > 0$ can be omitted if Theorem 4.13 is used. Indeed, the identity $|\beta_-(t)| + C^{-1}\varrho(t) \equiv 0$ implies $\beta_-(t) \equiv 0$, $\varrho(t) \equiv 0$ and consequently, in view of (3.7), (3.6), (ii), we have $\lambda(t) \equiv 0$, $\kappa(t) \equiv 0$, $A(t) \equiv 0$, $B(t) \equiv 0$, $g(t, 0, 0) \equiv 0$. Thus the equation (3.1) has the trivial solution $z_0(t) \equiv 0$ in this case.

As a corollary of Theorem 4.13 we obtain sufficient conditions for the existence of a bounded solution of (3.1) or the existence of a solution $z_0(t)$ of (3.1) satisfying $\lim_{t\to\infty} z_0(t) = 0$.

Corollary 4.14. *Let the assumptions of* Theorem 4.13 *be satisfied. If*

$$\limsup_{t\to\infty}\left[\frac{1}{\gamma(t) - |c(t)|}\exp\left(\int_{T}^{t} \xi(s)\ \mathrm{d}s\right)\right] < \infty,$$

then there is a bounded solution $z_0(t)$ *of* (3.1). *If*

$$\lim_{t\to\infty}\left[\frac{1}{\gamma(t) - |c(t)|}\exp\left(\int_{T}^{t} \xi(s)\ \mathrm{d}s\right)\right] = 0,$$

then there is a solution $z_0(t)$ *of* (3.1) *such that*

$$\lim_{t\to\infty} z_0(t) = 0.$$

The following theorem is obtained by the combination of Theorem 4.10 and Theorem 4.13 and generalizes [3, Theorem 8].

Theorem 4.15. *Suppose that the hypotheses* (i), (ii), (ii$_n$), (iv), (iv$_n$), (v$_n$) *are fulfilled for* $\tau_n \geq T$ *and* $n \in \mathbb{N}$*, where* $R_n > 0$, $\inf_{n\in\mathbb{N}} R_n = 0$*. Let* $\tilde{\Lambda}$ *be a continuous function satisfying the inequality* $\tilde{\Lambda}(t) \leq \Theta(t)$ *a. e. on* $[T,\infty)$*, where* Θ *is defined by* (3.10)*. Assume that* $\xi : [T-r,\infty) \to \mathbb{R}$ *is a continuous function such that*

$$\tilde{\Lambda}(t) + \beta_-(t)\exp\left[-\int_{t-r}^{t}\xi(s)\ \mathrm{d}s\right] - \xi(t) > \varrho(t)C^{-1}\exp\left(-\int_{T}^{t}\xi(s)\ \mathrm{d}s\right) \tag{4.24}$$

for $t \in [T,\infty)$ *and some constant* $C > 0$*. Suppose*

$$\limsup_{t\to\infty}\left[\int_T^t (\Lambda_n(s) - \xi(s))\ \mathrm{d}s + \ln\frac{\gamma(t) - |c(t)|}{\gamma(t) + |c(t)|}\right] = \infty, \tag{4.25}$$

$$\lim_{t\to\infty}\left[\beta_n(t)\max_{s\in[t-r,t]}\frac{\exp\left[\int_T^s \xi(\sigma)\,d\sigma\right]}{\gamma(s) - |c(s)|}\int_{t-r}^{t}(\gamma(s) + |c(s)|)\ \mathrm{d}s\right] = 0, \tag{4.26}$$

$$\inf_{\tau_n\leq s\leq t<\infty}\left[\int_s^t \Lambda_n(\sigma)\,d\sigma - \ln(\gamma(t) + |c(t)|)\right] \geq \nu \tag{4.27}$$

for $n \in \mathbb{N}$*, where* $\nu \in (-\infty,\infty)$*. Then there exists a solution* $z_0(t)$ *of* (3.1) *such that*

$$\lim_{t\to\infty}\min_{s\in[t-r,t]}|z_0(s)| = 0. \tag{4.28}$$

Proof. For the proof see [3]. □

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