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ON SPATIAL DECAY ESTIMATES FOR DERIVATIVES OF VORTICITIES OF THE TWO DIMENSIONAL NAVIER-STOKES FLOW

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Abstract. We are concerned with the spatial decay estimates for derivatives of vorticities solving the two dimensional vorticity equations equivalent to the Navies-Stokes equations. As an application we derive asymptotic behaviors of derivatives of vorticities at time infinity. It is well-known by now that the vorticity behaves asymptotically as the Oseen vortex provided that the initial vorticity is integrable. We show that each derivative of the vorticity also behaves asymptotically as that of the Oseen vortex.

Key words. Two dimensional Navier-Stokes equations, vorticity equations, large time behavior.

AMS subject classifications. 35Q30, 35Q35, 76D05

1. Introduction. We are interested in the two dimensional flow of a viscous incompressible fluid. The velocity of the fluid is described by the Navier-Stokes equations:

$$\begin{cases} u_t - \Delta u + (u,\nabla)u + \nabla p = 0 & \text{for} \quad t>0,\ x\in\mathbb{R}^2, \\ \nabla\cdot u = 0 & \text{for} \quad x\in\mathbb{R}^2, \end{cases} \tag{1.1}$$

where $u = u(x,t) \in \mathbb{R}^2$ is the fluid velocity, $p(x,t) \in \mathbb{R}$ is the pressure, $\nabla = (\partial/\partial x_1, \partial/\partial x_2)$, $\Delta = (\partial/\partial x_1)^2 + (\partial/\partial x_2)^2$ and $u_t = \partial_t u = \partial u/\partial t$. The kinematic viscosity has been rescaled to be 1. We are concerned with the vorticity $\omega =\text{rot } u = \partial u_2/\partial x_1 - \partial u_1/\partial x_2$ when initial vorticity is integrable. For this purpose, instead of (1.1), we consider an equation for the vorticity which is obtained by taking the curl of (1.1):

$$\omega_t - \Delta\omega + (u,\nabla)\omega = 0, \qquad t>0, \quad x\in\mathbb{R}^2. \tag{1.2}$$

The velocity u is obtained in terms of ω via the Biot-Savart law

$$u(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2}\omega(y,t)\,\mathrm{d}y, \qquad t>0, \quad x\in\mathbb{R}^2, \tag{1.3}$$

where $x^\perp = (-x_2, x_1)$. The equation (1.2)–(1.3) are formally equivalent to (1.1).

The global well-posedness of the two dimensional vorticity equations in $L^1(\mathbb{R}^2)$ is first obtained by Y. Giga, T. Miyakawa and H. Osada [10]. In fact they constructed a global solution even when initial data is a finite measure. This result is extended by various authors for example by M. Ben-Artzi [1], H. Brezis [2], and T. Kato [13]. Although the uniqueness of solution was known by [10] when the point mass part of the initial data is small, it is quite recent that the uniqueness is proved for a general measure by I. Gallagher and Th. Gallay [5].

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2. Spatial decay estimates for derivatives of vorticities. In the past papers, several estimates for vorticities have been established. For example, we already know L^p estimates of vorticities and velocities as follows.

Let $p \in [1,\infty]$ and $q \in (2,\infty]$. Let $|f|_p$ denotes the norm of f in L^p; if f is a vector (f_1, f_2), by $|f|_p$ we mean $|(|f_1|^2 + |f_2|^2)^{\frac{1}{2}}|_p$. Then, we have

$$|\partial_t^b \partial_x^\beta \omega(\cdot,t)|_p \le \frac{W_1}{t^{b+\frac{|\beta|}{2}+1-\frac{1}{p}}} |\omega_0|_1, \tag{2.1}$$

$$|\partial_t^b \partial_x^\beta u(\cdot,t)|_q \le \frac{W_2}{t^{b+\frac{|\beta|}{2}+\frac{1}{2}-\frac{1}{q}}} |\omega_0|_1, \tag{2.2}$$

where $W_1 = W_1(b,\beta,p,|\omega_0|_1)$ and $W_2 = W_2(b,\beta,q,|\omega_0|_1)$. Here, $\partial_x^\beta = \partial_{x_1}^{\beta_1}\partial_{x_2}^{\beta_2}$ for multi-index $\beta = (\beta_1,\beta_2) \in \mathbb{N}_0 \times \mathbb{N}_0$, where $\partial_{x_i} = \partial/\partial x_i$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, the set of all nonnegative integers.

The above estimates (2.1), (2.2) were proved by T. Kato [13] for $p \in (1,\infty)$ by using an interpolation method, and by Y. Giga and M.-H. Giga [9] for $p \in [1,\infty]$ by a Gronwall-type argument (see also Y. Giga [11], or Y. Giga and O. Sawada [12]). In this paper we establish spatial decay estimates for derivatives of vorticities. Our main result is

Theorem 2.1 ([14]). *Assume that $p \in [1,\infty]$, $q \in (2,\infty]$. Let ω be the solution of (1.2)–(2.2) with initial vorticity $\omega_0 \in L^1(\mathbb{R}^2)$ and u be the velocity field associated with ω via the Biot-Savart law. Then, there exists a positive constant $W_3 = W_3(b,\beta,p,|\omega_0|_1)$ such that for all $R \ge 1$ and $t > 0$,*

$$|\partial_t^b \partial_x^\beta \omega(\cdot,t)|_{p,2R} \le \frac{W_3}{t^{1-\frac{1}{p}+b+\frac{|\beta|}{2}}} \left\{ \frac{t^{\frac{1}{4}}}{R^{\frac{1}{2}}} + |\omega_0|_{1,R} \right\}, \tag{2.3}$$

where $|\omega(\cdot,t)|_{p,R} := \left(\int_{|x|>R} |\omega(x,t)|^p \, \mathrm{d}x\right)^{\frac{1}{p}}$, $\quad |\omega(\cdot,t)|_{\infty,R} := \operatorname{ess.sup}_{|x|>R} |\omega(x,t)|$.

When $b = 0$ and $|\beta| = 0$, the spatial decay estimates similar to (2.3) are obtained by A. Carpio [4] and by Y. Giga and M.-H. Giga [9]. In order to establish the estimate (2.3), we need three spatial decay estimates as follows.

Let $p,\ q,\ \tilde{q} \in [1,\infty]$ with $q,\ \tilde{q} \le p$. Then, we have

$$|\omega(\cdot,t)|_{p,2R} \le \frac{C}{t^{1-\frac{1}{p}}} \left(\frac{t^{\frac{1}{4}}}{R^{\frac{1}{2}}} + |\omega_0|_{1,R}\right), \tag{2.4}$$

$$|u(\cdot,t)|_{\infty,2R} \le \frac{M_1'}{R^{\frac{1}{2}}} |\omega(\cdot,t)|_4 + M_2' |\omega(\cdot,t)|_{1,R}^{\frac{1}{2}} |\omega(\cdot,t)|_{\infty,R}^{\frac{1}{2}}\ , \tag{2.5}$$

$$|\partial_x^\beta e^{t\Delta} f|_{p,2R} \le \frac{M_3}{t^{\frac{1}{q}-\frac{1}{p}+\frac{k}{2}-\frac{1}{4}} R^{\frac{1}{2}}} |f|_q + \frac{M_4}{t^{\frac{1}{\tilde{q}}-\frac{1}{p}+\frac{k}{2}}} |f|_{\tilde{q},R}, \qquad \forall f \in C_0(\mathbb{R}^2)\ . \tag{2.6}$$

Here, $e^{t\Delta}$ is the heat semigroup. The estimate (2.4) is for ω itself, which is obtained by using the pointwise estimate for the fundamental solution of the perturbed heat equation, $\omega_t - \Delta\omega + (u,\nabla)\omega = 0$; as for this pointwise estimate, see [3]. The estimte (2.5) is for the velocity u. Since u is represented by ω via the Biot-Savart law (1.3), it suffices to estimate the well-known Reisz potential. The last estimate (2.6) is for the solution of the heat equation. This estimate is established by using the representation

$$e^{t\Delta} f = \int_{\mathbb{R}^2} \frac{1}{4\pi t} \, \mathrm{e}^{-\frac{|x-y|^2}{4t}} \, f(y) \, \mathrm{d}y.$$

Collecting these estimates, one can derive the estimate (2.3) from the integral equation

$$\omega(x,t) = \mathrm{e}^{t\Delta}\,\omega_0 - \int_0^t \mathrm{e}^{(t-s)\Delta}(u(s),\nabla)\omega(s)ds. \tag{2.7}$$

We omit the details here.

3. Application to large time behaviors of the derivatives of vorticities. As an application of Theorem 2.1, we study the large time behaviors of derivatives of vorticities. It is well-known that the vorticity itself behaves like a constant multiple of the Gauss kernel $g(x,t) = (4\pi t)^{-1}\exp(-\frac{|x|^2}{4t})$ at time infinity. Let us recall its precise form:

Theorem 3.1 ([8], [4], [6]). *Assume that $p \in [1,\infty]$, $q \in (2,\infty]$. Let ω be the solution of (1.2)–(2.2) with initial vorticity $\omega_0 \in L^1(\mathbb{R}^2)$. Let $m = \int_{\mathbb{R}^2}\omega_0(x)\,\mathrm{d}x$, and $g(x,t) = \frac{1}{4\pi t}\,\mathrm{e}^{-\frac{|x|^2}{4t}}$. Then*

$$\begin{aligned} \lim_{t\to\infty} t^{1-\frac{1}{p}}|\omega(\cdot,t) - mg(\cdot,t)|_p &= 0\ , \\ \lim_{t\to\infty} t^{\frac{1}{2}-\frac{1}{q}}|u(\cdot,t) - mv^g(\cdot,t)|_q &= 0\ . \end{aligned} \tag{3.1}$$

Here v^g is the velocity field associated with g via the Biot-Savart law (1.3).

The above theorem shows that the vorticity behaves asymptotically as mg which is called the Oseen vortex. Note that the Gauss kernel is a solution of (1.2)–(2.2) with a Dirac mass as the initial data. The quantity $m = \int_{\mathbb{R}^2}\omega_0(x)\,\mathrm{d}x$ is called "total circulation" and it is preserved by the semi-flow defined by (1.2)–(2.2) in $L^1(\mathbb{R}^2)$;

$$\int_{\mathbb{R}^2}\omega(x,t)\,\mathrm{d}x = \int_{\mathbb{R}^2}\omega_0(x)\,\mathrm{d}x, \quad t \geq 0. \tag{3.2}$$

Y. Giga and T. Kambe [8] first proved Theorem 3.1 when the Reynolds number $\int_{\mathbb{R}^2}|\omega_0(x)|\,\mathrm{d}x$ is sufficiently small by giving the delicate estimates of the bilinear form of the integral equation associated with (1.2). Later A. Carpio [4] proved Theorem 3.1 under the assumption that $|m|$ is small by rescaling solutions: $\omega_k(x,t) = k^2\omega(kx,k^2t)$, $u_k(x,t) = ku(kx,k^2t)$ for $k > 0$. Recently, Th. Gallay and C. E. Wayne [6] proved for a general initital vorticity in $L^1(\mathbb{R}^2)$ by introducing entropy-like Lyapunov function for a renormalized equation. After this work was completed, the auther was informed of a recent work of I. Gallagher, Th. Gallay and P.-L. Lions [7] which give another proof for Theorem 3.1 using the rearrangement argument.

With spatial decay estimates for derivatives of vorticities, we shall prove that each derivative of vorticities behaves asymptotically as that of the Oseen vortex. That is, we have the following theorem.

Theorem 3.2 ([14]). *Assume that $p \in [1,\infty]$, $q \in (2,\infty]$, $b \in \mathbb{N}_0$ and β is a multi-index. Let ω be the solution of (1.2)–(2.2) with initial vorticity $\omega_0 \in L^1(\mathbb{R}^2)$, $m = \int_{\mathbb{R}^2}\omega_0(x)\,\mathrm{d}x$, and $g(x,t) = \frac{1}{4\pi t}\,\mathrm{e}^{-\frac{|x|^2}{4t}}$. Then, we have*

$$\lim_{t\to\infty} t^{b+\frac{|\beta|}{2}+1-\frac{1}{p}}|\partial_t^b\partial_x^\beta\omega(\cdot,t) - \partial_t^b\partial_x^\beta mg(\cdot,t)|_p = 0, \tag{3.3}$$

$$\lim_{t\to\infty} t^{b+\frac{|\beta|}{2}+\frac{1}{2}-\frac{1}{q}}|\partial_t^b\partial_x^\beta u(\cdot,t) - \partial_t^b\partial_x^\beta mv^g(\cdot,t)|_q = 0\ . \tag{3.4}$$

Let us give the outline of the proof of THEOREM 3.2 for $\alpha = 0$, $|\beta| = 1$. First we consider the same rescaling as was used in A. Carpio [4]. We shall see that the convergence of $\partial_x \omega(x,t)$ as time goes to infinity is equivalent to the convergence of the rescaled functions $\partial_x \omega_k(x,1)$ as k goes to infinity. Once we obtain THEOREM 2.1, we can apply Ascoli-Arzelà type compactness theorem in L^p to the family of rescaled functions $\{\partial_x \omega_k(x,1)\}_{k \geq 1}$. So every subsequence of $\{\partial_x \omega_{k(l)}(x,1)\}_{l=1}^{\infty}$ ($k(l) \to \infty$ as l goes to infinity) has a convergent subseqence in L^p. THEOREM 3.1 implies that the limit function is unique, so we obtain THEOREM 3.2. By the induction we see that THEOREM 3.2 also holds for higher derivatives of the solution; see [14] for details.

4. Alternative method. In fact, to prove the convergence results on derivatives in THEOREM 3.2, there is an alternative method by appealing interpolation together with the convergence results of the vorticity ω itself and global estimates on derivatives (2.1). In particular, spatial decay estimates in THEOREM 2.1 are not involved. We shall show the proof only for the case $b = 0$ and $|\beta| = 1$.

First, note that we have the interpolation inequalities such as

$$|f|_{1,p} \leq C|f|_p^{\frac{1}{2}} |f|_{2,p}^{\frac{1}{2}}, \qquad \text{for all } f \in W^{2,p}(\mathbb{R}^n), \tag{4.1}$$

where C depends only on n and $p \in [1, \infty]$. So we see

$$\begin{aligned} |\omega_k - mg|_{1,p} &\leq C|\omega_k - mg|_p^{\frac{1}{2}} |\omega_k - mg|_{2,p}^{\frac{1}{2}} \\ &\leq C|\omega_k - mg|_p^{\frac{1}{2}} (|\omega_k|_{2,p} + |mg|_{2,p})^{\frac{1}{2}} \\ &\leq C|\omega_k - mg|_p^{\frac{1}{2}}, \end{aligned}$$

where C depends only on p and $|\omega_0|_1$. Here, the last inequality follows from the global estimates (2.1). Since we already have $\lim_{k\to\infty} |\omega_k(\cdot,1) - mg(\cdot,1)|_p = 0$, the desired convergence follows.

This proof is very simple compared with the proof using compactness argument together with spatial decay estimates for derivatives of vorticities (2.3). However, the above interpolation method has a disadvantage if there is an inhomogeneous term in the vorticity equations of the form

$$\omega_t - \Delta\omega + (u, \nabla)\omega = f. \tag{4.2}$$

Before seeing this, note that we can prove the global existence and uniqueness of the solution of (4.2) under appropriate conditions on f. Moreover, we can also show the large time behaviors of solutions similar to those in THEOREM 3.1. Let us state the typical results for the inhomogeneous case without proofs.

THEOREM 4.1. *Assume that a function $f \in L^1(\mathbb{R}^2 \times (0,\infty))$ satisfies that $tf(\cdot,t) \in L^\infty(0,\infty; L^1(\mathbb{R}^2))$. Let $\omega_0 \in L^1(\mathbb{R}^2)$. Then, there exists a unique solution $\omega \in C([0,\infty); L^1(\mathbb{R}^2))$ of (4.2) with initial vorticity ω_0. The vorticity ω satisfies that for $p \in [1,\infty)$ and $q \in [1,2)$,*

$$\sup_{t>0} t^{1-\frac{1}{p}} |\omega(\cdot,t)|_p \leq C, \tag{4.3}$$

$$\sup_{t>0} t^{\frac{3}{2}-\frac{1}{q}} |\partial_x \omega(\cdot,t)|_q \leq C, \tag{4.4}$$

$$|\omega(\cdot,t)|_{p,2R} \le \frac{C}{t^{1-\frac{1}{p}}}\left(\frac{t^{\frac{1}{4}}}{R^{\frac{1}{2}}}+|\omega_0|_{1,R}\right) \tag{4.5}$$
$$+C\int_0^t \frac{1}{(t-s)^{1-\frac{1}{p}}}\left(\frac{(t-s)^{\frac{1}{4}}}{R^{\frac{1}{2}}}|f(\cdot,s)|_1+|f(\cdot,s)|_{1,R}\right)\mathrm{d}s,$$

where C *depends only on* $p,\ q,\ |\omega_0|_1$*, and* $C_f := \int_0^\infty \int_{\mathbb{R}^2} |f(x,t)|\,\mathrm{d}x\,\mathrm{d}t + \text{ess.sup}_{t>0} t|f(\cdot,t)|_1$. *Moreover, we have*

$$\lim_{t\to\infty} t^{1-\frac{1}{p}}|\omega(\cdot,t)-(m+m_f)g(\cdot,t)|_p = 0. \tag{4.6}$$

Here, $m = \int_{\mathbb{R}^2}\omega_0(x)\,\mathrm{d}x$ *and* $m_f = \int_0^\infty\int_{\mathbb{R}^2} f(x,t)\,\mathrm{d}x\,\mathrm{d}t$.

If we use interpolation inequalities such as (4.1) in order to derive the large time behaviors of derivatives of vorticities solving (4.2), we are forced to assume unnecessary regularity conditions on the inhomogeneous term f. On the other hand, by arguing as in Section 3 with spatial decay estimates for derivatives of solutions of (4.2), we can derive the large time behavior of derivatives of solutions without irrelevant regularity assumptions on f. Precisely, we have

Theorem 4.2. *Assume that a function* f *satisfies the conditions in* Theorem 4.1. *Let* $\omega_0 \in L^1(\mathbb{R}^2)$ *and* $q \in [1,2)$. *Then, the solution* ω *satisfies that*

$$|\partial_x\omega(\cdot,t)|_{q,2R} \le \frac{C}{t^{\frac{3}{2}-\frac{1}{q}}}\left(\frac{t^{\frac{1}{4}}}{R^{\frac{1}{2}}}+|\omega_0|_{1,R}\right) \tag{4.7}$$
$$+C\int_0^t \frac{1}{(t-s)^{\frac{3}{2}-\frac{1}{q}}}\left(\frac{(t-s)^{\frac{1}{4}}}{R^{\frac{1}{2}}}|f(\cdot,s)|_1+|f(\cdot,s)|_{1,R}\right)\mathrm{d}s,$$

where C *depends only on* $q,\ |\omega_0|_1$*, and* C_f. *Moreover, we have*

$$\lim_{t\to\infty} t^{\frac{3}{2}-\frac{1}{q}}|\partial_x\omega(\cdot,t)-\partial_x(m+m_f)g(\cdot,t)|_q = 0. \tag{4.8}$$

The proof of the above theorem is quite similar to that of the homogeneous case, we omit the details here.

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