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ON CONSTRUCTING A SOLUTION OF A BOUNDARY VALUE PROBLEM FOR FUNCTIONAL DIFFERENTIAL EQUATIONS*

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Abstract. Consider the boundary value problem for functional differential equation

$$x'(t) = F(x)(t), \quad h(x) = 0$$

upon closed interval $I \subset R$, where $F : C(I; R^n) \longrightarrow L(I; \mathbb{R}^n)$ is a regular operator (fulfiling Caratheodory conditions) and $h : C(I; \mathbb{R}^n) \longrightarrow R^n$ is a continuous functional. The sequence of the solution of certain corresponding problems, which converges to the solution of the consider problem, is used for constructing this solution.

This article extends and complements the results of I. Kiguradze ([4], [5]) dealing with Cauchy-Nicoletti's problems for a system of ordinary differential equations, for a system of differential equations with deviating arguments and for a system of functional differential equations. The bound of a generalized method of theoretic approximation is determined by Cauchy-Nicoletti's problems, among others. Methods and results are illustrated by examples (including numeric solutions to definite problems).

Key words. Functional differential equation, Cauchy-Nicoletti's problems, differential equation with deviating argument, theoretic approximation of the solution

AMS subject classifications. 34K10

1. Introduction.

Systematic study the special case of the functional differential equation (in particular ordinary differential equation with deviating arguments) was studied since 50th of last century. After work of professor Myshkis (e.g. [1]) appeared other authors as

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Azbelev, Rachmatulina (e.g. [2]), Hale (e.g. [3]), etc. Together with theory of differential equations with deviating argument has started study integro-differential equations with deviating argument and another, which came to the continuous generalizing till the functional differential equations.

One of the basic question of qualitative theory these equations is existence, uniqueness and construction of the solution of the boundary value problem. In these work we use a methods for the theoretic approximation of the solution boundary value problem for the ordinary differential equations, which published in 1987 professor Kiguradze ([4]). The same technique was used in [6] for solving the Cauchy-Nicolett's problem for functional differential equations.

In the paper is used following basic notation: I – [a,b], $\mathbb{R}^n$ – the space of n-dimensional column vectors $x = (x_i)_{i=1}^n$ with the elements $x_i \in \mathbb{R}$ $(i = 1, \ldots, n)$ and the norm $\|x\| = \sum_{i=1}^n |x_i|$, $C(I;\mathbb{R}^n)$ – the space of continuous vector functions $x : I \to \mathbb{R}^n$ with the norm $\| x \| = \max\{\| x(t) \| : t \in I\}$, $L(I;R^n)$ – the space of summable vector functions $x : I \to \mathbb{R}^n$ with the norm $\|x\|_L = \int_a^b \|x(t)\|$ dt.

2. Statement of the problem.

We consider the problem

$$x'(t) = p(x)(t) + f(x)(t), \tag{2.1}$$

$$\ell(x) = h(x), \tag{2.2}$$

where $p : C(I;\mathbb{R}^n) \to L(I;\mathbb{R}^n)$, $\ell : C(I;\mathbb{R}^n) \to \mathbb{R}^n$ be linear and $f : C(I;\mathbb{R}^n) \to L(I;\mathbb{R}^n)$, $h : C(I;\mathbb{R}^n) \to \mathbb{R}^n$ be generally nonlinear continuous operators.

Under a solution of the problem (2.1), (2.2) we uderstand an absolutely continuous vector function x which almost everywhere on I satisfies (2.1) and which satisfies the condition (2.2).

We would like to construct a sequence of vector function $\{x_m\}$ such that for $m \to \infty$ holds $\| x_m - x \| \to 0$ and x_m are solutions of corresponding boundary value problems to the problem (2.1), (2.2).

Special cases of the boundary value problem (2.1), (2.2) are following boundary value problems:

2.1. Cauchy-Nicoletti linear boundary value problem for systems of linear ordinary differential equations. Consider the equation (2.1), where

$$p(x)(t) = \left(\sum_{k=1}^{n} p_{ik}(t)x(t)\right)_{i=1}^{n} \qquad \text{and} \qquad f(x)(t) = (q_i(t))_{i=1}^{n}$$

i.e. the system of linear ordinary differential equations

$$x_i'(t) = \sum_{k=1}^{n} p_{ik}(t)x_k(t) + q_i(t) \qquad (i = 1, \dots, n) \tag{2.3}$$

with boundary value condition (2.2), where

$$\ell(x) = (x_i(t_i))_{i=1}^{n} \qquad \text{and} \qquad h(x) = (\ell_i(x_1, \dots, x_n) + c_i)_{i=1}^{n}$$

i.e. the Cauchy-Nicoletti linear boundary value condition

$$x_i(t_i) = \ell_i(x_1, \dots, x_n) + c_i \qquad (i = 1, \dots, n) \tag{2.4}$$

where $p_{ij} \in L(I;\mathbb{R})$, $q_i \in L(I;\mathbb{R})$, $\ell_i : C(I;\mathbb{R}^n) \to \mathbb{R}$ are a linear continuous functionals and $c_i \in R$.

2.2. Cauchy-Nicoletti nonlinear boundary value problem for systems of nonlinear ordinary differential equations. Consider the equation (2.1), where

$$p(x)(t) \equiv 0 \qquad \text{and} \qquad f(x)(t) = (f_i(t, x_1, \dots, x_n))_{i=1}^{n}$$

i.e. the system of nonlinear ordinary differential equations

$$x_i'(t) = f_i(t, x_1, \dots, x_n) \qquad (i = 1, \dots, n) \tag{2.5}$$

with boundary value condition (2.2), where

$$\ell(x) = (x_i(t_i))_{i=1}^n \qquad \text{and} \qquad h(x) = (\varphi_i(x_1, \dots, , x_n))_{i=1}^n$$

i.e. with the Cauchy-Nicoletti nonlinear boundary value condition

$$x_i(t_i) = \varphi_i(x_1, \dots, , x_n) \qquad (i = 1, \dots, n), \tag{2.6}$$

where $t_i \in I$ and $\varphi_i : C(I; \mathbb{R}^n) \to \mathbb{R}$, $f_i : I \times \mathbb{R}^n \to \mathbb{R}$ are nonlinear continuous functionals and f_i satisfies Caratheodory conditions:

(i) $f(\cdot, x)$ is measurable for any $x \in \mathbb{R}^n$

(ii) $f(t, \cdot)$ is continuous for almost all $t \in I$

(iii) $\forall r > 0$, $\exists q_r \in L(I; \mathbb{R}_+)$ so that $\| f(t, x) \| \le q_r(t)$ for almost all $t \in I$ and $x \in \mathbb{R}^n$, that $\| x \| \le r$.

2.3. Cauchy-Nicoletti nonlinear boundary value problem for systems of functional differential equations. Consider the equation (2.1), where

$$p(x)(t) \equiv 0 \qquad \text{and} \qquad f(x)(t) = (f_i(x_1, \dots, x_n)(t))_{i=1}^n$$

i.e. the system of nonlinear functional differential equations

$$x_i'(t) = f_i(x_1, \dots, x_n)(t) \qquad (i = 1, \dots, n) \tag{2.7}$$

with the condition (2.6), where $f_i : C(I; \mathbb{R}^n) \to L(I; \mathbb{R})$ are continuous operators.

2.4. General nonlinear boundary value problem for systems of functional differential equations. Consider the system of nonlinear functional differential equations (2.1) with general boundary value condition (2.2).

Remark 2.1. The problems (2.3), (2.4); (2.5), (2.6) and (2.7)(2.6) are special cases of the problem (2.1), (2.2).

Remark 2.2. Another special case of the problem (2.1), (2.2) are periodic, antiperiodic, linear 2-points, boundary value problems with deviating argument,etc. Important case is boundary value problems with deviating argument

$$x_i'(t) = f_i(t, x_1(\tau_1), \dots, x_n(\tau_n)) \qquad (i = 1, \dots, n),$$
$$x_i(t_i) = \varphi_i(x_1, \dots, x_n) \qquad (i = 1, \dots, n),$$

where $f_i : C(I;\mathbb{R}^n) \to L(I;\mathbb{R})$ satisfies Caratheodory conditions, $\tau_i : I \to \mathbb{R}$ are measurable and solution $(x_i)_{i=1}^n$ for $t \in \mathbb{R} - I$ is given by a continuous function .

3. Main results. In cited literature are known following theorems.

Theorem 3.1. ([5]) *Let $p_{ik} \in L(I;\mathbb{R})$, $\ell_i : C(I;\mathbb{R}^n) \to R$, $t_i \in I$ and the following system of differential inequalities*

$$x_i'(t)\operatorname{sgn}(t - t_i) \le \sum_{k=1}^{n} p_{ik}(t)x_k(t) \qquad (i = 1, \dots, n)$$

has no nonzero nonnegative solution, which satisfies conditions

$$x_i(t_i) \le \ell_i(x_1, \dots, x_n) \qquad (i = 1, \dots, n).$$

Then the problem (2.3), (2.4) has the unique solution $(x_i)_{i=1}^n$ for arbitrary $c_i \in \mathbb{R}$, $q_i \in L(I;\mathbb{R})$ and for any $(x_{i0})_{i=1}^n \in C(I;\mathbb{R}^n)$ there exists unique sequence $\{(x_{im})_{i=1}^n\}_{m=1}^\infty \subset C(I;\mathbb{R}^n)$, such that for each natural number m and $i \in \{1, \dots, n\}$, the function x_{im} is a solution of the problem

$$x_{im}'(t) = p_{ii}(t)x_{im}(t) + \sum_{k=1}^{n} (1 - \delta_{ik})p_{ik}(t)x_{km-1}(t) + q_i(t)$$
$$x_{im}(t_i) = \ell_i(x_{1m-1}, \dots, x_{nm-1}) + c_{0i},$$

and

$$\|x_i - x_{im}\|_C \to 0 \qquad \text{for} \qquad m \to \infty.$$

THEOREM 3.2.([4]) *Let in* $I \times \mathbb{R}^n$

$$[f_i(t,x_1,\dots,x_n) - f_i(t,y_1,\dots,y_n)]\,\mathrm{sgn}[(t-t_i)(x_i-y_i)] \le \sum_{k=1}^{n} p_{ik}(t)|\,x_k - y_k\,| \qquad (i=1,\dots,n)$$

and in $C(I;R^n)$

$$|\,\varphi_i(x_1,\dots,x_n) - \varphi_i(y_1,\dots,y_n)| \le \varphi_{0i}(|x_1-y_1|,\dots,|x_n-y_n|) \qquad (i=1,\dots,n)$$

where $(\varphi_{0i})_{i=1}^n : C(I;\mathbb{R}_+^n) \to \mathbb{R}_+^n$ *are nondecreasing functionals and* $(p_{ik})_{i,k=1}^n \in L(I;\mathbb{R}^{n\times n})$ *satisfies condition*

$$p_{ik}(t) \ge 0 \qquad \text{for } a<t<b, \quad i \ne k$$

and the problem

$$x_i'(t)sgn(t-t_i) \le \sum_{k=1}^{n} p_{ik}(t)x_k(t) \qquad (i=1,\dots,n),$$

$$x_i(t_i) \le \varphi_{0i}(|\,x_1\,|,\dots,|\,x_n\,|) \qquad (i=1,\dots,n)$$

has only the zero solution. Then the problem (2.5), (2.6) *has the unique solution* $(x_i)_{i=1}^n$ *and for any* $(x_{i0})_{i=1}^n \in C(I;\mathbb{R}^n)$ *there exists unique sequence* $\{(x_{im})_{i=1}^n\}_{m=1}^\infty \subset C(I;\mathbb{R}^n)$*,* *such that for each natural number m and $i \in \{1, \dots, n\}$, the function x_{im} is solution of the problem*

$$x'_{im}(t) = f_i(t, x_{1m-1}(t), \dots, x_{i-1m-1}(t), x_{im}(t), x_{i+1m-1}(t), \dots, x_{nm-1}(t)) \tag{3.1}$$

$$x_{im}(t_i) = \varphi_i(x_{1m-1}, \dots, x_{nm-1}) \tag{3.2}$$

and

$$\|x_i - x_{im}\|_C \to 0 \qquad for \qquad m \to \infty.,$$

Theorem 3.3.([6]) *Let in $C(I;\mathbb{R}^n)$ the inequalities*

$$[f_i(x_1, \dots, x_n)(t) - f_i(y_1, \dots, y_n)(t) - h_i(t)(x_i(t) - y_i(t))]\mathrm{sgn}[(t - t_i)(x_i(t) - y_i(t))] \le f_{0i}(|x_1 - y_1|, \dots, |x_n - y_n|) \qquad (i = 1, \dots, n)$$

and

$$|\,\varphi_i(x_1, \dots, x_n) - \varphi_i(y_1, \dots, y_n)\,| \le \varphi_{0k}(|x_1 - y_1|, \dots, |x_n - y_n|) \qquad (i = 1, \dots, n)$$

be fulfilled, where $h_i \in L(I;\mathbb{R}), f_{0i} : C(I;\mathbb{R}^n_+) \to L(I;\mathbb{R}_+)$ and $\varphi_{0i} : C(I;\mathbb{R}^n) \to \mathbb{R}_+$ be positively homogeneous continuous nondecreasing operators and functionals such that the system of differential inequalities

$$|x'_i(t) - h_i(t)x_i(t)| \le f_{0i}(|x_1|, \dots, |x_n|) \qquad (i = 1, \dots, n)$$

with the boundary conditions

$$|\,x_i(t_i)\,| \le \varphi_{0i}(|\,x_1\,|, \dots, |\,x_n\,|) \qquad (i = 1, \dots, n)$$

has only the zero solution. Then the problem (2.7), (2.6) has a unique solution $(x_i)_{i=1}^n$. *Then for any* $(x_{i0})_{i=1}^n \in C(I;\mathbb{R}^n)$ *there exists unique sequence* $\{(x_{im})_{i=1}^n\}_{m=1}^\infty \subset C(I;\mathbb{R}^n)$, *such that for each natural number* m *and* $i \in \{1,\dots,n\}$, *the function* x_{im} *is a solution of the problem*

$$x'_{im}(t) = f_i(x_{1m-1},\dots,x_{i-1m-1},x_{im},x_{i+1m-1},\dots,x_{nm-1})(t) \tag{3.3}$$

with the boundary value condition (3.2) *and*

$$\| x_i - x_{im} \|_C \to 0 \ \ for \ \ m \to \infty.$$

4. Results. Previous theorems are special cases of the following theorem. To the formulation following theorem we will need definition.

Definition 4.1. We say that a pair (p,l), where $p : C(I;\mathbb{R}^n) \to L(I;\mathbb{R}^n)$ is a linear strongly bounded operator (i.e. there exists a summable function $\eta : I \to \mathbb{R}_+$ such that $\|p(x)(t)\| \le \eta(t)\|x\|_C$ for $t \in I, x \in C(I;\mathbb{R}^n)$) and $l : C(I;\mathbb{R}^n) \to \mathbb{R}^n$ is a linear bounded operator, belong to the set $M_I^{\sigma_1,\dots,\sigma_n}$ if homogeneous problem

$$x'(t) = p(x)(t),$$

$$\ell(x) = 0,$$

has only a trivial solution, and for any $c \in \mathbb{R}_+^n$ and $q \in L(I;\mathbb{R}^n)$ satisfying the condition

$$\mathrm{diag}(\sigma_1,\dots,\sigma_n)q \in L(I;\mathbb{R}_+^n)$$

a solution x of problem

$$x'(t) = p(x)(t) + q(t),$$

$$\ell(x) = c,$$

is nonnegative, i.e., $x(t) \in R^n_+$ for $t \in I$.

THEOREM 4.2. *Let $(p, l) \in M_I^{\sigma_1,\dots,\sigma_n}$ and let exists $\rho > 0$ such that for all $\lambda \in [0, 1]$ and arbitrary solution x of the problem*

$$x'(t) = p(x)(t) + \lambda f(x)(t)$$
$$\ell(x) = \lambda h(x)$$

satisfies the estimate

$$||\, x \,||_C \leq \rho.$$

Then the problem (2.1), (2.2) has at least one solution. Then for any x_0 there exists sequence $\{x_m\}_{m=1}^{\infty}$ of the solution of the problem

$$x'_m = p(x_m)(t) + f(x_{m-1})(t)$$
$$\ell(x_m) = h(x_{m-1}),$$

such that x for which

$$\|\, x - x_m \,\|_C \to 0 \;\; for \;\; m \to \infty.$$

be a solution of the problem (2.1), (2.2).

5. Examples. For ilustration previous theorems we use the solution of the boundary value problem for $n = 1$. Following examples were computed in MAPLE.

First example shows convergency sequential aproximation process in $I = [0, 2]$ of the problem

$$x'(t) = x\left(\frac{t}{2} + 1\right) + 1, \qquad x(0) = -x(2) + 1.$$

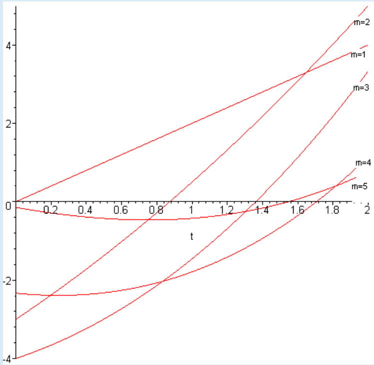

Fig. 5.1. *Iterations: 1-5*

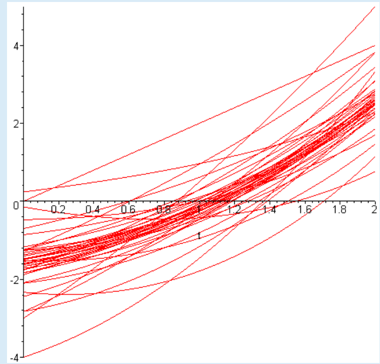

Fig. 5.2. *Iterations: 1-30*

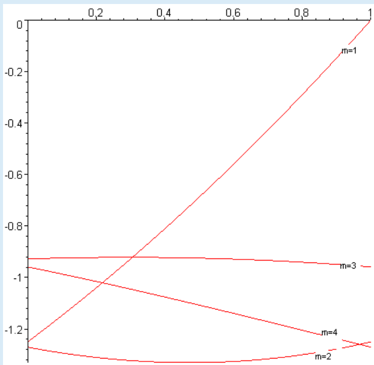

Fig. 5.3. *Iterations: 1-4*

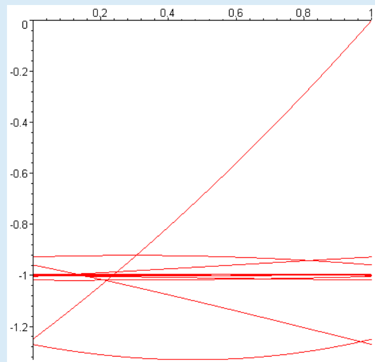

Fig. 5.4. *Iterations: 1-10*

Second and third example shows sensitivity of aproximation process on the periodical boundary condition. Consider in $I = [0, 1]$ the problem

$$x'(t) = x\left(\frac{t}{2}\right) + 1, \quad x(1) = x(0).$$

Consider in $I = [0, 1]$ the problem

$$x'(t) = x\left(\frac{t}{2}\right) + 1, \quad x(0) = x(1).$$

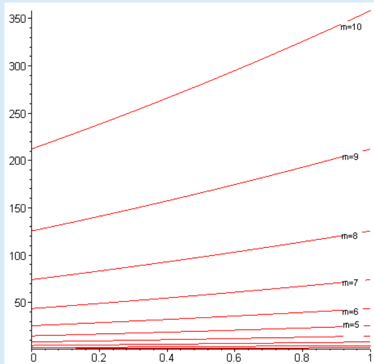

FIG. 5.5. *Iterations: 1-10*

Fourth example shows approximation process on the problem, which is partly defined out the interval I. Consider in $I = [-\pi, \pi]$ the problem

$$x'(t) = x\left(\frac{\pi}{2} - t\right), \qquad x\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

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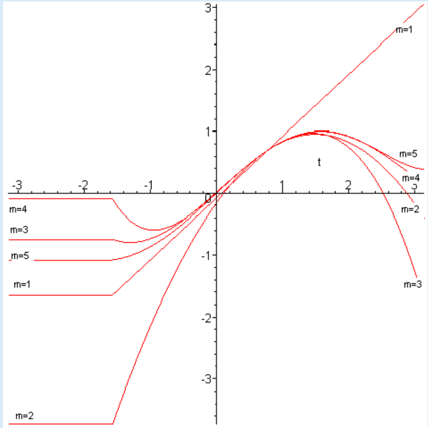

FIG. 5.6. *Iterations: 1-5*

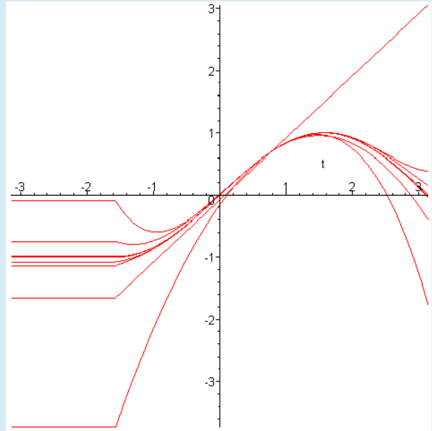

FIG. 5.7. *Iterations: 1-10*

REFERENCES

[1] A. D. Myshkis, *Linear differential equations with a retarded argument*, Moscow, Nauka, 1972.

[2] N. V. Azbelev, Maximov and Rachmatulina, *Introduction to FDE*, Moscow, 1991.

[3] J. Hale and S. M. V. Lunel, *Introduction to FDE*, 1993.

[4] I. Kiguradze, *Boundary value problems for systems of ordinary differential equations*, Itogi Nauki Tekh., Ser. Sovrem. Provl. Mat., Novejshie Dostizh. **30** (1987).

[5] I. Kiguradze, *Initial and boundary value problems for systems of ordinary differential equations*, Metsniereba, Tbillisi 1997.

[6] Sh. Gelashvili and I. Kiguradze, *On multi-point boundary value problems for systems of functional differential and difference equations*, Mem.Differential Equations Math. Phys. **5** (1995), 1–113.

[7] P. Marušiak and R. Olach, *Funkcionálne diferenciálne rovnice*, Žilina, 2000