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NUMERICAL BLOW-UP FOR THE P-LAPLACIAN EQUATION WITH A NONLINEAR SOURCE

ARTURO DE PABLO*, MAYTE PÉREZ-LLANOS†, AND RAÚL FERREIRA‡

Abstract. We study numerical approximations of nonnegative solutions of the p-laplacian equation with a nonlinear source,

$$\begin{cases} u_t = (|u_x|^{p-2}u_x)_x + |u|^{q-2}u, & (x,t) \in (-L,L) \times (0,T), \\ u(-L,t) = u(L,t) = 0, & t \in [0,T), \\ u(x,0) = \varphi(x) > 0, & x \in (-L,L), \end{cases} \tag{0.1}$$

where $p > 2$, $q > 2$ and $L > 0$ are parameters. We describe in terms of p, q and L when solutions of a semidiscretization in space exist globally in time and when they blow up in a finite time. We also find the blow-up rates and the blow-up sets by means of the discrete self-similar profiles.

Key words. numerical blow-up, parabolic p-laplacian, eigenvalue problem

AMS subject classifications. 35B40, 65M12

1. Introduction. In the following work we analyze numerically some of the features of the blow-up phenomena arising from a quasilineal parabolic equation with the p-Laplacian operator. More precisely, we study numerical approximations of positive solutions of the problem

$$\begin{cases} u_t = (|u_x|^{p-2}u_x)_x + |u|^{q-2}u, & (x,t) \in (-L,L) \times (0,T), \\ u(-L,t) = u(L,t) = 0, & t \in [0,T), \\ u(x,0) = \varphi(x), & x \in (-L,L), \end{cases} \tag{1.1}$$

will behave in rather different ways.

2. The numerical scheme. We build a numerical scheme based on a discretization of the spatial variable, using piecewise linear finite elements with mass lumping in a uniform mesh. With such semisdiscretization we translate the analysis of a PDEs problem into the study of the following ODEs system:

$$\begin{cases} MU' = h^{-p}\mathcal{D}_p\, U + M|U|^{q-2}U, \\ u_{-N} = u_N = 0, \\ U(0) = \varphi^I, \end{cases} \tag{2.1}$$

where by $U = U(t) = (u_{-N}(t), \cdots, u_N(t))$ we denote the value of the numerical approximation at the nodes $x_i = ih \quad (h = L/N)$ at time t.

We observe that the operator

$$\mathcal{D}_p\, U = D_+|D_-U|^{p-2}D_-U, \tag{2.2}$$

*Dpto. de Matemáticas, U. Carlos III de Madrid, 28911 Leganés, Madrid, Spain (arturop@math.uc3m.es)
† Dpto. de Matemáticas, U. Carlos III de Madrid, 28911 Leganés, Madrid, Spain (mtperez@math.uc3m.es)
‡Dpto de Matemática Aplicada, U. Complutense de Madrid, 28040 Madrid, Spain (raul_ferreira@mat.ucm.es)

being D_+ y D_- the stiffness matrices and M the mass matrix, is nonlinear. This fact makes that our problem differs widely from others that involve as a diffusion operator the Laplacian or even the porous media operator.

By φ^I we denote the Lagrange interpolation of the initial condition φ, which for simplicity is supposed to be symmetric and monotonously decreasing in $[0, L]$. It is not difficult to show that our scheme preserves the symmetry and monotonicity properties of the initial datum and that it is also provided with a comparison principle. Moreover, we develop an intersection theory for the proposed scheme known as Sturm Comparison Theory in the continuous background.

If we denote by $u_h(x, t)$ the linear interpolation of the values at the nodes, and by T_h the maximal time of existence for this numerical solution, we obtain the following uniform convergence result, in sets of the form $[-L, L] \times [0, T_h - \tau]$, for every $\tau > 0$.

Theorem 2.1. *Let $u \in C^{1+\alpha,\frac{1}{2}}([-L, L] \times [0, T_h - \tau])$, $0 < \alpha < 1$, be the solution of the continuous problem. Then $\exists C = C(\tau, \|u\|_\infty) > 0$ such that $\forall h > 0$,*

$$\max_{0\le t\le T-\tau} \left\{ \max_{-L\le x\le L} |u(x,t) - u_h(x,t)| \right\} \le Ch^\alpha,$$
$$\max_{0\le t\le T-\tau} \left\{ \max_{-L\le x\le L} |u_x(x,t) - (u_h)_x(x,t)| \right\} \le Ch^{2\alpha/p},$$

Moreover, if $u(\cdot, t) \in W^{2,2}([-L, L])$ we can take $\alpha = 1$ in the estimates above.

However, it is not possible to extend this result up to time T_h, due to the singularity developed by the solutions at this instant.

Once we have checked the converge of the method, we analyze next wether our scheme is efficient to reproduce the asymptotic behaviour of the continuous solutions.

3. Blow-up for the numerical scheme. Constructing suitable sub- an super-solutions by means of the corresponding discrete eigenvalue problem, we show that the conditions assuring the blow-up occurrence are identical to the conditions for the continuous problem, as we state in the theorem below. By $\lambda_1(L, h)$ we denote the first discrete eigenvalue, whose convergence to the first continuous eigenvalue, $\lambda_1(L)$, is deduced as a consequence of the density of our approximation space in the Sobolev space.

Theorem 3.1. *For the problem (2.1) it holds:*

i) *If $q < p$ every solution is global.*
ii) *If $q = p$, the solution blows up in finite time, whenever $\lambda_1(L, h) < 1$, whereas if $\lambda_1(L, h) \ge 1$ every solution of (2.1) is global.*
iii) *If $q > p$ there are solutions that blow up in finite time.*

Despite of the fact that from the previous space inclusion it is also deduced that, at least for $q = p$, there exist continuous blowing up solutions, whose numerical approximation is global for certain values of h , we show that for h sufficiently small, the blow-up occurrence for the first implies blow-up for the numerical solution. We assure in this sense that every blowing up solution is included in our analysis.

Theorem 3.2. *If $q \ge p$ it holds that, whenever u blows up there exists h sufficiently small such that U_h blows up.*

4. Blow-up rate and blow-up time. As it would be expected, we obtain the same blow-up rate corresponding to the continuous case.

Theorem 4.1. *Let be $q \geq p$ and $\alpha = \dfrac{1}{q-2}$. Then we have*

$$C_1(T_h - t)^{-\alpha} \leq \|U_h(t)\|_\infty \leq C_2(T_h - t)^{-\alpha}.$$

The lower inequality is obtained in both cases by a comparison of intersections with the solution spatially constant, and for the case $q > p$ the upper bound is deduced from the equation verified by the central node.

However, the nonlinearity of the operator $\mathcal{D}_p$ makes more difficult to obtain the upper estimate in the case $q = p$, which leads us to study the self-similar solutions. We consider solutions of the form

$$V_h(t) = (T_h - t)^{1/(p-2)} W_h,$$

and W_h verifying the equation

$$0 = h^{-p}\mathcal{D}_p\, W_h + W_h^{p-1} - \frac{1}{p-2} W_h.$$

By a shooting method from the origin we show that, for a sufficiently large initial datum, there must exist a node at which such discrete profiles become negative. As can be appreciated in Fig. 4.1, we also prove that the larger this condition is, the closer to L_p they cross the axis, where L_p denotes the length for which the first eigenvalue is one. For any initial condition it is always possible to find one of those profiles that intersects our datum as it is shown in Fig. 4.2.

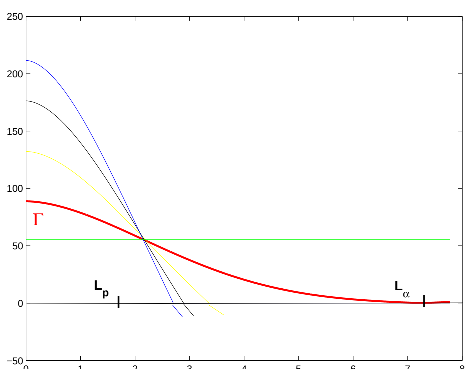

Fig. 4.1. *Stationary profiles obtained by shooting method*

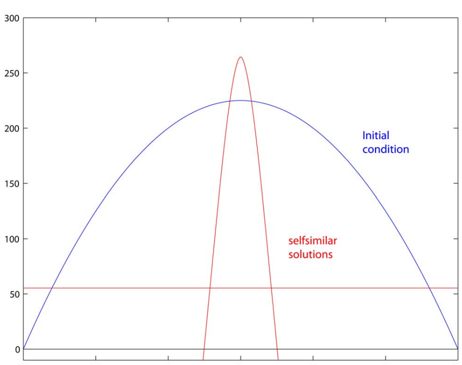

Fig. 4.2. *The initial condition intersected by one of the previous profiles*

It is at this stage that the intersection theory developed for our scheme plays a fundamental role, since it states that the number of intersections between two solutions cannot increase among the nodes lying at the interior of the support of such profile. Since we look for positive solutions, they cannot increase at the extremes either.

On the other hand, by the comparison principle we know that two solutions cannot be ordered. If so, they would explode at different times. Therefore, it proves the upper bound for the rate if $q = p$, whose constant is independent of h, by the convergence of these profiles to the corresponding continuous profiles.

We remark that the constants appearing in the rates are independent of h in both cases. This fact allows us to deduce immediately from the upper bound the convergence of the blow-up times.

THEOREM 4.2. *Let be $q \geq p$, then*

$$\lim_{h\to 0} T_h = T.$$

5. Blow-up sets. Finally, we describe the blow-up sets for the discrete problem

THEOREM 5.1. *Let U_h be a blowing up solution of* (2.1).

i) *If $q > p$ the blow-up set consists on the finite number of nodes, $B(U) = [-Kh, Kh]$, where $K \equiv K(p, q)$.*

ii) *If $q = p$ the blow-up set is the whole interval, $B(U) = [-L, L]$.*

However, it is known that if $q > p$, the continuous solutions blow up only at the origen and that if $q = p$, for large enough values of L the blow-up is regional.

This result seems to contradict the accuracy of our method to reproduce the blow-up sets corresponding to the continuous case. Nevertheless, let us show that there is not such a contradiction.

If $q > p$, since K is independent on h, in the limit as h tends to zero we recover the single point blow-up set of the continuous case:

$$B(U) \to \{0\} = B(u), \quad \text{as}\, h \searrow 0.$$

Moreover, it is only at the central node that the discrete solution blows up with the *correct* rate, being that rate at the remaining K nodes smaller, as we move away from the central node.

For the case $q = p$, from the convergence of the solution U_h towards a strictly positive self-similar solution as t approaches the blow-up time, it follows that the blow-up is always global. However, for L sufficiently large, the convergence of the self-similar profiles to the continuous compact supported self-similar profile, implies that these discrete profiles, despite being positive, are tending to zero at the nodes that lie out of such support, and the regional blow-up is recovered in this sense.

6. Numerical experiments. We conclude by showing some numerical experiments which illustrate some of the previous results.

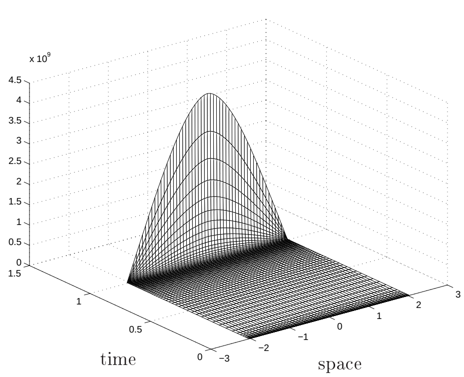

FIG. 6.1. *Evolution of the numerical solution ($q = p$, $L < L_1$)*

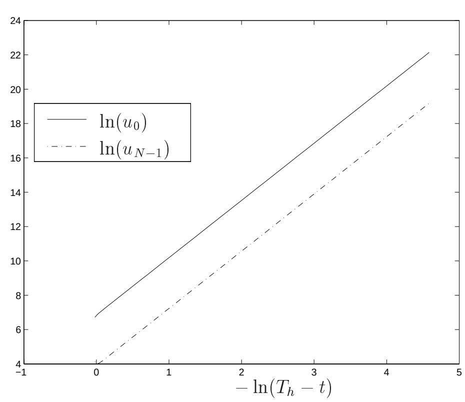

FIG. 6.2. *Blow-up rates ($q = p$)*

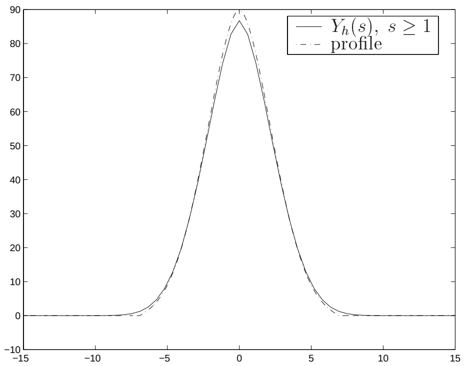

FIG. 6.3. *The rescaled solution near T_h and the self-similar profile*

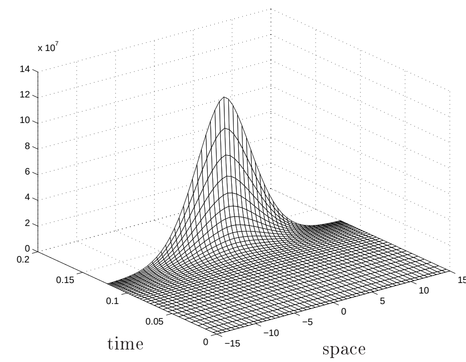

FIG. 6.4. *Evolution of the numerical solution ($q = p$, $L > L_1$)*

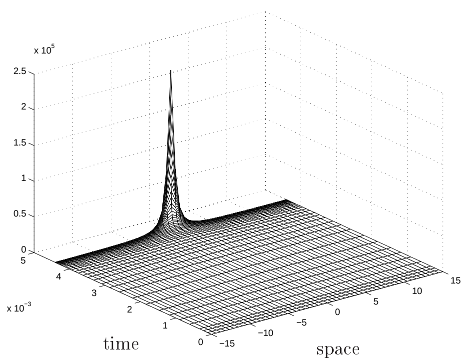

FIG. 6.5. *Evolution of the numerical solution ($q > p$)*

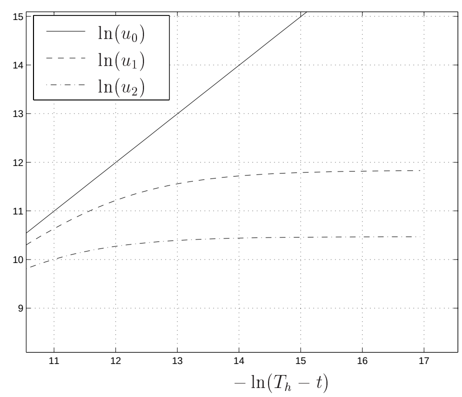

FIG. 6.6. *If $3 = q > p = 2.3$ the blow-up set consist on $K = 2$ nodes*

REFERENCES

[1] J.W. Barrett and W.B. Liu. *Finite Element Approximation of the Parabolic p-Laplacian.* SIAM J. on Numer. Anal., **31** (1994), 413–428.

[2] C. Cortázar, M. del Pino and M. Elgueta. *The problem of uniqueness of the limit in a semilinear heat equation.* Comm. Partial Differential Equations, **24** (1999), 2147–2172.

[3] R. Ferreira, P. Groisman and J. D. Rossi. *Numerical blow-up for the porous medium equation with a source* Numer. Methods Partial Differential Equations, **20** (2004), 552–575.

[4] A. Fujii and M. Ohta. *Asymptotic Behavior of Blowup Solutions of a Parabolic Equation with the p-Laplacian.* Publ. RIMS. Kyoto Univ., **32** (1996), 503–515.

[5] V.A. Galaktionov and S. A. Posashkov. *Single point blow-up for N-dimensional quasilinear equations with gradient diffusion and source.* Indiana Univ. Math. J. **40** (1991), 1041–1060.

[6] Y. Li, and C. Xie. *Blow-up for p-Laplacian parabolic equations.* Electron. J. Differential Equations, **2003** (2003), 1–12.

[7] S. A. Messaoudi. *A note on blow up solutions of a quasilinear heat equation with vanishing initial energy.* J. Math. Anal. Appl., **273** (2002), 243–247.

[8] J. D. Rossi. *Approximation of the Sobolev Trace Constant.* Divulg. Mat. **11** (2003), 109–113.