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MATHEMATICAL THEORY FOR THE GINZBURG-LANDAU APPROXIMATION IN SEMILINEAR PATTERN FORMING SYSTEMS WITH TIME-PERIODIC FORCING APPLIED TO ELECTRO-CONVECTION IN NEMATIC LIQUID CRYSTALS*

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Abstract. Electro-convection in nematic liquid crystals and the Faraday problem are paradigms for pattern formation in systems with external time-periodic forcing. Close to the first instability the bifurcating solutions can be described via perturbation analysis by a Ginzburg-Landau equation. This formal procedure can be justified mathematically through approximation and attractivity theorems. In this paper this theory is explained for a regularized standard model describing electro-convection in nematic liquid crystals.

Key words. amplitude equations, approximation, attractivity, time periodic forcing

AMS subject classifications. 35B40, 35Q35, 37L10, 76D05

1. Introduction.

In the experiments for electro-convection in nematic liquid crystals a thin layer of such a material is contained in between two spatially extended electrode plates. When an alternating current is applied to the electrodes an electro-hydrodynamic instability can occur if the voltage is above a certain threshold. The trivial spatially homogeneous solution becomes unstable and bifurcates into non-trivial pattern [4, 12]. This experiment together with the Faraday problem is a paradigm for pattern formation in systems with external time-periodic forcing.

The mathematical description of the dynamics of the bifurcating patterns is based very often on the reduction of the governing partial differential equations to finite or infinite-dimensional amplitude equations. The most famous amplitude equation occurring in such a setup is the so called Ginzburg-Landau equation (GLe)

$$\partial_T A = c_0 A + c_1 \partial_X^2 A + c_2 A|A|^2 \tag{1.1}$$

with $A = A(X,T) \in \mathbb{C}$ depending on $X \in \mathbb{R}$ and $T \geq 0$ and with coefficients $c_0, c_1, c_2 \in \mathbb{C}$. It is derived by multiple scaling analysis and describes slow modulations in time and space of the amplitude of the linearly most unstable modes. Our interest is in the justification of GLes for pattern forming systems with time periodic forcing.

The GLe has been derived for example for reaction-diffusion systems and hydrodynamical stability problems, as the Bénard and the Taylor-Couette problem. For these examples the GLe has been justified as an amplitude equation by a number of mathematical results: so called approximation and attractivity theorems have been established by a number of authors for model problems, but also for general systems including the

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Navier-Stokes equation, cf. [3, 26, 6, 15, 13, 16, 22]. Nowadays the theory is a well established mathematical tool which can be used to prove stability results [25, 21], upper semi-continuity of attractors [10, 20] and global existence results [14, 19]. Equations of Ginzburg-Landau type have also been used extensively to describe pattern formation in nematic liquid crystals [23, 12, 28, 1].

However, the literature cited above about the mathematical justification of GLes is restricted so far to autonomous systems and is not covering the situation of nematic liquid crystals due to the time-periodic forcing which has to be applied in the experiment in order to avoid the destruction of the experiment through electrolysis. In [2] we restricted the discussion of the validity question for time-periodic systems to a scalar model equation. Here we improve the results from [2] in such a way that all approximation and attractivity results from the autonomous case transfer almost one to one to the time-periodic case. As a consequence the analyticity of the solutions of the GLe as in [2] is no longer needed. The main steps of the theory are explained for the standard model describing electro-convection in nematic liquid crystals. However, we circumvent the problem of the local existence and uniqueness of solutions of the standard model by considering a regularized version. Moreover, to avoid some additional functional analytic difficulties with the Navier-Stokes equations over domains with more than one unbounded space directions, which are due to the non-differentiability of the symbol of the Helmholtz projection in that case, in the following the problem is considered in an infinitely extended strip.

The plan of this paper is as follows. In Section 2 we describe the standard model. In Section 3 this (fully nonlinear) evolutionary system is modified by some regularizing terms to obtain a semilinear system. In Section 4 we explain the spectral situation necessary for a Ginzburg-Landau approximation. Section 5 contains an approximation and an attractivity result for the Ginzburg-Landau approximation and some consequences of these results. In Section 6 we explain in an abstract way how the ideas from the autonomous case transfer to the time-periodic case, while in Section 7 we show in some detail how to derive the autonomous GLe from the time-periodic system. In Section 8 we discuss the Faraday problem as another pattern forming system with time-periodic forcing.

Notation. The space $H^m_{l,u}$ of m-times weakly differentiable uniformly local Sobolev-functions $\mathbb{R}\times\Sigma\to\mathbb{R}$ is equipped with the norm

$$\|u\|_{H^m_{l,u}(\mathbb{R}\times\Sigma)} = \sup_{x_1\in\mathbb{R}} \sum_{|j|=0}^{m} \|\partial_x^j u\|_{L^2((x,x+1)\times\Sigma)} \qquad \text{with} \qquad \|u\|^2_{L^2(\Omega)} = \int_\Omega |u(x)|^2\,\mathrm{d}x.$$

Throughout the paper we denote possibly different constants C with the same symbol if they can be chosen independent of the small bifurcation parameter $0<\varepsilon\ll 1$.

2. The standard model. There are essentially two models for the mathematical description of electro-convection in nematic liquid crystals. These are the standard model ([29] and the references therein) and the weak electrolyte model. The latter has been introduced by Kramer and Treiber in [24, 23] to overcome some insufficiencies of the standard model, which, however, will not concern us here. Thus, for simplicity we restrict ourselves to the standard model. The following presentation and non-dimensionalization of this model is similar to [5].

The continuum theory of Ericksen [7] and Leslie [9] treats nematic liquid crystals as incompressible fluids with the average molecular axis described locally by a director field n of unit vectors. For a layer of nematic liquid crystals in between two horizontal plates, the Leslie-Erickson equations for n and the generalized Navier-Stokes equations for the fluid velocity v and the pressure p in the presence of an electric field E are given by

$$(\partial_t + v \cdot \nabla)n = \omega \times n + \delta^{\perp}(\lambda A n - h)\,, \tag{2.1}$$

$$P_2(\partial_t + v \cdot \nabla)v = -\nabla p - \nabla \cdot (T^{visc} + \Pi) + \pi^2 \rho E\,, \tag{2.2}$$

$$\nabla \cdot v = 0\,, \tag{2.3}$$

where $\omega = (\nabla \times v)/2$ is the vorticity. As explained above, here we neglect the second unbounded space direction and thus consider the infinitely extended strip $(x,z) \in \mathbb{R} \times (0,\pi)$. The molecular field h is given by

$$h = 2\left(\frac{\partial f}{\partial n} - \nabla \cdot \frac{\partial f}{\partial \nabla n}\right) - \varepsilon_a \pi^2 (n \cdot E)E \tag{2.4}$$

where

$$2f = (\nabla \cdot n)^2 + K_2[n \times (\nabla \times n)]^2 + K_3[n \cdot (\nabla \times n)]^2\,, \tag{2.5}$$

is the elastic energy density describing splay, twist (K_2), and bend (K_3) deformations. We refer to [5] for a physical interpretation of the constants P_2, λ, K_2, K_3, and ε_a. The electric field $E = E(x,z,t) \in \mathbb{R}^2$ is considered to be quasistationary, i.e. rot $E = 0$. It is then split into an external forcing and some potential part, i.e.

$$E = \frac{\sqrt{2}}{\pi} E_0 \cos \omega_0 t \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \nabla \phi\,, \tag{2.6}$$

The tensors A, and T^{visc} are, respectively, the shear flow tensor

$$A_{ij} = (\partial_i v_j + \partial_j v_i)/2 \tag{2.7}$$

and the viscous stress tensor

$$\begin{aligned} -T^{visc}_{ij} = \sum_{k=1}^{3} (&\alpha_1 n_i n_j n_k n_l A_{kl} + \alpha_2 n_j m_i + \alpha_3 n_i m_j \\ &+ \alpha_4 A_{ij} + \alpha_5 n_j n_k A_{ki} + \alpha_6 n_i n_k A_{kj}), \end{aligned} \tag{2.8}$$

with $m = \delta^{\perp}(\lambda A n - h)$ and coefficients $\alpha_1, \ldots, \alpha_6$. The tensor Π is the nonlinear Ericksen stress tensor

$$\Pi_{ij} = \sum_{k=1}^{3} \frac{\partial f}{\partial n_{k,j}} n_{k,i}. \tag{2.9}$$

The projection tensor $\delta^{\perp}_{ij} = \delta_{ij} - n_i n_j$ in (2.1) guarantees that $|n| = 1$ as long as the solution exists.

In the standard model for electro-convection the continuum theory of Ericksen and Leslie is combined with the quasi-static Maxwell equations under the assumption of an ohmic resistivity, i.e.

$$P_1(\partial_t + v \cdot \nabla)\rho = -\nabla \cdot (\sigma E \sigma) \tag{2.10}$$

for the charge density ρ. Finally the system is closed by Poisson's law

$$\rho = \nabla \cdot (\varepsilon E)\,. \tag{2.11}$$

The dielectric tensor ε and conductivity tensor σ are given by $\varepsilon_{ij} = \varepsilon_\perp \delta_{ij} + \varepsilon_a n_i n_j$ and $\sigma_{ij} = \sigma_\perp \delta_{ij} + \sigma_a n_i n_j$, respectively. The parameters P_1 and P_2 are Prandtl-type time scale ratios. Again we refer to [5] for a physical interpretation of the constants P_1, σ_{ij}, ε_{ij}, α, r.

We assume rigid vertical boundary conditions derived from ideal conducting plate conditions, rigid anchoring for the director, and finite viscosity. This means

$$n_2 = v_1 = v_2 = \phi = 0 \tag{2.12}$$

at $z = 0, \pi$, i.e. the coordinate system is chosen such that $n = (1, 0)$ at the upper and lower plates located at $z = 0, \pi$. The model is invariant under arbitrary translations in x and under the reflection

$$(x, n_2, v_1) \to -(x, n_2, v_1).$$

3. The regularized standard model. Using Poisson's law, E resp. ϕ can be expressed in terms of ρ and so (2.1)–(2.3) and (2.10) can be rewritten as a system of dynamical equations for n, v, and ρ. Since $n_1^2 + n_2^2 = 1$ for our purposes it is sufficient to consider n_2. System (2.1)–(2.3) and (2.10) for n_2, v, and ρ is fully nonlinear and a mixture of different types of PDEs as quasilinear parabolic equations and balance laws. We are not aware of any local existence and uniqueness result for this system in the literature. Since such a theorem is fundamental for any approximation result we consider a regularized version of the standard model. In order to obtain a semilinear system, we add artificially a regularizing differential operator $-\beta\Delta^2$. For small $\beta > 0$ the regularized system and the original system show qualitatively the same bifurcation behavior. Thus we consider

$$\partial_t n_2 = \langle e_2, -(v \cdot \nabla)n + \omega \times n + \delta^\perp(\lambda A - h)\rangle - \beta\Delta^2 n_2, \tag{3.1}$$

$$\partial_t v = P_2^{-1} Q(-(v \cdot \nabla)v - \nabla \cdot (T^{visc} + \Pi) + \pi^2 \rho E) - \beta Q \Delta^2 v, \tag{3.2}$$

$$\partial_t \rho = -v \cdot \nabla\rho - P_1^{-1} \nabla \cdot (\mu E \sigma) - \beta\Delta^2 \rho\ , \tag{3.3}$$

where Q is the projection on the divergence-free vector fields $\{v \mid \nabla \cdot v = 0\}$, cf. [13, 19], and where E is defined through (2.6) and (2.11) in terms of ρ, n, and E_0. The extension of Q by identity to the ρ and n variables is also denoted by Q. The system is equipped with the boundary conditions from the non-regularized system

$$n_2 = v_1 = v_2 = \phi = 0 \tag{3.4}$$

for $z = 0, \pi$, and additional artificial boundary conditions due to the regularization

$$\partial_z^2 n_2 = \partial_z^2 v_1 = \partial_z^2 v_2 = \rho = \partial_z^2 \rho = 0, \tag{3.5}$$

for $z = 0, \pi$. In the following (3.1)–(3.3) is abbreviated as

$$\partial_t V = M(t)V + \tilde{N}(t, V) \tag{3.6}$$

where $M(t)V$ stands for the linear and $\tilde{N}(t, V)$ for the nonlinear terms with respect to $V = (n_2, v_1, v_2, \rho)$.

4. Linear stability analysis. In order to analyze the stability of the trivial solution V=0 in (3.6) we consider the linearized system

$$\partial_t V = M(t)V. \tag{4.1}$$

Due to the translational invariance of the problem the solutions are given by Floquet-Fourier modes

$$V = \hat{\varphi}_m(k, z, t)\, e^{ikx}\, e^{\lambda_m(k)t} \tag{4.2}$$

with $k \in \mathbb{R}$, $m \in \mathbb{N}$, and $\hat{\varphi}_m$ periodic in t, i.e.

$$\hat{\varphi}_m(\cdot, \cdot, t) = \hat{\varphi}_m(\cdot, \cdot, t + 2\pi/\omega_0).$$

For $V = 0$ asymptotically stable, we have for all $m \in \mathbb{N}$ and $k \in \mathbb{R}$ that $\mathrm{Re}\,\lambda_m(k) < 0$. If $V = 0$ becomes unstable through increasing E_0, then there exists one curve of eigenvalues λ_1 satisfying $\mathrm{Re}\lambda_1(k_c) = 0$ if the amplitude E_0 of the external alternate current equals a critical value $E_{0,crit}$. Due to the fact that we have a real-valued problem we also have $\mathrm{Re}\lambda_1(-k_c) = 0$. We assume that for k close to k_c the curve of eigenvalues λ_1 is simple. Due to the reflection symmetry for $x \to -x$ this implies $\lambda_1(k) = \lambda_1(-k)$ and so $\mathrm{Im}\lambda_1(k) = 0$ for all wave numbers k where λ_1 is simple. For $E_0 = E_{0,crit}$ we assume that all Floquet exponents possess a real part strictly less than $-\sigma_0$ for a $\sigma_0 > 0$, except of $\lambda_1(k)$ for k in small neighborhoods of $\pm k_c$. Since there is no possibility of confusion with the dielectric tensor we denote the bifurcation parameter as usual by ε. It is defined by $\varepsilon^2 = E_0 - E_{0,crit}$. Then by continuity for $\varepsilon > 0$ we have that the spectrum is only changed slightly, cf. FIG. 4.1.

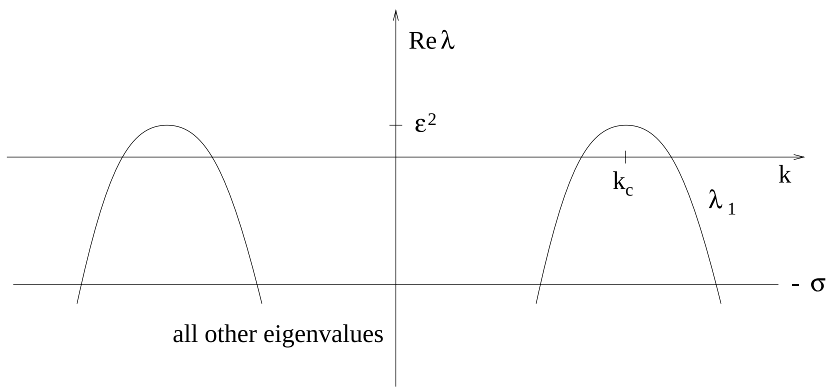

FIG. 4.1. *The real part of the spectrum as a function over the Fourier wave numbers k.*

5. Mathematical theory for the Ginzburg-Landau approximation. The ansatz for the derivation of the GLe is

$$\varepsilon\psi_A(x, z, t) = \varepsilon A(X, T) e^{ik_c x} \hat{\varphi}_1(k_c, z, t) + \text{c.c.} + \mathcal{O}(\varepsilon^2), \tag{5.1}$$

where

$$X = \varepsilon x \qquad \text{and} \qquad T = \varepsilon^2 t,$$

and $\hat{\varphi}_1$ is the critical mode belonging to $m = 1$ in (4.2). Inserting (5.1) into (3.6) shows that A has to satisfy the GLe (1.1), see Sec. 7 for details.

In the following we formulate an approximation and an attractivity result for the Ginzburg-Landau approximation and explain the consequences of the validity of such results. In the subsequent sections we explain how to conclude these theorems from the autonomous case.

5.1. An approximation result. The formal approximation (5.1) is only useful if the dynamics known for (1.1) can be found approximately in the original system (3.6), too. This means that for $T \in [0, T_0]$ or $t \in [0, T_0/\varepsilon^2]$, respectively. the error (in THEOREM 5.1 of order $\mathcal{O}(\varepsilon^2)$) should be much smaller than the approximation $\varepsilon\psi_A$ and the solution V which are both of order $\mathcal{O}(\varepsilon)$.

THEOREM 5.1. *Let $m \geq 8$ and $A = A(X,T)$ be a solution of the GLe (1.1) for $T \in [0, T_0]$, satisfying*

$$\sup_{T \in [0,T_0]} \|A(T)\|_{H^m_{l,u}} < \infty.$$

Then there are $\varepsilon_0 > 0$ and $C > 0$, such that for all $\varepsilon \in (0, \varepsilon_0)$ we have solutions V of (3.6) satisfying

$$\sup_{t \in [0,T_0/\varepsilon^2]} \sup_{(x,z) \in \mathbb{R} \times (0,\pi)} |V(x,z,t) - \varepsilon\psi_A(x,z,t)| \leq C\varepsilon^2.$$

We remark that there are other amplitude equations [17] which although derived by a formal perturbation analysis do not reflect the true dynamics of the original equations. Moreover, the proof of THEOREM 5.1 is not trivial since solutions of order $\mathcal{O}(\varepsilon)$ have to be bounded on a time interval of length $\mathcal{O}(1/\varepsilon^2)$. THEOREM 5.1 can be improved in a number of directions. The error can be made smaller by adding higher order terms to the approximation. However the time scale cannot be extended [26]. By a more involved analysis [14] less regularity for the solutions of the GLe is needed.

5.2. An attractivity result. The following attractivity theorem shows that solutions to order $\mathcal{O}(\varepsilon)$ initial conditions develop in such a way that after a time $\mathcal{O}(1/\varepsilon^2)$ they can be approximated by the solutions of the GLe (1.1). Thus, the GLe describes the solutions in the attracting set of the system, i.e. the interesting dynamics of the standard model close to the threshold of the first instability.

THEOREM 5.2. *Let $s \geq 4$. For every $m \geq 0$, $C_1 > 0$ there exist $T_0 > 0$, $\varepsilon_0 > 0$ and $C_2 > 0$ such that the following is true. For all $\varepsilon \in (0, \varepsilon_0)$ and all $U_0 \in H^s_{l,u}$ with $\|U_0\|_{H^s_{l,u}} \leq C_1\varepsilon$ the associated solution V of (3.6) at time $t = T_0/\varepsilon^2$ can be written as*

$$V(x,z,T_0/\varepsilon^2) = \varepsilon A(X)\, \mathrm{e}^{ik_c x}\, \hat{\varphi}_1(k_c,z,t) + \text{c.c.} + \varepsilon^2 R(x,z)$$

where $\|\hat{A}\|_{H^m_{l,u}} \leq C_2$ and $\|R\|_{H^s_{l,u}} \leq C_2$.

This is only one possible version of such an attractivity theorem. See [6, 16, 19] for other more advanced versions of attractivity theorems.

5.3. Global existence and upper semi-continuity of attractors. As already said the above versions of the approximation and of the attractivity theorem can be improved such that the outcome from the attractivity theorem can be used as input for the approximation theorem. The combination of the two theorems allows for instance to transfer the global existence of solutions from the GLe to the original system, cf. [14, 19]. Moreover, the upper semi-continuity of attractors holds, cf. [10, 20]. The proofs of these results are based only on suitable approximation and attractivity theorems. Therefore the global existence and upper semi-continuity of attractors also hold in the time-periodic case. Hence, the GLe really gives a proper description of these systems near the bifurcation point also in case of a time-periodic forcing.

We summarize this as follows:

Abstract theorem. *Suppose that the assumptions (A1)–(A3), (B1), (B3), and (C1) and either (B2) I or (B2) II of [19] hold for* (3.6) *with the following modifications. The operator $M(t)$ is a sum of the sectorial operator Λ from (A2) and a time-periodic operator $B(t) : Z \to Z^*$ where Z and Z^* are the Banach spaces from (A1). Moreover, (B2) I or (B2) II hold for the Floquet exponents of $M(t)$. Then the approximation and attractivity result from [19] remain valid if the Fourier modes in the approximation are replaced by the Floquet-Fourier modes.*

6. How to transfer the ideas from the autonomous to the time-periodic case. In the following we sketch all modifications from the autonomous case to the time-periodic case such that the reader will be able to check the validity of the above approximation resp. attractivity result by reading parallel for instance [13, 2] or [19].

The main problem in the proofs of the approximation results is the long time scale $\mathcal{O}(1/\varepsilon^2)$ which is much longer than $\mathcal{O}(1/\varepsilon)$ which can be obtained by a simple application of Gronwall's inequality due to the $\mathcal{O}(\varepsilon)$ magnitude of the solutions. Only by a separation of the modes with positive or slightly negative growth rates from the ones with strictly negative growth rates in the linearized system the long time scale can be approached. However, there is no spectral gap and so like in the autonomous case it turns out that it is essential for the mathematical analysis to consider the Fourier transformed system with respect to the unbounded spatial variable. In Fourier space (3.6) yields

$$\partial_t \hat{V}(k,t) = \hat{M}(k,t)\hat{V}(k,t) + \hat{N}(\hat{V})(k,t), \tag{6.1}$$

with $k \in \mathbb{R}$ and $\hat{V}(k,t)$ a vector-valued function of z. For fixed wave number $k \in \mathbb{R}$ close to k_c there is a spectral gap and so by [8, Theorem 7.2.3] (which is applicable due to our regularization), a separation into so called critical and non-critical modes is possible. Using again [8, Theorem 7.2.3] shows that the non-critical part of the evolution operator associated to $\hat{M}(k,t)$ is damping with some exponential rate. Moreover, [8, Theorem 7.2.3] allows to transform the one-dimensional critical part of $\hat{M}(k,t)$ with some bounded transformation into an autonomous operator, i.e. into a multiplication with λ_1. Since λ_1 is simple the associated semigroup shows growth rates of order $\mathcal{O}(e^{\varepsilon^2 t})$. Using the multiplier theorem in $H^m_{l,u}$-spaces from [13] shows that the associated evolution operators has this growth rate in physical space in the $H^m_{l,u}$-spaces, too. Since the estimates for the nonlinear terms are exactly the same in the autonomous and in the time-periodic case the proof of the approximation result then goes along the lines of the autonomous case, cf. [13, 19]. Here, the nonlinearity is a Lipschitz continuous mapping from some interpolation space $\mathcal{X}^\alpha$ with $\alpha \in (3/4, 1)$ into $\mathcal{X} = H^0_{l,u} \cap \{V = QV\}$, where $\mathcal{X}^1$ is the domain of definition of $-\beta Q \Delta^2$. The error is then bounded in $\mathcal{X}^\alpha$ using Gronwall's inequality, now in the system for the critical and noncritical modes. $\mathcal{X}^\alpha$ can be embedded by [8, Theorem 1.6.1] into $H^3_{l,u}$ which can be embedded by Sobolev's embedding theorem into the space C^0_b of uniformly bounded continuous functions.

Similarly the proof of the attractivity result also goes along the lines of the autonomous case, cf. [19].

7. Derivation of the Ginzburg-Landau equation. For the subsequent analysis it is sufficient that the critical Floquet exponents λ_1 near k_c of $\hat{M}(k,t)$ are simple. However, in order to make things less abstract we assume that the linear operator $\hat{M}(k,t)$ with $\hat{M}(k,t) = \hat{M}(k, t + 2\pi/\omega_0)$ yields for every $k \in \mathbb{R}$ and $t \in [0, 2\pi/\omega_0)$ a Floquet Schauder basis $(\hat{\varphi}_j(k,t))_{j\in\mathbb{N}}$ of $L^2((0,\pi), \mathbb{C}^4)$ of $2\pi/\omega_0$-periodic functions $\hat{\varphi}_j(k,t) = \hat{\varphi}_j(k, t+2\pi/\omega_0)$

solving

$$\partial_t \hat{\varphi}_j(k,t) = \hat{M}(k,t)\hat{\varphi}_j(k,t) - \lambda_j(k)\hat{\varphi}_j(k,t),$$

i.e. the Floquet functions $e^{\hat{\lambda}_j(k)t}\,\hat{\varphi}_j(k,t)$ are solution of $\partial_t \hat{V}(k,t) = \hat{M}(k,t)\hat{V}(k,t)$ and $\lambda_j(k)$ are the associated Floquet exponents. This means that we assume that there are no Jordan blocks in the monodromy operator for $\hat{M}(t)$. The functions $\hat{\varphi}_j$ are normalized by setting $\|\hat{\varphi}_j(k,0)\|_{L^2}=1$. For defining projections onto the $\hat{\varphi}_j(k,t)$ we consider the adjoint problem $-\partial_t \hat{V}(k,t) = \hat{M}^*(k,t)\hat{V}(k,t)$. Consequently also this problem has for every $k \in \mathbb{R}$ and $t \in [0, 2\pi/\omega_0)$ a Floquet Schauder basis $(\hat{\varphi}^*_j(k,t))_{j\in\mathbb{N}}$ of $L^2((0,\pi),\mathbb{C}^4)$ of $2\pi/\omega_0$-periodic functions $\hat{\varphi}^*_j(k,t) = \hat{\varphi}^*_j(k,t+2\pi/\omega_0)$ solving

$$-\partial_t \hat{\varphi}^*_j(k,t) = \hat{M}^*(k,t)\hat{\varphi}^*_j(k,t) - \overline{\lambda_j(k)}\hat{\varphi}^*_j(k,t),$$

and satisfying the orthogonality

$$\langle \hat{\varphi}^*_i, \hat{\varphi}_j \rangle = \delta_{ij}. \tag{7.1}$$

A solution $\hat{V}(k,t)$ of (6.1) is expanded in terms of the Floquet functions $\hat{\varphi}_j(k,t)$, i.e.

$$\hat{V}(k,t) = \sum_{j\in\mathbb{N}} \hat{a}_j(k,t)\hat{\varphi}_j(k,t) \quad \text{with} \quad \hat{a}_j(k,t) \in \mathbb{C}, \tag{7.2}$$

such that

$$\begin{aligned}\partial_t \left(\sum_{j\in\mathbb{N}} \hat{a}_j(k,t)\hat{\varphi}_j(k,t) \right) &= \sum_{j\in\mathbb{N}} ((\partial_t \hat{a}_j(k,t))\hat{\varphi}_j(k,t) + \hat{a}_j(k,t)\partial_t \hat{\varphi}_j(k,t)) \\ &= \sum_{j\in\mathbb{N}} \hat{a}_j(k,t)\hat{M}(k,t)\hat{\varphi}_j(k,t) + \hat{N}(\hat{V})(k,t).\end{aligned}$$

In order to find the equations for the coefficient functions $\hat{a}_j(k,t)$ we apply the adjoint eigenfunction $\hat{\varphi}^*_j(k,t)$ and find

$$\partial_t \hat{a}_j(k,t) = \hat{\lambda}_j(k)\hat{a}_j(k,t) + \langle \hat{\varphi}^*_j(k,t), \hat{N}(k,t) \rangle \tag{7.3}$$

for $j \in \mathbb{N}$. We used (7.1) and

$$\begin{aligned}-\langle \hat{\varphi}^*_j(k,t), \partial_t \hat{\varphi}_i(k,t) \rangle + \langle \hat{\varphi}^*_j(k,t), \hat{M}(k,t)\hat{\varphi}_i(k,t) \rangle \\ = \langle \hat{\varphi}^*_j(k,t), \hat{\lambda}_j(k)\hat{\varphi}_i(k,t) \rangle = \hat{\lambda}_j(k)\delta_{ij}.\end{aligned}$$

Our derivation of the GLe is now based on (7.3). For notational simplicity we avoid the explicit notation of the small parameter ε in the following. We make the ansatz

$$\begin{aligned}a_1(x,t) &= \varepsilon A_1(X,T)\,e^{ik_c x} + \varepsilon^2 A_{2,1}(X,T)\,e^{2ik_c x} + \varepsilon^2 A_{0,1}(X,T) + \text{c.c.}, \\ a_j(x,t) &= \varepsilon^2 A_{2,j}(X,T)\,e^{2ik_c x} + \varepsilon^2 A_{0,j}(X,T) + \text{c.c.}\end{aligned}$$

where $j \in \mathbb{N} \setminus \{1\}$, $X = \varepsilon x$ and $T = \varepsilon^2 t$. With this ansatz we derive formally a GLe with time periodic coefficients. We write the nonlinearity of (3.6) in the form

$$N(V) = B(t,V,V) + C(t,V,V,V) + \mathcal{O}(V^4), \tag{7.4}$$

with bilinear and trilinear symmetric terms B and C and introduce the abbreviations

$$\begin{aligned}&\hat{B}_j(t,k,k-m,m) = \mathrm{e}^{-ikx}\, B(t,\hat{\varphi}_1(k-m,t)\,\mathrm{e}^{i(k-m)x},\hat{\varphi}_j(m,t)\,\mathrm{e}^{imx}),\\ &\hat{C}(t,k,k-l_1,l_1-l_2,l_2)\\ &\qquad = \mathrm{e}^{-ikx}\, C(t,\hat{\varphi}_1(k-l_1,t)\,\mathrm{e}^{i(k-l_1)x},\hat{\varphi}_1(l_1-l_2,t)\,\mathrm{e}^{i(l_1-l_2)x},\hat{\varphi}_1(l_2,t)\,\mathrm{e}^{il_2x}).\end{aligned}$$

For $\varepsilon^2 e^{0ix}$ in the j-the equation we obtain

$$\lambda_j(0,0)A_{0,j} = -2\langle\hat{\varphi}_j^*,\hat{B}_1(t,0,k_c,-k_c)\rangle|A_1|^2, \tag{7.5}$$

and for $\varepsilon^2 e^{2ik_cx}$ in the j-th equation

$$\lambda_j(2k_c,0)A_{2,j} = -\langle\hat{\varphi}_j^*,\hat{B}_1(t,2k_c,k_c,k_c)\rangle A_1^2. \tag{7.6}$$

For $\varepsilon^3 e^{ik_cx}$ in the equation for $j=1$ we obtain

$$\begin{aligned}\partial_T A_1 &= d_0A_1 + d_1\partial_X^2 A_1 \\ &\quad + 2\langle\hat{\varphi}_1^*,\sum_{j\in\mathbb{N}\setminus\{1\}}\hat{B}_j(t,k_c,k_c,0)\rangle A_1A_{0,j}\\ &\quad + 2\langle\hat{\varphi}_1^*,\sum_{j\in\mathbb{N}\setminus\{1\}}\hat{B}_j(t,k_c,-k_c,2k_c)\rangle A_{-1}A_{2,j}\\ &\quad + 3\langle\hat{\varphi}_1^*,\hat{C}(t,k_c,k_c,k_c,-k_c)\rangle A_1|A_1|^2,\end{aligned} \tag{7.7}$$

with $d_0 = \partial_{\varepsilon^2}\lambda_1(k_c,0)$ and $2d_1 = \partial_k^2\lambda_1(k_c,0)$. In (7.7) we replace $A_{0,j}$ through (7.5) and $A_{2,j}$ through (7.6) and obtain the GLe

$$\partial_T A_1 = d_0(\varepsilon)A_1 + d_1(\varepsilon)\partial_X^2 A_1 + \gamma(t,\varepsilon)A_1|A_1|^2, \tag{7.8}$$

with a time-periodic coefficient $\gamma(t,\varepsilon)$. Since all coefficients d_j and γ depend smoothly on ε^2 we have the existence of limits c_j and $\gamma_0(t)$ with

$$d_j(\varepsilon) = c_j + \mathcal{O}(\varepsilon^2) \qquad \text{and} \qquad \gamma(t,\varepsilon) = \gamma_0(t) + \mathcal{O}(\varepsilon^2).$$

In the limit $\varepsilon^2 \to 0$ we obtain a GLe

$$\partial_T A_1 = c_0A_1 + c_1\partial_X^2 A_1 + \gamma_0(T/\varepsilon^2)A_1|A_1|^2. \tag{7.9}$$

Averaging over the highly oscillating cubic coefficient $\gamma_0(T/\varepsilon^2)$ shows that for the dynamics only the mean value c_2 is essential in lowest order. Thus we finally have the autonomous GLe

$$\partial_T A_1 = c_0A_1 + c_1\partial_X^2 A_1 + c_2A_1|A_1|^2. \tag{7.10}$$

8. Another example. When a container of fluid is shaken vertically with sufficient strength, pattern develop on the the free surface. This pattern forming system is known as the Faraday problem. If this problem is considered in an infinitely extended strip the trivial solution, i.e. the flat surface, becomes unstable exactly as described in Section 4, cf. [11, 27]. The first pattern to appear is sub-harmonic with half the external frequency. One model to describe the Faraday problem are the Zhang-Vinals equations [27] which are derived in the limit of weak damping and a deep container and which are given in case of two unbounded dimensions by

$$\partial_t h = \gamma\Delta h + D\phi - \nabla\cdot(h\nabla\phi) + \frac{1}{2}\nabla^2(h^2 D\phi) - D(hD\phi) \tag{8.1}$$

$$+D\big(hD(hD\phi) + \frac{1}{2}h^2\Delta\phi\big), \tag{8.2}$$

$$\partial_t\phi = \gamma\Delta\phi + \Gamma_0\Delta h - G(t)h + \frac{1}{2}(D\phi)^2 - \frac{1}{2}(\nabla\phi)^2 \tag{8.3}$$

$$-(D\phi)(h\Delta\phi + D(hD\phi)) - \frac{1}{2}\Gamma_0\nabla\cdot((\nabla h)(\nabla h)^2),$$

where $h(x,t)$ is the surface height and $\phi(x,t)$ a velocity potential, and the symbol of D in Fourier space is $\hat{D}(k) = |k|$. The external forcing is given by $G(t) = G_0\cos(\omega_0 t)$ and the parameters γ and Γ_0 correspond to viscosity and surface tension respectively [27]. In case of a strip we have $\nabla \to \partial_x$ and $\Delta \to \partial_x^2$. The Zhang-Vinals equations are fully nonlinear and so our theory again only applies to a regularized version, i.e. if $-\beta\Delta^2 h$ and $-\beta\Delta^2\phi$, with a small $\beta > 0$, are added to the right hand side of (8.1) and (8.3), respectively.

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