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EXTREMAL EQUILIBRIA FOR PARABOLIC NON-LINEAR REACTION-DIFFUSION EQUATIONS*

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Abstract. We prove the existence of two extremal equilibria for a wide class of parabolic reaction-diffusion equations (RD) whose non-linear term satisfies a suitable structure condition. Moreover, these equilibria are ordered. We also obtain some stability property for the extremal equilibria as well as uniform bounds for the asymptotic behaviour of the solutions in terms of the extremal equilibria. In fact, the attractor for (RD) is contained in the order interval defined by the extremal equilibria.

Key words. Extremal equilibia, reaction diffusion, attractors,

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1. Introduction.

Let consider the following model problem

$$\begin{cases} u_t - \Delta u = f(x,u) & \text{in } \Omega \\ u(0) = u_0 & \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain. We pose the problem in $X = C(\overline{\Omega})$. We assume that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is such that there exists a unique local solution of problem (1.1) which we denote by $u(t,x;u_0)$. These solutions define a non-linear semigroup $S(t) : u_0 \in X \to S(t)u_0 = u(t,x;u_0)$.

For problems like (1.1) the following monotonicity properties with respect to the initial data and the nonlinear term hold:

1. Given two ordered initial data, the corresponding solutions remain ordered as long as they exist.
2. Given two nonlinear terms, f and g, we denote by u_f and u_g the solutions of problem (1.1) with right hand side f and g, respectively. If $f(t,x,s) \leq g(t,x,s)$ for all $t \geq 0$, $s \in \mathbb{R}$ a.e. $x \in \Omega$ then $u_f(t,x;u_0) \leq u_g(t,x;u_0)$ a.e. $x \in \Omega$ as long as they exist.

Our goal here is to show that for a large class of nonlinear terms, problem (1.1) has the following remarkable dynamical property: For these problems we obtain two special equilibria which are extremal in the sense that they are maximal and minimal in the ordering sense. Moreover they provide uniform bounds for the asymptotic behaviour of the solutions of (1.1). In particular, they give a bound for the global attractor of (1.1), and in fact they act as the caps of the attractor.

The outline of this note is as follows. In Section 2 we state a general results for the existence of the extremal equilibria and some associated dynamical properties. In Section 3 we make our result precise for the case of positive solutions. In particular, we prove a uniqueness result for positive equilibria in Section 4. Finally, we apply the general results

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to logistic equations, in Section 5. Some other examples are presented in Sections 6 and 7 including the case of nonlinear boundary conditions.

2. A general result for extremal equilibria. We start with some general resuts on the existence of extremal equilibria, see [9, 10] for details. The main assumption we make is that f satisfies the following structure condition

$$s\ f(x,s) \le C(x)s^2 + D(x)|s| \quad \forall s \in \mathbb{R} \tag{2.1}$$

with

$$C \in L^p(\Omega), \quad p > N/2, \quad D \in L^r(\Omega), \quad r > N/2.$$

These conditions imply, in particular, that solutions of problem (1.1) are globally defined since, in such case, using comparison we have

$$|u(t,x;u_0)| \le v(t,x;|u_0|) \tag{2.2}$$

where v solves the following linear problem

$$\begin{cases} v_t - \Delta v = C(x)v + D(x) & \text{in } \Omega \\ v(0) = u_0 & \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

We will also assume in what follows that the associted semigroup to the linear equation above satisfies

$$S_{\Delta+C}(t) \quad \text{has exponential decay} \tag{2.4}$$

that is, 0 is globally asymptotically exponentially stable for

$$\begin{cases} w_t - \Delta w = C(x)w & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega \end{cases}$$

This assumption and the regularity of D implies the existence of a unique equilibrium for (2.3) which belongs to $L^\infty(\Omega)$ and is globally asymptotically stable for (2.3).

With these, we have

Theorem 2.1. *Assume* (2.1) *and* (2.4). *Then, there exist two extremal equilibria of* (1.1), $\varphi_m \le \varphi_M$, *which are minimal and maximal respectively, such that any other equilibrium* ψ *satisfies*

$$\varphi_m \le \psi \le \varphi_M$$

and

$$\varphi_m(x) \le \liminf_{t\to\infty} u(t,x;u_0) \le \limsup_{t\to\infty} u(t,x;u_0) \le \varphi_M(x)$$

uniformly in $x \in \Omega$ *and* u_0 *in bounded sets of* X.

Moreover, φ_m *is stable from below and* φ_M *from above. Also, there exists an attractor for the problem satisfying*

$$\mathcal{A} \subset [\varphi_m, \varphi_M], \quad \varphi_m, \varphi_M \in \mathcal{A}$$

and the order interval $[\varphi_m, \varphi_M]$ *is positively invariant.*

We now give an sketch of the proof. First, notice that we can compare solutions of (1.1) with those of (2.3), i.e. (2.2) holds. Now, taking limits in this inequality as $t \to \infty$, we have the following estimate

$$\limsup_{t\to\infty} |u(t,x;u_0)| \leq \phi(x) \quad \text{uniformly in } x \tag{2.5}$$

where ϕ is the unique equilibrium of (2.3). Notice that ϕ is a supersolution of (1.1). We take $\eta = \phi + \delta$ for some positive constant $\delta > 0$. Taking η as initial data in (1.1) and using (2.5) we have that there exists $T > 0$ such that $S(T)\eta \leq \eta$. Now, using monotonicity, we can construct a decreasing sequence $\{S(nT)\eta\}_n$ which is bounded from below (by $-\eta$). Thus, it converges to some function $\varphi_M(x)$. Actually, it can be shown that $S(t)\eta \to \varphi_M$ uniformly in Ω (by compactness properties of (1.1)). Using the continuity of $S(t)$, $t > 0$, we have that φ_M is an equilibrium. From (2.5) we have

$$\limsup_{t\to\infty} u(t,x;u_0) \leq \varphi_M(x) \quad \text{uniformly in } x. \tag{2.6}$$

The result for the minimal equilibrium follows in an analogous way.

Notice that the extremal equilibira constructed above might be sign-changing.

3. Positive solutions. Sometimes, in applications it is interesting to consider only non-negative solutions. We have a first result

THEOREM 3.1. *Consider problem* (1.1)*. Suppose that* $f(x,0) \geq 0$*. Then, either there exists a minimal* ***non-negative*** *equilibria* φ_m^+*; or the dynamics of the system goes to infinity, that is,* $||u(t)||_{L^\infty(\Omega)} \to \infty$*, in finite or infinite time.*

If it exists, the minimal equilibrium is either $\varphi_m^+ \equiv 0$ *(if* $f(x,0) = 0$ *a.e.* $x \in \Omega$*), or* $\varphi_m^+(x) > 0$ *for all* $x \in \Omega$ *(if* $f(x_0,0) > 0$ *for some* $x_0 \in \Omega$*).*

Moreover, if $\varphi_m > 0$ *then it is globally asymptotically stable from below.*

A more interesting case occurs when zero is an equilibrium and we look for the minimal positive solutions. In that direction, we have

THEOREM 3.2. *Assume* $f(x,0) \equiv 0$ *and there exists* $m \in L^p(\Omega)$ *with* $p > N/2$ *such that*

$$f(x,s) \geq m(x)s \quad \text{a.e. } x \in \Omega, \quad 0 \leq s \leq s_0. \tag{3.1}$$

Also assume that m *is such that* 0 *is unstable for problem*

$$\begin{cases} v_t - \Delta v = m(x)v & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases} \tag{3.2}$$

i.e. we assume $\lambda_1(-\Delta - m(x)) < 0$ *where we denote by* $\lambda_1(-\Delta - m(x))$ *the first eigenvalue of* $-\Delta - m$ *in* Ω *with Dirichlet boundary conditions.*

Then, 0 *is an isolated equilibrium point. Moreover, either all the dynamics goes to infinity (in finite or infinite time), or there exists a minimal* ***positive*** *equilibrium* φ_m^+ *which is asymptotically stable from below.*

Notice that condition (3.1) is nothing but an instability condition for $v = 0$. Indeed, that condition holds if f satisfies e.g.

$$\liminf_{s\to 0} \frac{f(x,s)}{s} > \lambda_1 \quad \text{uniformly in } x.$$

where λ_1 is the first eignvalue of the Laplacian with Dirichlet boundary conditions.

As a consequence we have

COROLLARY 3.3. *Suppose $f(x,s)$ satisfies the assumptions of the theorem above. Also assume that f satisfies*

$$f(x,s) \leq C(x)s + D(x) \tag{3.3}$$

with $C \in L^p(\Omega)$, $p > N/2$, and $D \in L^r(\Omega)$, $r > N/2$, such that the semigroup generated by $\Delta + C(x)$ has exponential decay.

Then, there exists two ordered extremal positive equilibria $0 < \varphi_m^+ \leq \varphi_M$ (which may coincide). Moreover, φ_m^+ is stable from below and φ_M it is so from above.

Furthermore, there exists an attractor for positive solutions $\mathcal{A}_+$ which satisfies

$$\mathcal{A}_+ \subset [\varphi_m^+, \varphi_M]$$

and $\varphi_m^+, \varphi_M \in \mathcal{A}_+$.

4. A uniqueness result for positive solutions. We present now a uniqueness result of positive equilibria for (1.1). See [4] for a related result.

THEOREM 4.1. *Suppose that there exists the maximal positive equilibrium solution for* (1.1)*. Assume in adition that, either*

$$\frac{f(x,s)}{s} \quad \textit{is decreasing in } s;$$

or

$$\frac{f(x,s)}{s} \quad \textit{is increasing in } s$$

strictly in a set of positive measure. Then, there exists a unique positive equilibrium of (1.1).

Proof. Let φ be the maximal positive solution of (1.1) and $\psi \leq \varphi$ any other solution. Then,

$$-\Delta\varphi = f(x,\varphi) \qquad -\Delta\psi = f(x,\psi).$$

Multiplying the first equation by ψ, the second one by φ, substracting and integrating in Ω, we have

$$0 = \int_\Omega \frac{f(x,\varphi)}{\varphi}\varphi\psi - \int_\Omega \frac{f(x,\psi)}{\psi}\varphi\psi = \int_\Omega \left(\frac{f(x,\varphi)}{\varphi} - \frac{f(x,\psi)}{\psi}\right)\varphi\psi.$$

Now, since $\psi \leq \varphi$ using the assumption on $f(x,s)/s$ we have that

$$\frac{f(x,\varphi)}{\varphi} - \frac{f(x,\psi)}{\psi}$$

does not change sign and is non zero in a set of positive measure. Therefore, we must have $\psi \equiv 0$. □

Observe that the result above combined with COROLLARY 3.3 covers some cases not included in [4]. Also, in this case the unique positive equilbrium is globally asymptotically stable for non negative solutions of (1.1).

5. An example: logistic equations. Our aim in this section is to apply the previous results to a model equation: a logistic autonomous equation. That is, in (1.1), we take

$$f(x,s) = m(x)s - n(x)|s|^{\rho-1}s, \quad \rho > 1,$$

with

$$m \in L^p(\Omega) \quad \text{for some} \quad p > N/2$$

and

$$n(x) \geq 0 \quad \text{in} \quad \Omega \quad \text{is a continuous function.}$$

First, notice that

$$f(x,s)s = m(x)s^2 - n(x)|s|^{\rho+1}. \tag{5.1}$$

Suppose that there exists a decomposition of m in the form $m(x) = m_1(x) + m_2(x)$, $x \in \Omega$, with $m_2 \geq 0$. Then, at least formally, by Young inequality

$$f(x,s)s \leq m_1(x)s^2 + \beta \left[\frac{m_2(x)}{n^{1/\rho}(x)}\right]^{\rho'} |s| \tag{5.2}$$

for certain postive constant $\beta > 0$. Thus, f satisfies the structure condition (2.1) with

$$C(x) = m_1(x) \quad \text{and} \quad D(x) = \beta \left[\frac{m_2(x)}{n^{1/\rho}(x)}\right]^{\rho'}. \tag{5.3}$$

Hence, to apply Theorem 2.1 and Corollary 3.3 we need to choose $m_1 \in L^p(\Omega)$, for some $p > N/2$, such that the linear semigroup generated by $\Delta + m_1$ has exponential decay and m_2 such that $D(x)$ belongs to some $L^r(\Omega)$ with $r > N/2$.

In such a case note that the maximal equilibrium φ_M is non-negative and the minimal one φ_m is non-positive since $f(\cdot, 0) \equiv 0$. Moreover, if the maximal equilibrium is positive then the uniqueness follows from Theorem 4.1 since

$$\frac{f(x,s)}{s} = m(x) - n(x)|s|^{\rho-1}$$

is decreasing, strictly on a set of positive measure, since $n(x) \geq 0$ does not vanish identically. Also, note that since $m \in L^p(\Omega)$ then f does not satisfies the hypothesis of Brezis–Oswald in [4]. The same comments apply for the minimal equilibrium and non-negative solutions.

Therefore, it remain to check the appropiate conditions for C and D in (5.3). In this direction, notice that if $\lambda_1(\Delta + m) > 0$ then

$$f(x,s)s = m(x)s^2 - n(x)|s|^{\rho+1} \leq m(x)s^2.$$

Thus, we can take $C = m$ and $D = 0$ and get the existence of a unique equilibrium $\varphi_m = \varphi_M = 0$ which is globally asymptotically stable.

On the contrary, if $\lambda_1(\Delta - m) < 0$ then we can apply Theorem 3.1. In this case, as we will see below, we will distinguish the case in which $n(x) > 0$ in Ω or vanishes slowly in a small region from the case in which $n(x)$ vashines very fast or in a large set.

Hence, for the first case, we have

Proposition 5.1. *Suppose that either* $n(x) \geq \gamma > 0$ *in* $\overline{\Omega}$ *or* $1/n \in L^s(\Omega)$ *for* $s > \frac{N}{2(\rho-1)}$.

Then Theorem 2.1*,* Corollary 3.3 *and* Theorem 4.1 *apply. In particular, the unique positive equilbrium is globally asymptotically stable for non-negative solutions of* (1.1).

Proof. In this case we take $m_1(x) = m(x) - \lambda$ and $m_2(x) = \lambda$, with λ large enough so that the semigroup generated by $\Delta + m_1$ has exponential decay. Then, D in (5.3) is bounded above by

$$\beta \left[\frac{m_2(x)}{n^{1/\rho}(x)}\right]^{\rho'} \leq C(\beta, \lambda) \frac{1}{n(x)^{\rho'/\rho}}.$$

Thus, if $n(x) \geq \gamma > 0$ in $\overline{\Omega}$ then $D \in L^\infty(\Omega)$ while $D \in L^r(\Omega)$ for some $r > N/2$, in the other case. □

We now consider the case in which $n(x)$ vanishes in a general subset of Ω with no size or regularity assumptions on it. Then, we have

Proposition 5.2. *Let* $\Omega_0 = \{x \in \Omega : \ n(x) = 0\}$ *and* Ω_δ *a neighbourhood of* Ω_0 *such that* $n(x) \geq \delta > 0$ *for all* $x \in \Omega \setminus \overline{\Omega}_\delta$. *Suppose that the first eigenvalue of* $-\Delta - m$ *in* Ω_δ*, with Dirichlet boundary conditions,* $\lambda_1^{\Omega_\delta}(-\Delta - m)$*, is positive.*

Also assume that $m \in L^p(\Omega)$ *with* $p > N/2$ *is such that there exists a decomposition of the positive part of* m *of the form*

$$m^+(x) = m_0^+(x) + m_1^+(x), \quad x \in \Omega \setminus \Omega_\delta$$

with $0 \leq m_0^+ \in L^p(\Omega \setminus \Omega_\delta)$ *small enough and* $0 \leq m_1^+ \in L^s(\Omega \setminus \Omega_\delta)$ *with* $s > \rho' N/2$.

Then Theorems 2.1*,* Corollary 3.3 *and* Theorem 4.1 *apply. In particular, the unique positive equilbrium is globally asymptotically stable for non negative solutions of* (1.1).

Proof. By (5.1), if $x \in \Omega_\delta$ then

$$f(x,s) \leq m(x) s^2.$$

We take $C(x) = m(x)$ and $D(x) = 0$ if $x \in \Omega_\delta$.

On the other hand, if $x \in \Omega \setminus \Omega_\delta$ then, for A large enough, we write

$$m(x) = \big(m_0^+(x) - m^-(x) - A\big) + \big(m_1^+(x) + A\big) = m_1(x) + m_2(x), \quad x \in \Omega \setminus \Omega_\delta.$$

Then, we set $C(x) = m_1(x)$ for $x \in \Omega \setminus \Omega_\delta$.

Now, it can be shown that if m_0^+ is small then we can choose A large enough so that the linear semigroup generated by $\Delta + C(x)$ with Dirichlet boundary conditions has exponential decay.

Arguing and in (5.2) we have, for $x \in \Omega \setminus \Omega_\delta$,

$$0 \leq D(x) = \beta \left[\frac{m_2(x)}{n^{1/\rho}(x)}\right]^{\rho'} \leq \beta \gamma^{-\rho'/\rho} m_2^{\rho'}(x) \in L^r(\Omega \setminus \Omega_\delta), \quad r > N/2$$

where we have used that $m_1^+ \in L^p$, $p > N/2$. Therefore, $D \in L^r(\Omega)$, with $r > N/2$. □

Notice that results in this section allows to obtain some of the results in [6] with less regularity requirements.

6. Another model example. In this section we apply the techniques above to the model problem

$$\begin{cases} u_t - \Delta u = f(x,u) & \text{in } \Omega \\ u(0) = u_0 > 0 & \\ u = 0 & \text{on } \partial\Omega = \Gamma \end{cases} \tag{6.1}$$

with

$$f(x,s) = a(x)s^\rho - b(x)s$$

and $0 < \rho < 1$. Observe that only positive solutions are considered.

Assume $a(x) = a_1(x) - a_2(x)$ with $a_i(x) \geq 0$. Then we have

$$f(x,s) \leq a_1(x)u^q - b(x)u.$$

Hence, if $a_1(x) = 0$ then we take $C(x) = -b(x)$ and $D(x) = 0$.

On the other hand, if $a_1(x) > 0$, proceeding as above, Young's inequality yields for $\varepsilon(x) > 0$,

$$f(x,s) \leq \big(\varepsilon(x) - b(x)\big)s + \beta \left[\frac{a_1(x)}{\varepsilon^\rho(x)}\right]^{\frac{1}{1-\rho}} \tag{6.2}$$

for some constant $\beta > 0$. In summary

$$C(x) = \varepsilon(x)\mathcal{X}_{\{a_1>0\}} - b(x), \qquad D(x) = \beta \left[\frac{a_1(x)}{\varepsilon^\rho(x)}\right]^{\frac{1}{1-\rho}} \mathcal{X}_{\{a_1>0\}}.$$

Assume then that $b \in L^p(\Omega)$ with $p > N/2$ is such that the semigroup generated by $\Delta + b(x)I$ with Dirichlet boundary conditions, decays exponentially, see (2.4). In particular, from perturbation results, we know that if we chose $\varepsilon(x)$ with a small enough norm in $L^p(\Omega)$, then (2.4) is still satisfied for the $C(x)$ above.

In particular, we can always take $\varepsilon(x) = \varepsilon_0 > 0$, a sufficiently small constant and then we must have

$$a_1 \in L^s(\Omega), \quad \text{for} \quad s > \frac{N}{2(1-\rho)}$$

to have $D \in L^r(\Omega)$ for $r > \frac{N}{2}$. In such a case, we get THEOREM 2.1.

In particular, if $a \in L^\infty(\Omega)$ we can take a_1 a large constant and then the above conditions are satisfied. If a is not bounded, it is enough to assume that the positive part satisfies

$$a_1 = a^+ \in L^s(\Omega), \quad \text{for} \quad s > \frac{N}{2(1-\rho)}$$

to have $D \in L^r(\Omega)$ for $r > \frac{N}{2}$.

If moreover $a(x) \geq 0$, not identically zero, we also have THEOREM 4.1, since

$$\frac{f(x,s)}{s} = \frac{a(x)}{s^{1-\rho}} - b(x)$$

is decreasing in $s > 0$, striclty in a set of positive measure.

7. Non-linear boundary conditions. THEOREM 2.1 can be extended along the same lines above for parabolic problems with nonlinear boundary conditions of the form

$$\begin{cases} u_t - \Delta u = f(x,u) & \text{in } \Omega \\ u(0) = u_0 & \\ \dfrac{\partial u}{\partial \vec{n}} = g(x,u) & \text{on } \partial\Omega = \Gamma. \end{cases} \tag{7.1}$$

Indeed in this (2.1) reads now

$$\begin{aligned} uf(x,u) &\le -C_0(x)u^2 + C_1(x)|u| \\ ug(x,u) &\le -B_0(x)u^2 + B_1(x)|u| \end{aligned} \tag{7.2}$$

with $C_0 \in L^p(\Omega)$, $p > N/2$, $C_1 \in L^r(\Omega)$, $r > N/2$, $B_0 \in L^\sigma(\Gamma)$, $\sigma > N-1$, $B_1 \in L^\rho(\Gamma)$, $\rho > N-1$.

On the other hand, (2.3) must be replaced by

$$\begin{cases} v_t - \Delta v + C_0(x)v = C_1(x) & \text{in } \Omega \\ \dfrac{\partial v}{\partial \vec{n}} + B_0(x)v = B_1(x) & \text{on } \Gamma. \end{cases} \tag{7.3}$$

Finally (2.4) must be replaced by the assumption of the positivity of the first eigenvalue, λ_1, of the eigenvalue problem

$$\begin{cases} -\Delta u + C_0(x)u = \lambda u & \text{in } \Omega \\ \dfrac{\partial u}{\partial \vec{n}} + B_0(x)u = 0 & \text{on } \Gamma \end{cases} \tag{7.4}$$

Finally, note that the techniques above allow to recover in a unified way several results contained in [1, 3, 4, 5, 7, 8]; see [9, 10].

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