Proceedings of ALGORITMY 2009
pp. 71–80

CHECKERBOARD MODES AND WAVE EQUATION

STÉPHANE DELLACHERIE*

Abstract. Checkerboard modes are unphysical oscillations that sometime appear when the incompressible Navier-Stokes system is solved with a colocated scheme. In this paper, we study the rate of dissipation of these modes when the pressure and the velocity are solution of the linear wave equation solved with a Godunov scheme on a cartesian mesh. More precisely, we show that the checkerboard modes are the fastest diffused modes when we use the Godunov scheme in monodimensional geometry and that they are constant modes when the Godunov scheme is modified by centering the discretization of the pressure gradient. This study underlines that, on a cartesian mesh, the checkerboard modes do not exist at low Mach number when the compressible Navier-Stokes system is solved with a Godunov type scheme and may appear at large Reynolds number when the Godunov type scheme is modified to obtain an accurate scheme at low Mach number.

Key words. Checkerboard mode, linear wave equation, low Mach number flow, Godunov scheme.

AMS subject classifications. 35L05, 35Q30, 65M06.

1. Introduction.

Many numerical experiments show that Godunov type schemes applied to the numerical resolution of the compressible Euler or Navier-Stokes system are not accurate at low Mach number [12, 13, 14]. Some recent results show also that this inaccuracy is more important on a quadrangular mesh than on a triangular mesh [17]. Moreover, other colocated schemes suffer from this loss of accuracy at low Mach number [19]. In [6], we have shown that the inaccuracy of Godunov type schemes applied to the compressible Euler system at low Mach number can be explained by studying the equivalent equation solved on $[0, +\infty[\times\Omega$ (where $\Omega \subset \mathbb{R}^d$, $d \in \{1, 2, 3\}$)

$$\begin{cases} \partial_t q + \dfrac{L}{M} q = B_\kappa q, \\ q(t = 0, x) = q^0(x) \end{cases} \tag{1.1}$$

associated to the Godunov scheme applied to the linear wave equation $\partial_t q + \dfrac{L}{M} q = 0$ on a cartesian mesh. In (1.1), $q := \begin{pmatrix} r \\ u \end{pmatrix}$, $\dfrac{L}{M} q = \dfrac{a}{M} \begin{pmatrix} \nabla \cdot u \\ \nabla r \end{pmatrix}$. The constant a is a strictly positive constant of order 1 and $M \ll 1$ is the Mach number (a/M is the sound velocity). The unknows r and u are respectively the pressure perturbation and the velocity of the fluid. Moreover, the term $B_\kappa q$ is linked to the numerical diffusion. In 3D, it reads

$$B_\kappa q = \mathcal{K} \begin{pmatrix} \Delta r \\ \dfrac{\partial^2 u_1}{\partial x_1^2} \\ \dfrac{\partial^2 u_2}{\partial x_2^2} \\ \dfrac{\partial^2 u_3}{\partial x_3^2} \end{pmatrix} \qquad \text{with} \qquad \mathcal{K} = \begin{pmatrix} \nu_r & 0 & 0 & 0 \\ 0 & \nu_{u_1} & 0 & 0 \\ 0 & 0 & \nu_{u_2} & 0 \\ 0 & 0 & 0 & \nu_{u_3} \end{pmatrix} \tag{1.2}$$

*Commissariat à l'Énergie Atomique, DM2S/SFME, 91191 Gif sur Yvette, France (stephane.dellacherie@cea.fr)

where $\nu_r = \nu_{u_k} = \nu_{num} := a\frac{\Delta x}{2M}$ (ν_{num} is the numerical viscosity, Δx is the space step supposed to be identical in each direction and Δr is the laplacian operator applied to r). More precisely, we have linked the inaccuracy of Godunov scheme in the case of the linear wave equation to the following invariance property verified by the spaces

$$\begin{cases} \mathcal{E} = \left\{ q := \begin{pmatrix} r \\ u \end{pmatrix} \in (L^2(\mathbb{T}^d))^{1+d} \text{ such that } \nabla r = 0 \text{ and } \nabla \cdot u = 0 \right\}, \\ \mathcal{E}^\perp = \left\{ q := \begin{pmatrix} r \\ u \end{pmatrix} \in (L^2(\mathbb{T}^d))^{1+d} \text{ such that } \int_{\mathbb{T}^d} r dx = 0 \text{ and } \exists \phi \in H^1(\mathbb{T}^d), u = \nabla \phi \right\} \end{cases} \tag{1.3}$$

where the Hilbert space $(L^2(\mathbb{T}^d))^{1+d} := \left\{ q := \begin{pmatrix} r \\ u \end{pmatrix} \text{ such that } \int_{\mathbb{T}^d} r^2 dx + \int_{\mathbb{T}^d} ||u||^2 dx < +\infty \right\}$ is equipped with the classical inner product $\langle q_1, q_2 \rangle = \int_{\mathbb{T}^d} q_1 q_2 dx$:

LEMMA 1.1. *We have:*
1) when $\Omega = \mathbb{T}^{d=1}$: $\forall \mathcal{K} \geq 0$, the spaces $\mathcal{E}$ and $\mathcal{E}^\perp$ are invariant for the equation (1.1);
2) when $\Omega = \mathbb{T}^{d \in \{2,3\}}$: $\forall \mathcal{K} \geq 0$, the spaces $\mathcal{E}$ and $\mathcal{E}^\perp$ are invariant for the equation (1.1) if and only if $\nu_u = 0$.

In (1.3) and in the lemma 1.1, $\mathbb{T}^d$ is the torus $\mathbb{T}^d := [a_1, b_1] \times \ldots \times [a_d, b_d]$ in $\mathbb{R}^d$ (in other words, we apply periodic boundary conditions on $\partial\Omega$) and $\nu_u := (\nu_{u_k})_{k=1,\ldots,d}$. Let us note that the spaces $\mathcal{E}$ and $\mathcal{E}^\perp$ verify [3, 15]:

LEMMA 1.2.

$$\mathcal{E} \oplus \mathcal{E}^\perp = (L^2(\mathbb{T}^d))^{1+d} \qquad \textit{and} \qquad \mathcal{E} \perp \mathcal{E}^\perp.$$

In other words, any $q \in (L^2(\mathbb{T}^d))^{1+d}$ can be decomposed with

$$q = \widehat{q} + q^\perp \qquad \textit{where} \qquad (\widehat{q}, q^\perp) \in \mathcal{E} \times \mathcal{E}^\perp \tag{1.4}$$

and this decomposition is unique. Thus, we can define the projection $\mathbb{P}$ – named Hodge projection – with $\widehat{q} := \mathbb{P}(q)$.

The Hodge decomposition (1.4) allows to define the energies

$$\begin{cases} E & := & ||q||^2 & = & \text{total energy}, \\ E_{inc} & := & ||\widehat{q}||^2 & = & \text{incompressible energy}, \\ E_{ac} & := & ||q^\perp||^2 & = & \text{acoustic (or compressible) energy}. \end{cases} \tag{1.5}$$

Let us remark that $E = E_{inc} + E_{ac}$ since $\mathcal{E} \perp \mathcal{E}^\perp$. The lemma 1.1 shows that the 1D-case and the 2D(or 3D)-case are very different: this difference is due to the fact that, when $\nu_u \neq 0$, the velocity diffusive term in (1.2) is isotropic if and only if the space dimension is equal to one. The lemma 1.1 allows to write the following theorem:

THEOREM 1.3. *Let $q(t,x)$ be solution of (1.1) on $\Omega = \mathbb{T}^{d \in \{1,2,3\}}$. Then:*

$$||q^0 - \mathbb{P}(q^0)|| = \mathcal{O}(M) \quad \Longrightarrow \quad ||q - \mathbb{P}(q^0)||(t \geq 0) = \mathcal{O}(M) \tag{1.6}$$

if and only if one of the two following conditions are valid:
1) $\Omega = \mathbb{T}^{d=1}$ and $\mathcal{K} \geq 0$;
2) $\Omega = \mathbb{T}^{d \in \{2,3\}}$, $\mathcal{K} \geq 0$ and $\nu_u = 0$.

The theorem 1.3 means that if we modify the Godunov scheme applied to the linear wave equation by deleting the numerical diffusion on the velocity equation, this modified Godunov scheme should not create any spurious pressure waves of order $\mathcal{O}(M\Delta x)$. This condition is a necessary condition to obtain an accurate scheme at low Mach number [6, 12, 13, 14]. Then, we have proposed to extend the theorem 1.3 to the non-linear case and to any colocated scheme with the following conjecture [6]:

CONJECTURE 1.4. *Let X be a colocated scheme of Godunov type (X = Roe for example) or not (X = kinetic scheme for example [16]) applied to the compressible Euler system on any 2D (or 3D) mesh. We suppose that the scheme X is stable at low Mach number. Let us modify the scheme X in using the center differences:*
1) to discretize the momentum flux;
2) or to discretize only the pressure gradient in the momentum flux when it is possible (X = VFRoe [4, 11, 13, 14] or X = FDS [1, 5] for example).
Then, at low Mach number:
i) the modified X scheme remains stable;
ii) the modified X scheme does not create any spurious pressure wave of order $\mathcal{O}(M\Delta x)$ and, thus, is accurate at low Mach number.
This modified X scheme is named "low Mach X scheme".

In [6, 7], we have justified the conjecture 1.4 with numerical results. Nevertheless, we do not have studied the problem of checkerboard modes. These modes are not the spurious pressure waves mentioned in the conjecture 1.4. They are unphysical oscillations that sometimes appear when the incompressible Navier-Stokes system is solved with a colocated scheme [2, 9, 10]. We want to study the possible existence of checkerboard modes at low Mach number when the compressible Euler or Navier-Stokes system is solved with a X scheme of Godunov type or with a *low Mach X scheme*. Nevertheless, we again limit the analysis to the case of the linear wave equation $\partial_t q + \dfrac{L}{M} q = 0$ solved on a cartesian mesh with a Godunov scheme. We will consider the boundary condition (instead of periodic boundary conditions)

$$\begin{cases} \nabla r(t,x)\cdot n(x)|_{\partial\Omega} = 0, & \text{(a)} \\ \\ u(t,x)\cdot n(x)|_{\partial\Omega} = 0 & \text{(b)} \end{cases} \tag{1.7}$$

where $n(x)$ is the outer normal on the boundary $\partial\Omega$. Let us note that for regular solutions of $\partial_t q + \dfrac{L}{M} q = 0$, (1.7)(a) is a consequence of (1.7)(b).

2. Checkerboard modes on a cartesian mesh. We now clearly define the checkerboard modes. Let us underline that this notion has a sense only at the discrete level. This is not the case for the spurious pressure waves mentioned in the conjecture 1.4. This means that we have to define the spaces $\mathcal{E}$ and $\mathcal{E}^{\perp}$ at the discrete level, and that we cannot study the notion of checkerboard modes by studying the equation (1.1) at the continuous level.

2.1. Definitions and basic properties at the continuous level. As we solve the linear wave equation with the boundary condition (1.7), the definition (1.3) has to be replaced by

$$\mathcal{E} = \left\{ q := \begin{pmatrix} r \\ u \end{pmatrix} \in (L^2(\Omega))^{1+d} \text{ such that } \nabla r = 0, \nabla \cdot u = 0 \text{ and } u(x) \cdot n(x)|_{\partial\Omega} = 0 \right\} \tag{2.1}$$

and

$$\mathcal{E}^{\perp} = \left\{ q := \begin{pmatrix} r \\ u \end{pmatrix} \in (L^2(\Omega))^{1+d} \text{ such that } \int_{\Omega} r dx = 0 \text{ and } \exists \phi \in H^1(\Omega), u = \nabla\phi \right\}. \tag{2.2}$$

Let us note that $\mathcal{E} = KerL$. Of course, the Hodge decomposition – see lemma 1.2 – is valid when (1.3) is replaced by (2.1) and (2.2). At last, we recall the classical relation

$$\langle u, \nabla r \rangle = -\langle \nabla \cdot u, r \rangle + \int_{\partial\Omega} ru \cdot n d\sigma. \tag{2.3}$$

In the sequel, we will write the definitions (2.1)(2.2) and the relation (2.3) at the discrete level in the 1D-case. We will also obtain the discrete formulation of the lemma 1.1. Let us underline that when the mesh is cartesian, the study of the checkerboard modes in the 2D(or 3D)-case may be deduced from the study in the 1D-case.

2.2. Discretization of the spaces $\mathcal{E}$ and $\mathcal{E}^{\perp}$. The space $(L^2(\Omega))^{1+d}$ is replaced by $\mathbb{R}^{N,2}$ that is equipped with the inner product $\langle q_1, q_2 \rangle = \langle r_1, r_2 \rangle + \langle u_1, u_2 \rangle$ where $q := (r, u) \in \mathbb{R}^{N,2}$ and $\langle f_1, f_2 \rangle = \sum_{i=1}^{N} f_{1,i} f_{2,i}$ is the inner product in $\mathbb{R}^N$. The euclidian norm is noted $|| \cdot ||$ in $\mathbb{R}^{N,2}$ and in $\mathbb{R}^N$. The discrete version of (2.1) is given by

$$\mathcal{E} = \{ q := (r, u) \in \mathbb{R}^{N,2} \text{ such that } Dr = 0 \text{ and } D \cdot u = 0 \}. \tag{2.4}$$

The discrete divergence operator noted $D\cdot$ applied to $f = (f_i)_{i=1,\ldots,N} \in \mathbb{R}^N$ is given by

$$D \cdot f = \begin{pmatrix} \frac{f_2+f_1}{2\Delta x} \\ D_2 f \\ \vdots \\ D_{N-1} f \\ -\frac{f_N+f_{N-1}}{2\Delta x} \end{pmatrix} \tag{2.5}$$

where $D_k f = \frac{f_{k+1}-f_{k-1}}{2\Delta x}, k \in \{2, \ldots, N-1\}$. The boundary condition (1.7)(b) in (2.1) is taken into account in (2.5) through the boundary terms $\frac{f_2+f_1}{2\Delta x}$ and $-\frac{f_N+f_{N-1}}{2\Delta x}$. The discrete gradient operator D is defined with

$$Df = \begin{pmatrix} \frac{f_2-f_1}{2\Delta x} \\ D_2 f \\ \vdots \\ D_{N-1} f \\ \frac{f_N-f_{N-1}}{2\Delta x} \end{pmatrix}. \tag{2.6}$$

The boundary terms $\frac{f_2-f_1}{2\Delta x}$ and $\frac{f_N-f_{N-1}}{2\Delta x}$ in (2.6) are chosen so that the following relation be satisfied [8]

$$\forall (r, u) \in \mathbb{R}^{N,2} : \quad \langle u, Dr \rangle = -\langle D \cdot u, r \rangle. \tag{2.7}$$

The relation (2.7) is the discrete version of (2.3) with the boundary condition (1.7)(b). Let us note that the definition (2.4) is equivalent to the definition

$$\begin{aligned}\mathcal{E} &= \{q := (r,u) \in \mathbb{R}^{N,2} \text{ such that } r_i = C_1^{ste} \text{ and } u_{2i} = -u_{2i+1} = C_2^{ste}\} \\ &= Vect\{\mathbf{1}\} \times Vect\{\mathbf{e}\}\end{aligned}$$

where $\mathbf{1} = (1,1,1,1,\dots)^T$ and $\mathbf{e} = (-1,1,-1,1,\dots)^T$. Thus, the orthogonal of $\mathcal{E}$ in $\mathbb{R}^{N,2}$ is given by

$$\mathcal{E}^{\perp} = \{q := (r,u) \in \mathbb{R}^{N,2} \text{ such that } \sum_i r_i = 0 \text{ and } \sum_{2i} u_{2i} = \sum_{2i+1} u_{2i+1}\}. \tag{2.8}$$

The space $\mathcal{E}^{\perp}$ defined with (2.8) is the discrete version of (2.2). Indeed, we have [8]:

Lemma 2.1. *The definition (2.8) is equivalent to the definition*

$$\mathcal{E}^{\perp} = \{q := (p,u) \in \mathbb{R}^{N,2} \textit{ such that } \sum_i p_i = 0 \textit{ and } \exists \phi \in \mathbb{R}^N, u = D\phi\} \tag{2.9}$$

where D is the discrete gradient operator defined with (2.6).

By using the lemma 2.1, we easily obtain the discrete version of the Hodge decomposition $q = \widehat{q} + q^{\perp}$ and of the Hodge projection $\mathbb{P}$ (see lemma 1.2). This allows to define also the incompressible and acoustic discrete energies with (1.5).

2.3. Definition of the checkerboard modes. We now define

$$\left\{\begin{aligned}\mathcal{E}_r &= \{r \in \mathbb{R}^N \text{ such that } Dr = 0\} \\ &= \{r \in \mathbb{R}^N \text{ such that } r_i = C^{ste}\} \\ &= Vect\{\mathbf{1}\}, \\ \\ \mathcal{E}_r^{\perp} &= \{r \in \mathbb{R}^N \text{ such that } \sum_i r_i = 0\}\end{aligned}\right. \tag{2.10}$$

and

$$\left\{\begin{aligned}\mathcal{E}_u &= \{u \in \mathbb{R}^N \text{ such that } D \cdot u = 0\} \\ &= \{u \in \mathbb{R}^N \text{ such that } u_{2i} = -u_{2i+1} = C^{ste}\} \\ &= Vect\{\mathbf{e}\}, \\ \\ \mathcal{E}_u^{\perp} &= \{u \in \mathbb{R}^N \text{ such that } \sum_{2i} u_{2i} = \sum_{2i+1} u_{2i+1}\}.\end{aligned}\right. \tag{2.11}$$

Of course, we have $\mathcal{E} = \mathcal{E}_r \times \mathcal{E}_u$ and $\mathcal{E}^{\perp} = \mathcal{E}_r^{\perp} \times \mathcal{E}_u^{\perp}$. We propose the following definition:

Definition 2.2. *The space of checkerboard modes is defined with*

$$\mathcal{E}_{checkerboard} := \{0\} \times \mathcal{E}_u.$$

Moreover, we define the space of constant modes with

$$\mathcal{E}_{constant} := \mathcal{E}_r \times \{0\}.$$

Then, $\mathcal{E}_{constant} \oplus \mathcal{E}_{checkerboard} = \mathcal{E}$.

Let us note that, due to the discrete Hodge decomposition, any $q \in \mathbb{R}^{N,2}$ may be written with $q = (\widehat{r}, 0) + (0, \widehat{u}) + q^{\perp}$ where $(\widehat{r}, 0)$ is a constant mode, $(0, \widehat{u})$ is a checkerboard mode and $q^{\perp}$ is an acoustic mode. Let us also underline that r and u do not play symmetric roles because of the boundary condition $u(x) \cdot n(x)|_{\partial\Omega}$ (more precisely, $D\cdot \neq D$). If the boundary condition was periodic, the variables r and u would have played symmetric roles.

3. Time behavior of the checkerboard modes. We now study the time behavior of the checkerboard modes $q := (r, u) \in \mathcal{E}_{checkerboard} := \{0\} \times \mathcal{E}_u$ when the wave equation is discretized with the 1D colocated scheme

$$\begin{cases} \dfrac{d}{dt} r_i + \dfrac{a}{M} \dfrac{u_{i+1} - u_{i-1}}{2\Delta x} = \nu_r \cdot \dfrac{r_{i+1} - 2r_i + r_{i-1}}{\Delta x^2}, \\ \\ \dfrac{d}{dt} u_i + \dfrac{a}{M} \dfrac{r_{i+1} - r_{i-1}}{2\Delta x} = \nu_u \cdot \dfrac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} \end{cases} \tag{3.1}$$

where $i \in \{1, \ldots, N\}$ is the space subscript and Δx the space step. The boundary condition (1.7) is discretized with

$$\begin{cases} r_0 = r_1 \quad \text{and} \quad r_{N+1} = r_N, & \text{(a)} \\ u_0 = -u_1 \quad \text{and} \quad u_{N+1} = -u_N. & \text{(b)} \end{cases} \tag{3.2}$$

The Godunov scheme is obtained when $(\nu_r, \nu_u) = \nu_{num}(1, 1)$ where $\nu_{num} := \frac{a\Delta x}{2M}$. The *low Mach Godunov scheme* – deduced from the conjecture 1.4 – is obtained when $(\nu_r, \nu_u) = \nu_{num}(1, 0)$. The scheme (3.1) with the initial condition $q^0 := (r^0, u^0)$ can be rewritten as

$$\begin{cases} \dfrac{d}{dt} q + \dfrac{\mathbb{L}_\nu}{M} q = 0, \\ \\ q(t = 0) = q^0 \end{cases} \tag{3.3}$$

where $\nu := (\nu_r, \nu_u)$, $\mathbb{L}_\nu := \mathbb{L} - \mathbb{B}_\nu$ and

$$\begin{cases} \mathbb{L} q = a(D \cdot u, Dp), & \text{(a)} \\ \\ \mathbb{B}_\nu q = \dfrac{M}{\Delta x^2} \left(\nu_r \mathcal{B}_r r, \nu_u \mathcal{B}_u u \right). & \text{(b)} \end{cases} \tag{3.4}$$

In (3.4)(a), the discrete operators $D\cdot$ and D are defined with (2.5) and (2.6). The discrete operators $\mathcal{B}_r$ and $\mathcal{B}_u$ are the classical diffusion matrices in $\mathbb{R}^{N,N}$ that take into account the boundary condition (3.2). They are defined as

$$\mathcal{B}_r = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & & & & \\ 1 & -2 & 1 & 0 & 0 & & & & \\ 0 & 1 & -2 & 1 & 0 & & & & \\ & & \cdot & \cdot & \cdot & & & & \\ & & & \cdot & \cdot & \cdot & & & \\ & & & & \cdot & \cdot & \cdot & & \\ & & & & 0 & 1 & -2 & 1 & 0 \\ & & & & 0 & 0 & 1 & -2 & 1 \\ & & & & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \text{ and } \mathcal{B}_u = \begin{pmatrix} -3 & 1 & 0 & 0 & 0 & & & & \\ 1 & -2 & 1 & 0 & 0 & & & & \\ 0 & 1 & -2 & 1 & 0 & & & & \\ & & \cdot & \cdot & \cdot & & & & \\ & & & \cdot & \cdot & \cdot & & & \\ & & & & \cdot & \cdot & \cdot & & \\ & & & & 0 & 1 & -2 & 1 & 0 \\ & & & & 0 & 0 & 1 & -2 & 1 \\ & & & & 0 & 0 & 0 & 1 & -3 \end{pmatrix}.$$

3.1. Basic properties. We have the following classical property:

LEMMA 3.1. *The discrete operators* $\mathbb{L}$, $\mathcal{B}_r$, $\mathcal{B}_u$ *and* $\mathbb{B}_\nu$ *verify:*
1) the operator $\mathbb{L}$ *is antisymmetric;*
2) the symmetric operators $\mathcal{B}_r$ *and* $\mathcal{B}_u$ *verify for any* $(\widetilde{f}, f) \in \mathbb{R}^{N,2}$*:*

$$\left\{\begin{array}{lcll} \langle \widetilde{f}, \mathcal{B}_r f\rangle & = & -\displaystyle\sum_{i=1}^{N-1}(\widetilde{f}_{i+1}-\widetilde{f}_i)(f_{i+1}-f_i), & (a) \\ \langle \widetilde{f}, \mathcal{B}_u f\rangle & = & -\displaystyle\sum_{i=1}^{N-1}(\widetilde{f}_{i+1}-\widetilde{f}_i)(f_{i+1}-f_i) - 2(\widetilde{f}_N f_N + \widetilde{f}_1 f_1). & (b) \end{array}\right. \tag{3.5}$$

Thus, $Ker\mathcal{B}_r = \mathcal{E}_r$ *and* $Ker\mathcal{B}_u = \{0\}$*;*
3) the operator $\mathbb{B}_\nu$ *is symmetric negative-semidefinite when* $\nu \in \mathbb{R}^+ \times \mathbb{R}^+$ *and verifies for any* $q := (r,u) \in \mathbb{R}^{N,2}$*:*

$$\langle q, \mathbb{B}_\nu q\rangle = -\frac{M}{\Delta x^2}\left\{\nu_r \sum_{i=1}^{N-1}(r_{i+1}-r_i)^2 + \nu_u\left[\sum_{i=1}^{N-1}(u_{i+1}-u_i)^2 + 2(u_N^2+u_1^2)\right]\right\}. \tag{3.6}$$

The first point of lemma 3.1 is a direct consequence of (2.7). Let us note that $Ker\mathbb{L} = \mathcal{E}$ as in the continuous case. Nevertheless, $Ker\mathbb{L}_\nu \subseteq \mathcal{E}$.

3.2. Time behavior. We deduce from the lemma 3.1:

LEMMA 3.2. *Let* $q(t) := (r,u)(t)$ *be solution of (3.3). Then:*

$$\forall(\widetilde{q}, q^0) \in \mathcal{E}\times\mathbb{R}^{N,2}, \forall\nu\in\mathbb{R}^2 : \quad \langle \widetilde{q}, q\rangle(t) = \langle \widetilde{q}, q^0\rangle + \langle \widetilde{u}, u^0\rangle\left[\exp\left(-\frac{4\nu_u t}{\Delta x^2}\right) - 1\right] \tag{3.7}$$

where $\widetilde{q} := (\widetilde{r}, \widetilde{u})$*.*

By taking $q^0 = (\mathbf{1}, 0)$, $q^0 = (0, \mathbf{e})$ and $q^0 \in \mathcal{E}^\perp$, we deduce from this lemma:

COROLLARY 3.3. *For any* $\nu \in \mathbb{R}^2$*, the spaces* $\mathcal{E}_{constant}$*,* $\mathcal{E}_{checkerboard}$ *and* $\mathcal{E}^\perp$ *are invariant spaces for the equation (3.3).*

We now precise the result obtained in lemma 3.2:

THEOREM 3.4. *Let* $q(t)$ *be solution of (3.3). Then:*
1) the checkerboard mode energy is equal to

$$||\widehat{u}||^2(t=0)\exp\left(-\frac{8\nu_u t}{\Delta x^2}\right);$$

2) when $(\nu_r, \nu_u) \in \mathbb{R}^+\times\mathbb{R}^+$*, the acoustic energy is a decreasing function that is always greater than*

$$E_{ac}(t=0)\exp\left(-\frac{8\max(\nu_u,\nu_r)t}{\Delta x^2}\right);$$

3) when $\nu_u \geq \nu_r > 0$*, the checkerboard mode is the fastest diffused mode.*

This result shows that numerical diffusion prevents from detecting any checkerboard mode at low Mach number when the compressible Euler or Navier-Stokes systems is solved with a Godunov type scheme. However, it shows also that by using a *low Mach X scheme* (*cf.* conjecture 1.4), it could be possible to detect a chekerboard mode at large Reynolds number – *i.e.* when the physical diffusion is not large – although the scheme remains diffusive (it only diffuses the acoustic mode in that case).

Proof of lemma 3.2: For any $\widetilde{q} \in \mathbb{R}^{N,2}$, we have $M\frac{d}{dt}\langle \widetilde{q}, q\rangle = -\langle \widetilde{q}, \mathbb{L}q\rangle + \langle \widetilde{q}, \mathbb{B}_\nu q\rangle$. Then

$$M\frac{d}{dt}\langle \widetilde{q}, q\rangle = \langle \mathbb{L}\widetilde{q}, q\rangle + \frac{M}{\Delta x^2}\left(\nu_r\langle \widetilde{r}, \mathcal{B}_r r\rangle + \nu_u\langle \widetilde{u}, \mathcal{B}_u u\rangle\right).$$

On the other hand, we have

$$\forall \widetilde{q} \in \mathcal{E} : \qquad \begin{cases} \mathbb{L}\widetilde{q} = 0 & \text{(since } Ker\mathbb{L} = \mathcal{E}\text{)}, \\ \\ \langle \widetilde{r}, \mathcal{B}_r r\rangle = 0 & \text{(since } Ker\mathcal{B}_r = \mathcal{E}_r\text{)}. \end{cases}$$

Then

$$\forall \widetilde{q} \in \mathcal{E} : \qquad \frac{d}{dt}\langle \widetilde{q}, q\rangle = \frac{\nu_u}{\Delta x^2}\langle \widetilde{u}, \mathcal{B}_u u\rangle.$$

Moreover, we deduce from (3.5)(b) that $\langle \mathbf{e}, \mathcal{B}_u u\rangle = -4\langle \mathbf{e}, u\rangle$. Thus, $\forall \widetilde{u} \in \mathcal{E}_u$: $\langle \widetilde{u}, \mathcal{B}_u u\rangle = -4\langle \widetilde{u}, u\rangle$. This implies that $\forall \widetilde{q} \in \mathcal{E}$: $\frac{d}{dt}\langle \widetilde{q}, q\rangle = -\frac{4\nu_u}{\Delta x^2}\langle \widetilde{u}, u\rangle$. We conclude the proof by taking $\widetilde{q} \in \mathcal{E}_r \times \{0\}$ and $\widetilde{q} \in \{0\} \times \mathcal{E}_u$.□

Proof of theorem 3.4: Let us define $E_{inc}^{checkerboard}(t) := ||\widehat{u}(t)||^2$ (checkerboard mode energy). Taking $\widetilde{q} = (0, \mathbf{e})$ and noting that the dimension of $\mathcal{E}_{checkerboard}$ is equal to one, we deduce from lemma 3.2 that $E_{inc}^{checkerboard}(t) = ||\widehat{u}^0||^2 \exp\left(-\frac{8\nu_u t}{\Delta x^2}\right)$. On the other hand, since $\frac{d}{dt}E(t) = 2\langle q, \frac{d}{dt}q\rangle$, we obtain, by using the lemma 3.1, that

$$\frac{d}{dt}E(t) = -\frac{2}{\Delta x^2}\left\{\nu_r\sum_{i=1}^{N-1}(r_{i+1} - r_i)^2 + \nu_u\left[\sum_{i=1}^{N-1}(u_{i+1} - u_i)^2 + 2(u_N^2 + u_1^2)\right]\right\}. \tag{3.8}$$

Thus, the inequality $\frac{d}{dt}E(t) \leq 0$ is satisfied by the acoustic energy $E_{ac}(t)$ since $\mathcal{E}^\perp$ is invariant (*cf.* corollary 3.3). Moreover, the relation (3.8) can be rewritten as

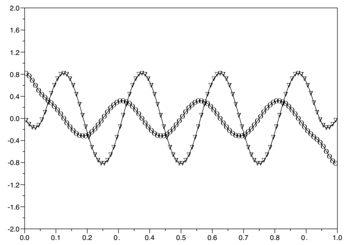
$$\frac{d}{dt}E(t) + \frac{8}{\Delta x^2}\left(\nu_r\sum_{i=1}^{N} r_i^2 + \nu_u\sum_{i=1}^{N} u_i^2\right) = \lambda$$

where $\lambda = \dfrac{2}{\Delta x^2}\left\{\nu_r\left[2(r_1^2 + r_N^2) + \sum_{i=1}^{N-1}(r_{i+1} + r_i)^2\right] + \nu_u\sum_{i=1}^{N-1}(u_{i+1} + u_i)^2\right\}$. Thus $\frac{d}{dt}E(t) + \frac{8}{\Delta x^2}\max(\nu_r, \nu_u)\left(\sum_{i=1}^{N} r_i^2 + \sum_{i=1}^{N} u_i^2\right) \geq \lambda$ that implies that

$$\frac{d}{dt}E(t) + \frac{8}{\Delta x^2}\max(\nu_r, \nu_u)E(t) \geq 0 \tag{3.9}$$

since $\lambda \geq 0$. The inequality (3.9) is valid for the acoustic energy $E_{ac}(t)$ since $\mathcal{E}^\perp$ is invariant (*cf.* corollary 3.3). Thus, by using the Grönwall's lemma, we obtain that $E_{ac}(t) \geq E_{ac}(t = 0)\exp\left(-\frac{8\max(\nu_u, \nu_r)t}{\Delta x^2}\right)$. Statement *3)* is a direct consequence of *1)* and *2)*.□

4. Numerical results. We choose the initial conditions $r^0 = (sin(8\pi i\Delta x))_{i=1,\dots,N}$ and $u^0 = \frac{\mathbf{e}}{2}$ with $\Omega = [0,1]$, $\Delta x = 1/N$ and $N = 100$ (thus, $(\widehat{r}^0, \widehat{u}^0) = (0, u^0)$ and $({r^\perp}^0, {u^\perp}^0) = (r^0, 0)$). The sound velocity a/M is equal to 1 and the time step is given by $\Delta t = 0,15 \times \Delta x$ (we use an explicit Euler scheme for the time discretization of (3.3)). The figures 1 and 2 show that the checkerboard mode $\frac{\mathbf{e}}{2}$ is diffused when $(\nu_r, \nu_u) = \nu_{num}(1,1)$ and is a constant mode when $(\nu_r, \nu_u) = \nu_{num}(1,0)$. Let us note that for these two test-cases, $\widehat{r}^{n\geq 0} = 0$ and that ${u^\perp}^{n>0} \neq 0$ although ${u^\perp}^0 = 0$. On the figures 3 and 4, we show the incompressible energy $E_{inc}(n)$, the acoustic energy $E_{ac}(n)$ (normalized by the initial condition) and the function $\psi(n) = \exp(-\frac{8\max(\nu_u,\nu_r)n\Delta t}{\Delta x^2})$. The figures 3 and 4 confirm the theorem 3.4.



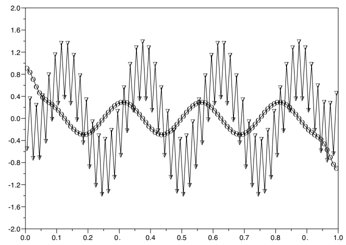
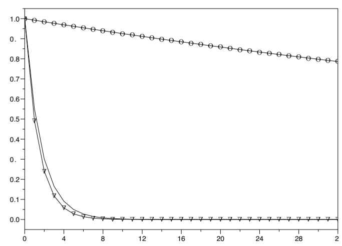


Fig. 1: $\nu_u = \nu_r = \nu_{num}$ Fig. 2: $\nu_u = 0$ and $\nu_r = \nu_{num}$

$u(n = 100, x)$ (∇) and $r(n = 100, x)$ ($\circ$)



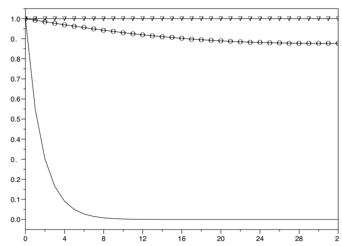


Fig. 3: $\nu_u = \nu_r = \nu_{num}$ Fig. 4: $\nu_u = 0$ and $\nu_r = \nu_{num}$

$E_{inc}(n)$ (∇), $E_{ac}(n)$ ($\circ$) and $\psi(n)$ ($-$) ($0 \leq n \leq 32$)

5. Conclusion. In [6, 7], we have proposed a class of colocated schemes that allow to solve the compressible Navier-Stokes system with accuracy at low Mach number. This class of colocated schemes is obtained by modifying Godunov type schemes in a simple way. The modification consists in centering the discretization of the pressure gradient in the velocity equation, the rest of the scheme remaining unchanged. Let us note that this modification may be applied to colocated schemes that are not of Godunov type. This method is justified by a theoretical study of the Godunov scheme applied to the linear wave equation and by numerical results in the non-linear case as well. Nevertheless, we have not studied in [6, 7] the possible existence of checkerboard modes at low Mach number although this question is classical in the field of colocated schemes solving the incompressible Navier-Stokes system. Therefore, we have studied in this paper the checkerboard modes in the case of the linear wave equation solved with a Godunov scheme on a cartesian mesh. We have shown that the checkerboard modes are the fastest modes diffused by the Godunov scheme and that these modes are constant modes when the Godunov scheme is modified by centering the discretization of the pressure gradient. This study underlines that, for any Reynolds number, it is impossible to detect any checkerboard mode at low Mach number when the compressible Navier-Stokes system is solved with a Godunov type scheme. Moreover, it shows that by using a *low Mach X scheme* at large Reynolds number on a cartesian mesh and at low Mach number, it could be possible to detect chekerboard modes although the scheme remains diffusive.

REFERENCES

[1] Beccantini A. – *Colella-Glaz splitting scheme for thermally perfect gases* – In *Godunov Methods: Theory and Application*, E.F. Toro (Ed.), Kluwer Academic/Plenum Publishers, New York, p. 89-95, 2001.

[2] Auteri F., Guermond J.L. and Parolini N. – *Role of the LBB Condition in Weak Spectral Projection Methods* – J. Comput. Phys., **174**, p. 405-420, 2001.

[3] Boyer F. and Fabrie P. – *Éléments d'analyse pour l'étude de quelques modèles d'écoulements de fluides visqueux incompressibles* – Mathématiques & Applications, Springer, 2006.

[4] Buffard T., Gallouët T. and Hérard J.M. – *A sequel to a rough Godunov scheme: application to real gases* – Computers and Fluids, **29**, p. 813-847, 2000.

[5] Colella P. and Glaz H. – *Efficient solution algorithms for the Riemann problem for real gases* – J. Comput. Phys., **59**, p. 264-289, 1985.

[6] Dellacherie S. – *Analysis of Godunov type schemes applied to the compressible Euler system at low Mach number* – Submitted. See also: Dellacherie S. – *Analyse à bas nombre de Mach du schéma de Roe appliqué à Euler compressible. PARTIE I : Construction d'une classe de schémas colocalisés précis à bas nombre de Mach* – CEA technical report, DM2S/SFME/LETR/RT/08-004/A, 2008.

[7] Dellacherie S. and Fillion P. – *On a low Mach Roe scheme applied to the compressible Navier-Stokes system* – Submitted. See also: Dellacherie S., Fillion P. and Mekkas A. – *Contribution à la validation du code de calcul FLICA-OVAP : simulations 2D et 3D d'écoulements compressibles monophasiques stationnaires à bas nombre de Mach* – CEA technical report, DM2S/SFME/LETR/RT/08-003/A, 2008.

[8] Dellacherie S. – *Analyse à bas nombre de Mach du schéma de Roe appliqué à Euler compressible. PARTIE II : Analyse monodimensionnelle des modes en échiquier* – CEA technical report, SFME/LETR/RT/08-005/A, 2008.

[9] Eymard R., Herbin R. and Latché J.C. – *On a stabilized colocated finite volume scheme for the Stokes problem* – Math. Model. and Num. Anal., **40**(3), p. 501-527, 2006.

[10] Faure S. – *Stability of a colocated finite volume scheme for the Navier-Stokes equations* – Num. Meth. for Partial Diff. Eq., **21**(2), p. 242-271, 2005.

[11] Faille I., Gallouët T. and Masella J.M. – *On an Approximate Godunov Scheme* – Int. J. of Comput. Fluid. Dyn., **12**, p. 133-149, 1999.

[12] Guillard H. and Viozat C. – *On the behavior of upwind schemes in the low Mach number limit* – Computers and Fluids, **28**, p. 63-86, 1999.

[13] Guillard H. and Murrone A. – *On the behavior of upwind schemes in the low Mach number limit: II. Godunov type schemes* – Computers and Fluids, **33**, p. 63-86, 2004.

[14] Guillard H. and Murrone A. – *Behavior of upwind scheme in the low Mach number limit: III. Preconditioned dissipation for a five equation two phase model* – Computers and Fluids, **37**(10), p. 1209-1224, 2008.

[15] Minion M.L. – *A projection method for locally refined grids* – J. Comput. Phys., **127**, p. 158-178, 1996.

[16] Perthame B. – *Boltzmann type schemes for gas dynamics and the entropy property* – SIAM J. Num. Anal., **27**, p. 1405-1421, 1990.

[17] Rieper F. – *Influence of cell geometry on the behaviour of the first-order Roe scheme in the low Mach number regime* – Proceedings of the 5th International Symposium on Finite Volumes for Complex Applications, Wiley, p. 625-632, 2008.

[18] Roe P.L. – *Approximate Riemann solvers, parameter vectors and difference schemes* – J. Comput. Phys., **43**, p. 357-372, 1981.

[19] Volpe G. – *Performance of compressible flow codes at low Mach number* – AIAA Journal, **31**(1), p. 49-56, 1993.