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OPTIMAL ERROR ESTIMATES IN THE DG METHOD FOR NONLINEAR CONVECTION-DIFFUSION PROBLEMS*

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Abstract. This paper is concerned with the analysis of the discontinuous Galerkin finite element method (DGFEM) applied to the space semidiscretization of a nonstationary convection-diffusion problem with nonlinear convection and nonlinear diffusion. Optimal estimates in the $L^\infty(L^2)$-norm are derived for the symmetric interior penalty (SIPG) scheme in two dimensions. The error analysis is carried out for nonconforming triangular meshes under the assumption that the exact solution of the problem and the solution of a linearised elliptic dual problem are sufficiently regular.

Key words. convection-diffusion equation, nonlinear diffusion, discontinuous Galerkin finite element method, optimal error estimates

AMS subject classifications. 65M15, 65M20, 65M60

1. Introduction. A natural generalization of the finite volume and finite element methods is the discontinuous Galerkin finite element method (DGFEM). This method uses advantages of FV as well as FE methods: it is based on piecewise polynomial but discontinuous approximations, where boundary fluxes are evaluated with the aid of a numerical flux. The use of discontinuous functions allows a precise capturing of discontinuities and steep gradients, while the use of higher degree polynomials ensures a higher order of approximation in regions, where the solution is smooth. Such properties are desirable in specific applications, among them the solution of compressible inviscid and viscous flows governed by the Navier-Stokes equations (cf. [2], [9]). This system of equations, when written in conservative form contain nonlinear convective as well as viscous (diffusive) terms. Therefore a theoretical analysis of the DGFEM applied to such problems is a very important topic.

In [3], [4] and [5] the order of convergence of the DGFEM is analyzed for a model scalar equation with linear diffusion. The presented paper generalizes results from [4], where a nonlinear convection-diffusion problem with linear diffusion is treated and results from [8], where $L^\infty(L^2)$-suboptimal error estimates are derived in the case of nonlinear diffusion. In this work we prove $L^\infty(L^2)$-optimal error estimates in the case of nonlinear convection as well as diffusion using the Aubin-Nitsche duality technique. Triangular elements are used and hanging nodes are allowed in the presented proof. The use of a linearised elliptic dual problem limits the result to Dirichlet problems on a convex domain. This work represents an overview of the paper [11], where all the proofs are carried out in detail.

2. Continuous problem. Let $\Omega \subset I\!R^2$ be a bounded open convex polygonal domain with Lipschitz-continuous boundary $\partial\Omega$ and $T > 0$. Let $Q_T := \Omega \times (0, T)$.

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We treat the following nonlinear problem:

$$\frac{\partial u}{\partial t}+\sum_{s=1}^{2}\frac{\partial f_s(u)}{\partial x_s}-\operatorname{div}\big(\beta(u)\nabla u\big)=g \quad \text{in } Q_T, \tag{2.1}$$

$$u|_{\partial\Omega\times(0,T)}=u_D, \tag{2.2}$$

$$u(x,0)=u^0(x), \quad x\in\Omega, \tag{2.3}$$

where $\beta\in C^2(I\!R)\cap W^{2,\infty}(I\!R)$ is bounded from below and Lipschitz continuous:

$$\beta: I\!R\to[\beta_0,\beta_1], \quad 0<\beta_0<\beta_1<\infty, \tag{2.4}$$

$$|\beta(u_1)-\beta(u_2)|\le L|u_1-u_2|, \quad \forall u_1,u_2\in I\!R. \tag{2.5}$$

Condition (2.5) implies $|\beta'|<L$. Let $g: Q_T\to I\!R$, $u_D:\partial\Omega\times(0,T)\to I\!R$ and $u^0:\Omega\to I\!R$ be given functions, and $f_1,f_2\in C^1(I\!R)$ be prescribed Lipschitz-continuous fluxes. Without loss of generality let $f_1(0)=f_2(0)=0$.

We shall use standard notation of function spaces. Let $G\subset I\!R^2$ be a bounded domain with a Lipschitz-continuous boundary ∂G. By $\overline{G}$ we denote the closure of G. Let $k\in\{0,1,2,\dots\}$ and $p\in[1,\infty]$. We use the well-known Lebesgue and Sobolev spaces $L^p(G)$, $L^p(\partial G)$, $W^{k,p}(G)$, $H^k(G)=W^{k,2}(G)$, $W^{k,p}(\partial G)$. By $H^1_0(G)$ we denote the space formed by all functions $v\in H^1(G)$ with zero traces on ∂G, i.e. $v|_{\partial G}=0$. Further, we use the Bochner spaces $L^p(0,T;X)$ of functions defined in $(0,T)$ with values in a Banach space X and the spaces $C^k([0,T];X)$ of k-times continuously differentiable mappings of the interval $[0,T]$ with values in X (see e.g. [12]). The symbols $\|\cdot\|_X$ and $|\cdot|_X$ will denote a norm and a seminorm in a space X, respectively. By $(\cdot,\cdot)$ we denote the standard $L^2(\Omega)-$scalar product.

3. Discretization. Let $\mathcal{T}_h$ be a partition of the closure $\overline{\Omega}$ of the domain Ω into a finite number of closed triangles with mutually disjoint interiors. We shall call $\mathcal{T}_h$ a triangulation of Ω. We do not require the standard conforming properties of $\mathcal{T}_h$ used in the finite element method. This means that we admit the so-called hanging nodes. We shall use the following notation. By ∂K we denote the boundary of an element $K\in\mathcal{T}_h$ and set $h_K=\operatorname{diam}(K)$, $h=\max_{K\in\mathcal{T}_h}h_K$. By ρ_K we denote the radius of the largest circle inscribed into K and by $|K|$ we denote the area of K.

Let $K,K'\in\mathcal{T}_h$. We say that K and K' are *neighbours*, if the set $\partial K\cap\partial K'$ has positive length. We say that $\Gamma\subset K$ is a *face* (or *edge* in $I\!R^2$) of K, if it is a maximal connected open subset either of $\partial K\cap\partial K'$, where K' is a neighbour of K, or of $\partial K\cap\partial\Omega$. By $\mathcal{F}_h$ we denote the system of all faces of all elements $K\in\mathcal{T}_h$. Further, we define the set of all inner faces by

$$\mathcal{F}_h^I=\{\Gamma\in\mathcal{F}_h;\ \Gamma\subset\Omega\}$$

and the set of all boundary faces by

$$\mathcal{F}_h^B=\{\Gamma\in\mathcal{F}_h;\ \Gamma\subset\partial\Omega\}.$$

Obviously, $\mathcal{F}_h=\mathcal{F}_h^I\cup\mathcal{F}_h^B$.

For each $\Gamma\in\mathcal{F}_h$ we define a unit normal vector $\mathbf{n}_\Gamma$. We assume that for $\Gamma\in\mathcal{F}_h^B$ the normal $\mathbf{n}_\Gamma$ has the same orientation as the outer normal to $\partial\Omega$. For each face $\Gamma\in\mathcal{F}_h^I$ the orientation of $\mathbf{n}_\Gamma$ is arbitrary but fixed. Finally, by $d(\Gamma)$ we denote the length of $\Gamma\in\mathcal{F}_h$.

3.1. Spaces of discontinuous functions. Over a triangulation $\mathcal{T}_h$ we define the *broken Sobolev spaces*

$$H^k(\Omega, \mathcal{T}_h) = \{v;\, v|_K \in H^k(K),\, \forall K \in \mathcal{T}_h\}$$

equipped with the seminorm

$$|v|_{H^k(\Omega,\mathcal{T}_h)} = \left(\sum_{K\in\mathcal{T}_h} |v|^2_{H^k(K)} \right)^{1/2}.$$

For each face $\Gamma \in \mathcal{F}_h^I$ there exist two neighbours $K_\Gamma^{(L)}, K_\Gamma^{(R)} \in \mathcal{T}_h$ such that $\Gamma \subset K_\Gamma^{(L)} \cap K_\Gamma^{(R)}$. We use the convention that $\mathbf{n}_\Gamma$ is the outer normal to the element $K_\Gamma^{(L)}$ and the inner normal to the element $K_\Gamma^{(R)}$. For $v \in H^1(\Omega, \mathcal{T}_h)$ and $\Gamma \in \mathcal{F}_h^I$ we introduce the following notation:

$$v|_\Gamma^{(L)} = \text{ the trace of } v|_{K_\Gamma^{(L)}} \text{ on } \Gamma, \quad v|_\Gamma^{(R)} = \text{ the trace of } v|_{K_\Gamma^{(R)}} \text{ on } \Gamma,$$
$$\langle v\rangle_\Gamma = \frac{1}{2}\big(v|_\Gamma^{(L)} + v|_\Gamma^{(R)}\big), \qquad [v]_\Gamma = v|_\Gamma^{(L)} - v|_\Gamma^{(R)}.$$

The value $[v]_\Gamma$ depends on the orientation of $\mathbf{n}_\Gamma$, but the values $\langle v\rangle_\Gamma$ and $[v]_\Gamma \mathbf{n}_\Gamma$ are independent of this orientation.

Now, let $\Gamma \in \mathcal{F}_h^B$ and $K_\Gamma^{(L)} \in \mathcal{T}_h$ be such an element that $\Gamma \subset \partial K_\Gamma^{(L)} \cap \partial\Omega$. For $v \in H^1(\Omega, \mathcal{T}_h)$ we set

$$v_\Gamma = v|_\Gamma^{(L)} = v|_\Gamma^{(R)} = \text{ the trace of } v|_{K_\Gamma^{(L)}} \text{ on } \Gamma.$$

If $[\cdot]_\Gamma$ and $\langle\cdot\rangle_\Gamma$ appear in an integral of the form $\int_\Gamma \ldots\, \mathrm{d}S$, we omit the subscript Γ and write simply $[\cdot]$ and $\langle\cdot\rangle$. For simplicity we shall use the following notation:

$$\int_{\mathcal{F}_h^I} \ldots\, \mathrm{d}S = \sum_{\Gamma\in\mathcal{F}_h^I} \int_\Gamma \ldots\, \mathrm{d}S$$

and similarly for $\mathcal{F}_h$ and $\mathcal{F}_h^B$.

Let $p \geq 1$ be an integer. The approximate solution will be sought in the space of discontinuous piecewise polynomial functions

$$S_h = \{v;\, v|_K \in P^p(K), \forall K \in \mathcal{T}_h\},$$

where $P^p(K)$ denotes the space of all polynomials on K of degree $\leq p$.

3.2. Discontinuous Galerkin space semidiscretization. We define the following forms defined for $u, v, \varphi \in H^2(\Omega, \mathcal{T}_h)$, which define the SIPG (*Symmetric Interior Penalty Galerkin*) version of the DG approximation.
Symmetric diffusion form:

$$\begin{aligned} a_h(u, v, \varphi) = &\sum_{K\in\mathcal{T}_h} \int_K \beta(u)\nabla v \cdot \nabla\varphi\, dx \\ &- \int_{\mathcal{F}_h^I} \langle\beta(u)\nabla v\rangle \cdot \mathbf{n}[\varphi]\, dS - \int_{\mathcal{F}_h^I} \langle\beta(u)\nabla\varphi\rangle \cdot \mathbf{n}[v]\, dS \\ &- \int_{\mathcal{F}_h^B} \beta(u)\nabla v \cdot \mathbf{n}\varphi\, dS - \int_{\mathcal{F}_h^B} \beta(u)\nabla\varphi \cdot \mathbf{n} v\, dS. \end{aligned}$$

Further we define the *interior and boundary penalty jump terms*:

$$J_h(u,\varphi) = \int_{\mathcal{F}_h^I} \sigma[u][\varphi]\, dS + \int_{\mathcal{F}_h^B} \sigma u\varphi\, dS \tag{3.1}$$

and the *symmetric right-hand side form*:

$$l_h(u,\varphi)(t) = \int_\Omega g(t)\varphi\, dx - \int_{\mathcal{F}_h^B} \beta(u)\nabla\varphi\cdot\mathbf{n}u_D(t)\, dS + \int_{\mathcal{F}_h^B} \sigma u_D(t)\varphi\, dS. \tag{3.2}$$

The parameter σ in (3.1) and (3.2) is constant on every edge and defined by

$$\sigma|_\Gamma = \frac{C_W}{d(\Gamma)}, \quad \forall\, \Gamma \in \mathcal{F}_h, \tag{3.3}$$

where $C_W > 0$ is a constant, which must be chosen large enough to ensure coercivity of the diffusion form – cf. Lemma 5.1.

Finally we define the *convective form*:

$$b_h(u,\varphi) = -\sum_{K\in\mathcal{T}_h}\int_K \sum_{s=1}^{2} f_s(u)\frac{\partial\varphi}{\partial x_s}\, dx + \int_{\mathcal{F}_h} H(u^{(L)},u^{(R)},\mathbf{n})[\varphi]\, dS.$$

The form b_h approximates convective terms with the aid of a numerical flux $H(u,v,\mathbf{n})$. We assume that H has the following properties:

Assumptions (H):

(H1) $H(u,v,\mathbf{n})$ is defined in $I\!R^2\times B_1$, where $B_1 = \{\mathbf{n}\in I\!R^2; |\mathbf{n}| = 1\}$, and is *Lipschitz-continuous* with respect to u, v:

$$|H(u,v,\mathbf{n}) - H(u^*,v^*,\mathbf{n})| \le C_L(|u-u^*| + |v-v^*|), \quad \forall u,\ v,\ u^*,\ v^* \in I\!R,\ \mathbf{n}\in B_1.$$

(H2) $H(u,v,\mathbf{n})$ is *consistent*:

$$H(u,u,\mathbf{n}) = \sum_{s=1}^{d} f_s(u)\, n_s, \quad \forall u\in I\!R,\ \mathbf{n} = (n_1,n_2)\in B_1.$$

(H3) $H(u,v,\mathbf{n})$ is *conservative*:

$$H(u,v,\mathbf{n}) = -H(v,u,-\mathbf{n}), \quad \forall u,\ v\in I\!R,\ \mathbf{n}\in B_1.$$

In virtue of assumptions (H1) and (H2), we have $2C_L \ge L_f$, where L_f is the Lipschitz-continuity constant of the functions $f_s,\ s = 1,2$.

DEFINITION 3.1. *We say that u_h is a DGFE solution of the convection-diffusion problem (2.1) - (2.3), if*

a) $u_h \in C^1([0,T];S_h)$,

b)
$$\begin{aligned} &\frac{d}{dt}\big(u_h(t),\varphi_h\big) + b_h\big(u_h(t),\varphi_h\big) + \beta_0 J_h\big(u_h(t),\varphi_h\big) + a_h\big(u_h(t),u_h(t),\varphi_h\big) \\ &\quad = l_h\big(u_h(t),\varphi_h\big)(t), \quad \forall\varphi_h\in S_h,\ \forall t\in(0,T), \end{aligned} \tag{3.4}$$

c) $u_h(0) = u_h^0$,

where u_h^0 denotes an S_h approximation of the initial condition u^0.

We can show that a sufficiently regular exact solution u of problem (2.1) satisfies

$$\frac{d}{dt}\big(u,\varphi_h\big) + b_h\big(u,\varphi_h\big) + \beta_0 J_h\big(u,\varphi_h\big) + a_h\big(u,u,\varphi_h\big) = l_h\big(u,\varphi_h\big), \quad \forall\varphi_h\in S_h, \tag{3.5}$$

which implies the *Galerkin orthogonality property* of the error.

4. Some necessary results and assumptions.

4.1. Regularity of the exact solution. We assume that the weak solution u is sufficiently regular, namely

$$\frac{\partial u}{\partial t} \in L^2\big([0,T]; H^{p+1}(\Omega)\big), \tag{4.1}$$

where $p \geq 1$ denotes the given degree of approximation. It is possible to show that, under these conditions, u satisfies equation (2.1) pointwise and $u \in C\big([0,T]; H^{p+1}(\Omega)\big)$.

To treat the nonlinear diffusion terms, we need additional regularity assumptions on the solution u: there exists a constant $C_R < \infty$ such that

$$\begin{aligned}
&\|\nabla u(t)\|_{L^\infty(\Omega)} \leq C_R, \quad \text{for all } t \in (0,T),\\
&\|u_t(t)\|_{L^\infty(\Omega)} = \left\|\frac{\partial u}{\partial t}(t)\right\|_{L^\infty(\Omega)} \leq C_R, \quad \text{for a.a. } t \in (0,T),\\
&\|\nabla u_t(t)\|_{L^\infty(\Omega)} \leq C_R, \quad \text{for a.a. } t \in (0,T).
\end{aligned}$$

4.2. Geometry of the mesh. Let us consider a system $\{\mathcal{T}_h\}_{h\in(0,h_0)}$, $h_0 > 0$, of partitions of the domain $\overline{\Omega}$ into a finite number of closed triangles K with mutually disjoint interiors. Let us assume that the system $\{\mathcal{T}_h\}_{h\in(0,h_0)}$ has the following properties:

Assumptions (A):

(A1) The system $\{\mathcal{T}_h\}_{h\in(0,h_0)}$ is *regular*: there exists a constant $C_1 > 0$ such that

$$\frac{h_K}{\rho_K} \leq C_1, \quad \forall K \in \mathcal{T}_h \quad \forall h \in (0,h_0).$$

(A2) There exists a constant $C_2 > 0$ such that

$$h_K \leq C_2\, d(\Gamma), \quad \forall K \in \mathcal{T}_h, \quad \forall \Gamma \subset \partial K,\ \Gamma \in \mathcal{F}_h \quad \forall h \in (0,h_0).$$

(A3) There exists a constant $C_3 > 0$ such that

$$h^p \leq C_3 h_K \quad \forall K \in \mathcal{T}_h, \quad \forall h \in (0,h_0).$$

Let us note that we do not require the usual conforming properties from the finite element method, particularly, hanging nodes are allowed. In the case of piecewise linear elements (i.e. $p = 1$), condition (A3) reduces to the standard *inverse assumption* of [1] and becomes weaker with growing p.

4.3. Some auxiliary results. Throughout this work we denote by C a generic constant independent of h. The derived error estimates rely on the following results:

Lemma 4.1 (Multiplicative trace inequality). *There exists a constant $C_M > 0$ independent of h, K such that for all $K \in \mathcal{T}_h$, $v \in H^1(K)$ and $h \in (0,h_0)$*

$$||v||^2_{L^2(\partial K)} \leq C_M\big(||v||_{L^2(K)}|v|_{H^1(K)} + h_K^{-1}||v||^2_{L^2(K)}\big).$$

Lemma 4.2 (Inverse inequalities). *There exists a constant $C_I > 0$ independent of h, K such that for all $K \in \mathcal{T}_h$ and $v \in P^p(K)$*

$$\begin{aligned}
|v|_{H^1(K)} &\leq C_I h_K^{-1}||v||_{L^2(K)},\\
\|v\|_{L^\infty(K)} &\leq C_I h_K^{-1}\|v\|_{L^2(K)}.
\end{aligned}$$

The proof of Lemma 4.1 can be found in [3]. Lemma 4.2 is proved in e.g. [1].

In the error estimates of the following sections, we will apply the following version of Gronwall's lemma:

LEMMA 4.3. *Let $y,\, q \in C([0,T])$, $y,\, q \geq 0$ in $[0,T]$, $Z,\, R \in I\!R$, $R \geq 0$ and*

$$y(t) + q(t) \leq Z + R \int_0^t y(s)\,\mathrm{d}S, \quad t \in [0,T].$$

Then

$$y(t) + q(t) \leq Z \exp(Rt), \quad t \in [0,T].$$

5. Properties of the diffusion terms. Throughout the following analysis we shall assume that the constant C_W from (3.3) satisfies

$$C_W \geq 4 \left(\frac{\beta_1}{\beta_0}\right)^2 C_M(1 + C_I), \tag{5.1}$$

where C_M and C_I are constants from Lemma 4.1 and 4.2, respectively.

Let us define the form

$$A_h(u,v,w) = a_h(u,v,w) + \beta_0 J_h(v,w), \quad \forall u,v,w \in H^2(\Omega,\mathcal{T}_h),$$

which is linear with respect to the second and third argument and nonlinear with respect to the first argument. We define the following norm in $H^1(\Omega,\mathcal{T}_h)$:

$$\|w\|_{DG} = \left(\frac{1}{2}\left(|w|^2_{H^1(\Omega,\mathcal{T}_h)} + J_h(w,w)\right)\right)^{1/2}.$$

The form A_h has the following properties proven in [11]:

LEMMA 5.1 (Coercivity and boundedness of A_h). *Let $w : \Omega \to I\!R$ be an arbitrary measurable function defined almost everywhere in Ω. Under assumption (5.1) on the constant C_W, we have*

$$\beta_0 \|\varphi_h\|^2_{DG} \leq A_h(w,\varphi_h,\varphi_h), \quad \forall \varphi_h \in S_h,\ h \in (0,h_0),$$
$$A_h(w,v_h,\varphi_h) \leq C\|v_h\|_{DG}\|\varphi\|_{DG}, \quad \forall v_h,\varphi_h \in S_h,\ h \in (0,h_0).$$

For each $h \in (0,h_0)$ and $t \in [0,T]$ we define the function $u^*(t)$ $(= u^*_h(t))$ as the "A_h-projection" of $u(t)$ on S_h, i.e. a function satisfying the conditions

$$u^*(t) \in S_h, \qquad A_h\big(u(t),u^*(t),\varphi_h\big) = A_h\big(u(t),u(t),\varphi_h\big) \quad \forall\, \varphi_h \in S_h. \tag{5.2}$$

For simplicity of notation, we shall omit the argument t, whenever the role of t is not crucial. The existence of u^* is a consequence of the Lax-Milgram theorem, by the coercivity and boundedness (Lemma 5.1) of the form A_h on the space S_h.

In [11] we derive estimates for the functions $\chi = u - u^*$ and $\chi_t = \frac{\partial\chi}{\partial t}$.

LEMMA 5.2. *There exists a constant $C > 0$ independent of h, such that*

$$\|\chi(t)\|_{DG} \leq C\, h^p |u(t)|_{H^{p+1}(\Omega)},$$
$$\|\chi_t(t)\|_{DG} \leq C\, h^p |u_t(t)|_{H^{p+1}(\Omega)}.$$

for all $h \in (0, h_0)$ *and for a.a.* $t \in (0,T)$.

Dual problem

In what follows, we shall consider the linearised elliptic dual problem: Given $z \in L^2(\Omega)$, find $\psi(t)$ such that for all $t \in (0,T)$

$$\begin{aligned} -\mathrm{div}\big(\beta(u(t))\nabla\psi(t)\big) &= z \quad \text{in } \Omega, \\ \psi|_{\partial\Omega} &= 0. \end{aligned} \tag{5.3}$$

The weak formulation of (5.3) reads: Find $\psi(t) \in H_0^1(\Omega)$ such that

$$\big(\beta(u(t))\nabla\psi(t),\, \nabla v\big) = (z, v), \quad \forall\, v \in H_0^1(\Omega).$$

In [11] we analyze the regularity of problem (5.3):

Lemma 5.3. *Problem (5.3) has a unique weak solution* $\psi(t)$. *Moreover,* $\psi(t) \in H^2(\Omega)$ *for* $t \in (0,T)$ *and* $\psi_t(t) = \frac{\partial\psi(t)}{\partial t} \in H^2(\Omega)$ *for a.a.* $t \in (0,T)$. *Furthermore, there exists a constant* $C > 0$ *independent of* z *such that*

$$\begin{aligned} \|\psi(t)\|_{H^2(\Omega)} &\le C\|z\|_{L^2(\Omega)}, \quad t \in (0,T), \\ \|\psi_t(t)\|_{H^2(\Omega)} &\le C\|z\|_{L^2(\Omega)}, \quad a.a.\ t \in (0,T). \end{aligned}$$

Let us note that $H^2(\Omega) \hookrightarrow C(\overline{\Omega})$. Let $\psi_h\,(= \psi_h(t))$ be the piecewise linear L^2-projection of the function ψ, i.e. $\psi|_K \in P^1(K)$ and

$$\big(\psi - \psi_h, \varphi_h\big)_{L^2(K)} = 0, \quad \forall \varphi_h \in P^1(K),\ \forall K \in \mathcal{T}_h.$$

Standard approximation results give us

Lemma 5.4. *There exists a constant independent of* h, *such that for all* $t \in (0,T)$

$$\|\psi - \psi_h\|_{DG} \le C\,h|\psi|_{H^2(\Omega)}.$$

Now we use the dual problem (5.3) to obtain L^2-optimal error estimates for χ and χ_t.

Lemma 5.5. *There exists a constant* $C > 0$ *such that for all* $h \in (0, h_0)$ *and* $t \in (0,T)$

$$\|\chi\|_{L^2(\Omega)} \le Ch^{p+1}|u|_{H^{p+1}(\Omega)}. \tag{5.4}$$

$$\|\chi_t(t)\|_{L^2(\Omega)} \le Ch^{p+1}|u_t(t)|_{H^{p+1}(\Omega)}. \tag{5.5}$$

Proof. We have

$$\|\chi\|_{L^2(\Omega)} = \sup_{z \in L^2(\Omega)} \frac{(\chi, z)}{\|z\|_{L^2(\Omega)}}.$$

The continuity of functions from the space $H^2(\Omega)$ yields

$$[\psi]_\Gamma = 0, \qquad \forall \Gamma \in \mathcal{F}_h^I. \tag{5.6}$$

Due to (5.3) and (5.6), for a fixed $z \in L^2(\Omega)$ we have by applying Green's theorem

$$\begin{aligned}(\chi, z) &= \int_\Omega z\chi \,\mathrm{d}x = -\int_\Omega \operatorname{div}\big(\beta(u)\nabla\psi\big)\chi \,\mathrm{d}x \\ &= \sum_{K\in\mathcal{T}_h}\int_K \beta(u)\nabla\psi\cdot\nabla\chi \,\mathrm{d}x - \int_{\mathcal{F}_h^I}\langle\beta(u)\nabla\psi\rangle\cdot\mathbf{n}\,[\chi]\,\mathrm{d}S - \int_{\mathcal{F}_h^B}\beta(u)\nabla\psi\cdot\mathbf{n}\,\chi\,\mathrm{d}S \\ &= A_h(u,\psi,\chi).\end{aligned}$$

Further, the symmetry of A_h and (5.2) give

$$A_h(u,\psi_h,\chi) = A_h(u,\chi,\psi_h) = A_h(u,u-u^*,\,\psi_h) = 0. \tag{5.7}$$

This and Lemmas 5.1 and 5.4 imply that for a.a. $t \in (0,T)$

$$\begin{aligned}(\chi, z) = A_h(u,\psi-\psi_h,\chi) &\le C\big(\|\psi-\psi_h\|_{DG} + h|\psi-\psi_h|_{H^2(\Omega,\mathcal{T}_h)}\big)\|\chi\|_{DG} \\ &\le Ch|\psi|_{H^2(\Omega)}h^p|u|_{H^{p+1}(\Omega)} \le Ch^{p+1}\|z\|_{L^2(\Omega)}|u|_{H^{p+1}(\Omega)}.\end{aligned}$$

Hence,

$$\|\chi\|_{L^2(\Omega)} = \sup_{z\in L^2(\Omega)}\frac{(\chi,z)}{\|z\|_{L^2(\Omega)}} \le C\,h^{p+1}|u|_{H^{p+1}(\Omega)},$$

which completes the proof of (5.4). Similarly it is possible to obtain (5.5). □

Now we state an important result from [11]. In the case of linear diffusion, the terms we estimate in (5.8) would be equal to zero. However in our case, these terms must be carefully estimated to obtain the optimal error estimates. The proof of Lemma 5.6 requires additional auxiliary results and the nonstandard condition (A3)

Lemma 5.6. *Let $\zeta := u^* - u_h \in S_h$. There exists a constant $C > 0$ such that for all $h \in (0,h_0)$ and a.a. $t \in (0,T)$*

$$\begin{aligned}&A_h(u,u^*,\zeta) - A_h(u_h,u^*,\zeta) - l_h(u,\zeta) + l_h(u_h,\zeta) \\ &\qquad \le Ch^{2(p+1)}|u|^2_{H^{p+1}(\Omega)} + C\|\zeta\|^2_{L^2(\Omega)} + \frac{\beta_0}{4}\|\zeta\|^2_{DG}.\end{aligned} \tag{5.8}$$

6. Properties of the convective term. Under assumptions (H) and (A) the convective form b_h is Lipschitz continuous in the following sense (cf. [11]):

Lemma 6.1. *Let u be the solution of the continuous problem (2.1), u_h the solution of the discrete problem (3.4), u^* be defined by (5.2), and ζ $(=\zeta_h) = u^* - u_h \in S_h$. Then there exists a constant $C > 0$, independent of $h \in (0,h_0)$, such that*

$$|b_h(u,\zeta) - b_h(u_h,\zeta)| \le \frac{\beta_0}{4}\|\zeta\|^2_{DG} + \frac{C}{\beta_0}\left(h^{2(p+1)}|u|^2_{H^{p+1}(\Omega)} + \|\zeta\|^2_{L^2(\Omega)}\right).$$

7. Error estimates. We state the main result:

Theorem 7.1 (Main theorem). *Let assumptions (H) and (A) be satisfied and let the constant C_W be chosen in such a way that (5.1) holds. Let u be the exact solution of problem (2.1) satisfying the regularity condition (4.1) and let u_h be the approximate solution defined by (3.4). Then the error $e_h = u - u_h$ satisfies the estimate*

$$\|e_h\|_{L^\infty(0,T;L^2(\Omega))} \le Ch^{p+1},$$

with a constant $C > 0$ independent of h.

Proof. Let u^* be the A_h projection defined by (5.2) and let χ and ζ be as in Lemmas 5.2 – 6.1, i.e. $\chi = u - u^*$, $\zeta = u^* - u_h$. Then $e_h = u - u_h = \chi + \zeta$. Let us subtract (3.4, b) from (3.5), substitute $\zeta \in S_h$ for φ_h and use the relation

$$\left(\frac{\partial \zeta(t)}{\partial t}, \zeta(t)\right) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\zeta(t)\|^2_{L^2(\Omega)}.$$

Then we get

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\zeta\|^2_{L^2(\Omega)} &+ A_h\big(u,u,\zeta\big) - A_h\big(u_h,u_h,\zeta\big)\\ &= \big[b_h\big(u_h,\zeta\big) - b_h\big(u,\zeta\big)\big] - \big(\chi_t,\zeta\big) + l_h\big(u,\zeta\big) - l_h\big(u_h,\zeta\big).\end{aligned} \tag{7.1}$$

The convective terms can be estimated by Lemma 6.1. For the second right-hand side term in (7.1), by the Cauchy and Young's inequalities and Lemma 5.5, we have

$$|(\chi_t,\zeta)| \le \frac{1}{2}\left(\|\chi_t\|^2_{L^2(\Omega)} + \|\zeta\|^2_{L^2(\Omega)}\right) \le \frac{1}{2}\left(C\,h^{2(p+1)}|u_t|^2_{H^{p+1}(\Omega)} + \|\zeta\|^2_{L^2(\Omega)}\right).$$

Further, we treat the diffusion terms in (7.1):

$$\begin{aligned}A_h\big(u,u,\zeta\big) &- A_h\big(u_h,u_h,\zeta\big)\\ &= A_h\big(u,\chi,\zeta\big) + A_h\big(u,u^*,\zeta\big) - A_h\big(u_h,u^*,\zeta\big) + A_h\big(u_h,\zeta,\zeta\big)\\ &\ge A_h\big(u,u^*,\zeta\big) - A_h\big(u_h,u^*,\zeta\big) + \beta_0\|\zeta\|_{DG},\end{aligned} \tag{7.2}$$

due to the coercivity of A_h – Lemma 5.1 – and the definition of u^*, cf. (5.2). Hence, by combining (7.1), (7.2) with Lemma 5.6 we obtain a.a. $t \in (0,T)$

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\zeta\|^2_{L^2(\Omega)} + \beta_0\|\zeta\|^2_{DG} \le C\,h^{2(p+1)}\left(|u|^2_{H^{p+1}(\Omega)} + |u_t|^2_{H^{p+1}(\Omega)}\right) + C\left(1+\frac{1}{\beta_0}\right)\|\zeta\|^2_{L^2(\Omega)}.$$

By integration from 0 to $t \in [0,T]$ we get

$$\begin{aligned}\|\zeta(t)\|^2_{L^2(\Omega)} &+ \beta_0\int_0^t \|\zeta(\vartheta)\|^2_{DG}\,\mathrm{d}\vartheta\\ &\le C\,h^{2(p+1)}\left(\int_0^t |u(\vartheta)|^2_{H^{p+1}(\Omega)}\,\mathrm{d}\vartheta + \int_0^t |u_t(\vartheta)|^2_{H^{p+1}(\Omega)}\,\mathrm{d}\vartheta\right)\\ &\quad + C\left(1+\frac{1}{\beta_0}\right)\int_0^t \|\zeta(\vartheta)\|^2_{L^2(\Omega)}\,\mathrm{d}\vartheta + C\,h^{2(p+1)}|u^0|^2_{H^{p+1}(\Omega)},\end{aligned}$$

since it is possible to prove

$$\|\zeta(0)\|^2_{L^2(\Omega)} \le C\,h^{2(p+1)}|u^0|^2_{H^{p+1}(\Omega)}.$$

Now we apply Gronwall's Lemma 4.3, which yields

$$\|\zeta(t)\|^2_{L^2(\Omega)} + \beta_0\int_0^t \|\zeta(\vartheta)\|^2_{DG}\,\mathrm{d}\vartheta \le C\,h^{2(p+1)}\,N(u)\,\exp\left(\tilde{C}\left(1+\frac{1}{\beta_0}\right)t\right), \tag{7.3}$$

where

$$N(u) = \|u\|^2_{L^2(0,T;H^{p+1}(\Omega))} + \|u_t\|^2_{L^2(0,T;H^{p+1}(\Omega))} + |u^0|^2_{H^{p+1}(\Omega)} < \infty.$$

(C and $\tilde{C}$ are constants independent of t and h). Since $e_h = \chi + \zeta$, it is sufficient now to combine (7.3) with the estimate of $\|\chi(t)\|_{L^2(\Omega)}$ from Lemma 5.5. ∎

8. Conclusion. This paper represents an overview of the results obtained in [11]. We are concerned with the analysis of the discontinuous Galerkin space semidiscretization of a nonstationary convection-diffusion problem with nonlinear diffusion and nonlinear convection, equipped with Dirichlet boundary conditions and an initial condition. We have proven optimal error estimates of order $O(h^{p+1})$ in the $L^\infty(0,T;L^2(\Omega))$-norm for the SIPG method under the assumptions that the piecewise polynomial approximation of degree p is used, the time derivative of the exact solution is sufficiently regular and the solution of a linearised elliptic dual problem possesses a sufficiently regular solution. This is true under additional conditions on the diffusive nonlinearity $\beta(\cdot)$ and the exact solution u, provided the polygonal domain Ω is convex.

The assumption of symmetry of the discretization of the diffusion terms is crucial in the presented proof. Namely, it enables us to exchange arguments in (5.7). This is the reason why we are unable to prove optimal error estimates for the nonsymmetric and incomplete variants of the DG scheme (cf. [3]) using the presented technique.

There are several open problems connected with the analysis of optimal error estimates of the DGFEM for convection-diffusion problems:

- Derivation of optimal error estimates in the case of a weaker regularity of the exact solution of the considered convection-diffusion problem and of the dual problem (the case of a polygonal nonconvex domain Ω and/or Neumann boundary conditions).
- The extension of the derived estimates to three spatial dimensions.
- The investigation of optimal error estimates for other variants of the DGFEM for the diffusion terms, such as the nonsymmetric and incomplete interior penalty Galerkin methods (NIPG and IIPG), where the presented technique cannot be applied.

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