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AN ITERATIVE SUBSTRUCTURING METHOD FOR THE DISCRETIZED STOKES EQUATIONS BY A STABILIZED FINITE ELEMENT METHOD*

ATSUSHI SUZUKI†

Abstract. A simple algorithm of iterative substructuring method as the same way of elasticity problem is proposed for a discretized Stokes equation by P1/P1 element and penalty stabilization technique. Owing to the stability term, solvabilities of local Dirichlet problem, of local Neumann problem for preconditioner, and of the coarse space problem are ensured. Conjugate gradient method with preconditioner constructed by a balancing technique is used to solve the linear system of the discretized Stokes equations whose matrix is symmetric but indefinite.

Key words. Stokes equations, stabilized finite element method, iterative substructuring method, balancing Neumann-Neumann preconditioner, CG method, indefinite matrix

AMS subject classifications. 76M10, 65F05, 65F10, 65Y05

1. Introduction. An iterative substructuring method with balancing Neumann-Neumann preconditioner [3, 7] is known as an efficient parallel algorithm for elasticity problem. This method is extended to the Stokes equation by Pavarino and Widlund [4] and to the Oseen equations by Lube et al.[2]. In contrast to elasticity equations whose coefficient matrix is symmetric positive definite, discretized Stokes equations consist of an indefinite matrix. Therefore, they use P2/P0-discontinuous pressure element to construct Schur complement system with “benign space” where incompressibility of the velocity is satisfied in discrete sense, and solve it by Conjugate Gradient (CG) method within the benign space. For construction of the coarse space, supplementary inf-sup condition is considered to ensure solvability of the Schur complement on the space.

In this paper, we introduce a simple algorithm, which is same as elasticity problem, for discretized Stokes equation with P1/P1 element for the velocity and the pressure unknowns and penalty stabilization technique [1]. Owing to the stability term, solvabilities of local Dirichlet problem for Schur complement system, of local Neumann problem for preconditioner, and of the coarse space problem are ensured. We employ CG method to solve linear system with symmetric indefinite matrix and here we also use a preconditioner whose matrix is indefinite. This solver plays a key role in the simple construction of the iterative substructuring method for the Stokes equations. We consider the case that a solution has some ambiguity, e.g. rigid body rotations and pressure constant.

The contents of this paper are as follows. First, we describe governing equations and finite element approximation. Second, we introduce non-overlapping domain decomposition, construct Schur complement system, and propose a direct solver for local problem. Finally we show a balancing Neumann-Neumann preconditioner with a numerical example.

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†Department of Mathematics, Faculty of Nuclear Sciences and Physical Engineering, Czech Technical University in Prague, Trojanova 13, 120 00 Prague, and Faculty of Mathematics, Kyushu University, Fukuoka 812-8581 Japan (asuzuki@math.kyushu-u.ac.jp).

2. Stabilized finite element approximation to Stokes equations. We consider the Stokes equations in spherical shell domain with slip boundary conditions which appears in numerical simulation of the Earth's mantle convection problem [6]. Here we consider the velocity and pressure in a function space with constraints to remove ambiguity of rigid body rotations and pressure constant. We show a finite element approximation and linear equations with stiffness matrix which is a main target of parallel solver.

2.1. Stokes equations with slip boundary conditions. Let Ω be a spherical shell domain, $\Omega := \{x \in \mathbb{R}^3\,;\, R_1 < |x| < R_2\}$ and $\Gamma_k (k = 1, 2)$ be its boundary, $\Gamma_k := \{x \in \partial\Omega\,;\, |x| = R_k\}$. We consider the Stokes equations with slip boundary conditions on the boundary:

$$-2\nabla \cdot D(u) + \nabla p = f, \tag{2.1a}$$
$$\nabla \cdot u = 0 \tag{2.1b}$$

in the domain and

$$u \cdot n_\Omega = 0, \tag{2.1c}$$
$$D(u)n_\Omega \times n_\Omega = 0 \tag{2.1d}$$

on the boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$. Here $D(u)$ is the strain rate tensor $[D(u)]_{k\,l} := (\partial u_k/\partial x_l + \partial u_l/\partial x_k)/2$ and n_Ω is an outer normal to the boundary.

We prepare function spaces where velocity and pressure are found,

$$V := \{v \in H^1(\Omega)^3\,;\, v \cdot n_\Omega = 0 \text{ on } \Gamma_1 \cup \Gamma_2,\ (v, v_k) = 0\ (k = 1, 2, 3)\},$$
$$Q := \{q \in L^2(\Omega)\,;\, \int_\Omega q\,dx = 0\}\,,$$

and bilinear forms

$$a(u, v) := 2\int_\Omega D(u) : D(v)\,dx \quad \text{for } u, v \in V,$$
$$b(u, p) := -\int_\Omega \nabla \cdot u\,p\,dx \quad \text{for } u \in V \text{ and } p \in Q\,.$$

Here v_k is a rigid rotation vector defined by $v_k(x) := e_k \times x$ with $[e_k]_l = \delta_{k\,l}$ $(1 \le k, l \le 3)$.

A weak formulation of (2.1) is to find $(u, p) \in V \times Q$ such that

$$a(u, v) + b(v, p) = (f, v) \qquad \text{for any } v \in V, \tag{2.2a}$$
$$b(u, q) = 0 \qquad \text{for any } q \in Q\,. \tag{2.2b}$$

2.2. Finite element approximation with penalty stabilization. We employ an equal order approximation to the velocity and pressure with $P1/P1$ element and a penalty stabilization [1].

Let Ω_h be a polyhedral approximation to Ω and $\mathcal{T}_h$ be a partition of $\bar{\Omega}_h$ by tetrahedra. Let S_h be the P1 finite element space defined by

$$S_h := \{v \in C^0(\bar{\Omega}_h)\,;\, v|_K \in \mathcal{P}^1(K), K \in \mathcal{T}_h\}\,,$$

where $\mathcal{P}^1$ is the set of polynomials of degree 1. In the following expression, we omit subscript h of the approximated domain Ω_h for simplicity. We prepare function spaces,

$$V_h := \{v \in S_h^3\,;\, v(P)\cdot n_\Omega(P) = 0,\ (v, v_k) = 0\ (k = 1, 2, 3)\},$$
$$Q_h := \{q \in S_h\,;\, \int_\Omega q\,dx = 0\}\,.$$

Here P is a node on $\Gamma_1 \cup \Gamma_2$, and n_Ω is the unit outer normal to the domain Ω not to Ω_h. As a stabilization technique for a remedy to insufficiency of satisfaction of the inf-sup condition, we use the following bilinear form,

$$d(p,\, q) := \sum_{K\in\mathcal{T}_h} h_K^2 \int_K \nabla p \cdot \nabla q\,dx\,.$$

A finite element approximation to (2.2) is to find $(u_h, p_h) \in V_h \times Q_h$ such that

$$a(u_h, v_h) + b(v_h, p_h) = (f, v_h) \qquad \text{for any } v_h \in V_h\,, \tag{2.3a}$$
$$b(u_h, q_h) - \delta\, d(p_h, q_h) = 0 \qquad \text{for any } q_h \in Q_h\,. \tag{2.3b}$$

Here $\delta > 0$ is called as the stabilization parameter.

2.3. Finite element equation with matrix and orthogonal projection. We use finite element bases, $\{\varphi_\alpha\}_{\alpha\in\Lambda_Y}$ for the velocity and $\{\psi_\mu\}_{\mu\in\Lambda_X}$ for the pressure to show a matrix form of (2.2). Here $\Lambda_X := \{1, 2, \cdots, n_X\}$ is an index set of the nodes P_μ in the domain and on the boundary, $P_\mu \in \Omega \cup \partial\Omega$ ($\mu \in \Lambda_X$). Let $\Lambda_\Gamma \subset \Lambda_X$ be an index set of the node on the boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$. $\Lambda_Y := \{1, 2, \cdots, n_Y\}$ with $n_Y = 3\,n_X$. We associate a pair $[\alpha_0,\, \alpha_1]$ of a node number and a component number with an index $\alpha \in \Lambda_Y$ and identify them

$$\alpha = [\alpha_0,\, \alpha_1] \qquad (\alpha_0 \in \Lambda_X,\ \alpha_1 \in \{1, 2, 3\})\,. \tag{2.4}$$

We assume that the association of $[\alpha_0,\, \alpha_1]$ with $\alpha \in \Lambda_Y$ is put as

$$\alpha_0 = \lfloor(\alpha - 1)/3\rfloor + 1, \qquad \alpha_1 = ((\alpha - 1) \bmod 3) + 1\,, \tag{2.5}$$

where $\lfloor\cdot\rfloor$ denotes the greatest integer less than or equal to the argument. This means the numbering of the degrees of freedom is done as $(3(\alpha_0-1)+1, 3(\alpha_0-1)+2, 3(\alpha_0-1)+3)$ at a node P_{α_0}. The basis φ_α satisfies that

$$[\varphi_\alpha(P_{\beta_0})]_{\beta_1} = \delta_{\alpha\,\beta} \qquad (\alpha,\, \beta = [\beta_0,\, \beta_1] \in \Lambda_Y)\,.$$

Let $\{\vec{n}_\mu\}_{\mu\in\Gamma_\Lambda} \subset \mathbb{R}^{N_Y}$ correspond to outer normals, $[\,\vec{n}_\mu\,]_\alpha = \delta_{\alpha_0\,\mu}[n_\Omega(P_\mu)]_{\alpha_1}$, and let $\vec{1} \in \mathbb{R}^{N_X}$ be defined by $[\vec{1}]_\mu = 1$ ($\mu \in \Lambda_X$). M_Y and M_X denote mass matrices on S_h^3 and S_h, respectively. We introduce subspaces of $\mathbb{R}^{N_Y}$ and $\mathbb{R}^{N_X}$ for finite element solutions,

$$\vec{V} := \{\vec{v} \in \mathbb{R}^{N_Y}\,;\, \vec{v}\cdot\vec{n}_\mu = 0,\ (\mu \in \Lambda_\Gamma),\ (M_Y\vec{v}, \vec{v}_k) = 0\ (k = 1, 2, 3)\},$$
$$\vec{Q} := \{\vec{q} \in \mathbb{R}^{N_X}\,;\, (M_X\vec{q}, \vec{1}) = 0\}\,,$$

where $(\cdot,\cdot)$ is ℓ^2-inner product of $\mathbb{R}^{N_Y}$ or $\mathbb{R}^{N_X}$. Let A, B and D be stiffness matrices defined by

$$[A]_{\alpha\,\beta} := a(\varphi_\beta, \varphi_\alpha) \qquad \text{for } \alpha,\, \beta \in \Lambda_Y\,,$$
$$[B]_{\mu\,\beta} := b(\varphi_\beta, \psi_\mu) \qquad \text{for } \mu \in \Lambda_X,\, \beta \in \Lambda_Y\,,$$
$$[D]_{\mu\,\nu} := d(\psi_\nu, \psi_\mu) \qquad \text{for } \mu,\, \nu \in \Lambda_X\,.$$

A matrix form of the finite element equations (2.3) is to find $(\vec{u},\vec{p}) \in \vec{V} \times \vec{Q}$ such that

$$\left(\begin{bmatrix} A & B^T \\ B & -\delta D \end{bmatrix} \begin{bmatrix} \vec{u} \\ \vec{p} \end{bmatrix}, \begin{bmatrix} \vec{v} \\ \vec{q} \end{bmatrix} \right) = \left(\begin{bmatrix} \vec{f} \\ 0 \end{bmatrix}, \begin{bmatrix} \vec{v} \\ \vec{q} \end{bmatrix} \right) .$$

Here $[\vec{f}]_\alpha := (f, \varphi_\alpha)$. This discrete variational form is expressed as a usual linear system with orthogonal projections. We prepare P_V from $\mathbb{R}^{N_Y}$ onto $\vec{V}$ and P_Q from $\mathbb{R}^{N_X}$ onto $\vec{Q}$. They are expressed as follows.

$$\begin{aligned} P_V \vec{u} &= \vec{u} - \sum_{\mu \in \Lambda_\Gamma} (\vec{v}, \vec{n}_\mu)\vec{n}_\mu - \sum_{1 \le k \le 3} (\vec{v}, \vec{m}_k)\vec{m}_k \, , \\ P_Q \vec{p} &= \vec{p} - (\vec{p}, \vec{m}_0)\vec{m}_0 \, , \end{aligned}$$

where $\{\vec{m}_k\}$ satisfy $\mathrm{span}[\, \{ \vec{m}_k \}_{1 \le k \le 3}, \{\vec{n}_\mu\}_{\mu \in \Lambda_\Gamma} \,] = \mathrm{span}[\, \{ M_Y \vec{v}_k \}_{1 \le k \le 3}, \{\vec{n}_\mu\}_{\mu \in \Lambda_\Gamma} \,]$ and $\{ \vec{m}_k \}_{1 \le k \le 3}, \{\vec{n}_\mu\}_{\mu \in \Lambda_\Gamma}$ are orthonormal. We set $\vec{m}_0 := M_X \vec{1}/||M_X \vec{1}||$. A matrix form with orthogonal projection of the finite element equations (2.3) is to find $(\vec{u},\vec{q}) \in \vec{V} \times \vec{Q}$ such that

$$\begin{bmatrix} P_V & 0 \\ 0 & P_Q \end{bmatrix} \begin{bmatrix} A & B^T \\ B & -\delta D \end{bmatrix} \begin{bmatrix} P_V & 0 \\ 0 & P_Q \end{bmatrix} \begin{bmatrix} \vec{u} \\ \vec{p} \end{bmatrix} = \begin{bmatrix} P_V & 0 \\ 0 & P_Q \end{bmatrix} \begin{bmatrix} \vec{f} \\ 0 \end{bmatrix} . \tag{2.6}$$

REMARK 1. A preconditioned conjugate gradient (PCG) method can solve (2.6), though the stiffness matrix is indefinite. When procedure of the PCG does not meet a breakdown, the solution is found in the largest Krylov subspace generated from initial residual and preconditioned coefficient matrix [5].

From coercivity of the bilinear forms $a(\cdot,\cdot)$ on $V_h \times V_h$ and $d(\cdot,\cdot)$ on $Q_h \times Q_h$, we obtain following properties on the stiffness matrices of the discretized Stokes equations with the penalty-type stabilization method.

PROPOSITION 1. *We have*

$$\begin{aligned} &A \in \mathbb{R}^{N_Y \times N_Y} \textit{ is symmetric and positive definite on } \vec{V} \, , \\ &D \in \mathbb{R}^{N_Y \times N_Y} \textit{ is symmetric and positive definite on } \vec{Q} \, . \end{aligned}$$

LEMMA 1. *Let $\vec{U} \subset \vec{V}$ and $\vec{R} \subset \vec{Q}$. The stiffness matrix of the discretized Stokes equations by the penalty-type stabilization method is regular on the product of subspaces, $\vec{U} \times \vec{R}$.*

Proof. Let $(\vec{u},\vec{p}) \in \vec{U} \times \vec{R}$. We have $(\vec{u},-\vec{p}) \in \vec{U} \times \vec{R}$. The positivity of the skewed stiffness matrix,

$$\left(\begin{bmatrix} A & B^T \\ B & -\delta D \end{bmatrix} \begin{bmatrix} \vec{u} \\ \vec{p} \end{bmatrix}, \begin{bmatrix} \vec{u} \\ -\vec{p} \end{bmatrix} \right) = (A\vec{u},\vec{u}) + \delta\,(D\vec{p},\vec{p}) > 0$$

leads to the result. □

3. Non-overlapping domain decomposition and iterative substructuring method. We describe an iterative substructuring method which is obtained by decomposition of the stiffness matrix according to non-overlapping domain decomposition. In case of the Stokes equations, since stiffness matrix is not positive definite, solvability of subproblems has to be considered.

3.1. Decomposition of domain and stiffness matrices. We decompose the domain Ω into a union of D non-overlapping subdomains,

$$\bar{\Omega} = \bigcup_{1\le i\le D} \bar{\Omega}^{(i)}, \quad \Omega^{(i)} \cap \Omega^{(j)} = \emptyset \; (1 \le i < j \le D) .$$

We introduce an interface among the subdomains by

$$\mathcal{F} := \bigcup_{1\le i\le j\le D} \partial\Omega^{(i)} \cap \partial\Omega^{(j)} .$$

Here $\mathcal{F}$ contains both original boundary $\partial\Omega$ and the artificial boundary induced from the non-overlapping domain decomposition. We define index sets $\Lambda_X^{(i)}$ $(1 \le i \le D)$ and $\Lambda_X^{(\mathcal{F})}$ by

$$\begin{aligned}
\Lambda_X^{(i)} &:= \left\{ \mu \in \Lambda_X \,;\, P_\mu \in \Omega^{(i)} \right\} \; (1 \le i \le D) , & n_X^{(i)} &:= \#\Lambda_X^{(i)} ,\\
\Lambda_X^{(\mathcal{F})} &:= \left\{ \mu \in \Lambda_X \,;\, P_\mu \in \mathcal{F} \right\} , & n_X^{(\mathcal{F})} &:= \#\Lambda_X^{(\mathcal{F})} ,\\
\Lambda_X^{(\mathcal{F}),i} &:= \left\{ \mu \in \Lambda_X \,;\, P_\mu \in \mathcal{F} \cap \partial\Omega^{(i)} \right\} , & n_X^{(\mathcal{F}),i} &:= \#\Lambda_X^{(\mathcal{F}),i} .
\end{aligned}$$

We note $\Lambda_\Gamma \subset \Lambda_X^{\mathcal{F}}$. $\Lambda_X^{(i)}$ corresponds to unknowns of Dirichlet problems and $\Lambda_X^{(i)} \cup \Lambda_X^{(\mathcal{F}),i}$ corresponds to unknowns of Neumann problems.

Remark 2. This decomposition way of nodes, where nodes in subdomains and ones on boundaries are separated, differs from usual way of iterative substructuring method. In this way, all subdomains are treated as floating subdomains [7] and original boundary conditions are separately treated from subproblems.

Index sets $\Lambda_Y^{(i)}$ $(1 \le i \le D)$ and $\Lambda_Y^{(\mathcal{F})}$ are defined from $\Lambda_X^{(i)}$ $(1 \le i \le D)$ and $\Lambda_X^{(\mathcal{F})}$ with the association rule (2.4), respectively. We have decomposition of indices without overlapping,

$$\begin{aligned}
\Lambda_X &= \Lambda_X^{(1)} \oplus \Lambda_X^{(2)} \oplus \cdots \oplus \Lambda_X^{(D)} \oplus \Lambda_X^{(\mathcal{F})} ,\\
\Lambda_Y &= \Lambda_Y^{(1)} \oplus \Lambda_Y^{(2)} \oplus \cdots \oplus \Lambda_Y^{(D)} \oplus \Lambda_Y^{(\mathcal{F})} .
\end{aligned}$$

We define restriction operators $R_X^{(i)} : \Lambda_X \to \Lambda_X^{(i)}$, $R_X^{(\mathcal{F})} : \Lambda_X \to \Lambda_X^{(\mathcal{F})}$, and $R_Y^{(i)}$ for the velocity unknowns by the same way. Now we decompose the stiffness matrix A,

$$\begin{aligned}
[\, A^{(i)} \,]_{\alpha\,\beta} &:= 2 \int_{\Omega^{(i)}} D(\varphi_\beta) : D(\varphi_\alpha) \, dx \quad \alpha, \beta \in \Lambda_Y^{(i)} \; (1 \le i \le D) ,\\
[\, A^{(i,\mathcal{F})} \,]_{\alpha\,\beta} &:= 2 \int_{\Omega^{(i)}} D(\varphi_\beta) : D(\varphi_\alpha) \, dx \quad \alpha \in \Lambda_Y^{(i)}, \beta \in \Lambda_Y^{(\mathcal{F}),i} \; (1 \le i \le D) ,\\
[\, A_i^{(\mathcal{F})} \,]_{\alpha\,\beta} &:= 2 \int_{\Omega^{(i)}} D(\varphi_\beta) : D(\varphi_\alpha) \, dx \quad \alpha, \beta \in \Lambda_Y^{(\mathcal{F}),i} \; (1 \le i \le D) ,\\
[\, A^{(\mathcal{F})} \,]_{\alpha\,\beta} &:= 2 \int_{\Omega} D(\varphi_\beta) : D(\varphi_\alpha) \, dx \quad \alpha, \beta \in \Lambda_Y^{(\mathcal{F})} .
\end{aligned}$$

We note that $[\, A^{(\mathcal{F})} \,]_{\alpha\,\beta} = \sum_{1\le i\le D} [A_i^{(\mathcal{F})}]_{\alpha\,\beta}$ $(\alpha, \beta \in \Lambda_Y^{(\mathcal{F})})$. Other matrices B and D are decomposed in the same way. We decompose the stiffness matrix of the Stokes equations,

$$K^{(i)} := \begin{bmatrix} A^{(i)} & {B^{(i)}}^T \\ B^{(i)} & -\delta D^{(i)} \end{bmatrix} , \; K^{(i,\mathcal{F})} := \begin{bmatrix} A^{(i,\mathcal{F})} & {B^{(i,\mathcal{F})}}^T \\ B^{(i,\mathcal{F})} & -\delta D^{(i,\mathcal{F})} \end{bmatrix} , \; K_i^{(\mathcal{F})} := \begin{bmatrix} A_i^{(\mathcal{F})} & {B_i^{(\mathcal{F})}}^T \\ B_i^{(\mathcal{F})} & -\delta D_i^{(\mathcal{F})} \end{bmatrix} .$$

We obtain decomposition of the stiffness matrix as usual way of iterative substructuring method,

$$\begin{bmatrix} K^{(1)} & & & & K^{(1,\mathcal{F})} \\ & K^{(2)} & & & K^{(1,\mathcal{F})} \\ & & \ddots & & \vdots \\ & & & K^{(D,\mathcal{F})} & K^{(D,\mathcal{F})} \\ K^{(\mathcal{F},1)} & K^{(\mathcal{F},2)} & \dots & K^{(\mathcal{F},D)} & \sum_{1\le i\le D} K_i^{(\mathcal{F})} \end{bmatrix}.$$

However, (2.6) contains the orthogonal projection. So we need to prepare a decomposition of the orthogonal projection into local projections on subdomains. Let $\vec{V}^{(i)}$ and $\vec{Q}^{(i)}$ be subspaces defined by

$$\vec{V}^{(i)} := \{\, \vec{v}^{(i)} \in \mathbb{R}^{N_Y^{(i)}}\,;\, (\vec{v}^{(i)}, R_Y^{(i)}\vec{m}_k) = 0\ (1 \le k \le 6)\,\},$$
$$\vec{Q}^{(i)} := \{\, \vec{q}^{(i)} \in \mathbb{R}^{N_X^{(i)}}\,;\, (\vec{q}^{(i)}, R_X^{(i)}\vec{m}_0) = 0\,\},$$

where $\{\vec{m}_k\}_{4\le k\le 6}$ are generated from rigid body displacement modes and mass-matrix weight as the same way of rigid body rotation modes. We introduce orthogonal projections $P_Y^{(i)} : \mathbb{R}^{N_Y^{(i)}} \to \vec{V}^{(i)}$, and $P_Q^{(i)} : \mathbb{R}^{N_X^{(i)}} \to \vec{Q}^{(i)}$. Let $P^{(i)}$ denote $\begin{bmatrix} P_Y^{(i)} & \\ & P_Q^{(i)} \end{bmatrix}$. Let $\{\vec{\tilde{m}}_k^{(i)}\}_{0\le k\le 6}$ be orthonormal vectors satisfying $\mathrm{span}[\{\vec{\tilde{m}}_k^{(i)}\}_{1\le k\le 6}] = \mathrm{span}[\{R_Y^{(i)}\vec{m}_k\}_{1\le k\le 6}]$, and $G^{(i)}$ be a constraint matrix defined by ${G^{(i)}}^T := [\vec{\tilde{m}}_0^{(i)}, \vec{\tilde{m}}_1^{(i)}, \cdots, \vec{\tilde{m}}_k^{(i)}]$. We introduce $(N_Y^{(i)} + N_X^{(i)} + 7) \times (N_Y^{(i)} + N_X^{(i)} + 7)$ matrix defined by

$$\begin{bmatrix} P^{(i)}K^{(i)}P^{(i)} & P^{(i)}K^{(i,\mathcal{F})} & P^{(i)}K^{(i)}{G^{(i)}}^T \\ K^{(\mathcal{F},i)}P^{(i)} & K_i^{(\mathcal{F})} & K^{(\mathcal{F},i)}{G^{(i)}}^T \\ G^{(i)}K^{(i)}P^{(i)} & G^{(i)}K^{(i,\mathcal{F})} & G^{(i)}K^{(i)}{G^{(i)}}^T \end{bmatrix},$$

which operates to $[\vec{x}^{(i)T}\ \ \vec{x}_i^{(\mathcal{F})T}\ \ \vec{x}_i^{(0)T}]^T \in (V^{(i)} \times Q^{(i)}) \oplus \mathbb{R}^{N_Y^{(i)}+N_X^{(i)}} \oplus \mathbb{R}^7$.

3.2. Local and global Schur complement matrices. Assuming regularity of $P^{(i)}K^{(i)}P^{(i)}$ on $V^{(i)} \times Q^{(i)}$, we can define local Schur complement,

$$\begin{aligned} S^{(i)} :=& \begin{bmatrix} K_i^{(\mathcal{F})} & K^{(\mathcal{F},i)}{G^{(i)}}^T \\ K^{(\mathcal{F},i)}{G^{(i)}}^T & G^{(i)}K^{(i)}{G^{(i)}}^T \end{bmatrix} \\ &- \begin{bmatrix} K^{(\mathcal{F},i)} \\ G^{(i)}K^{(i)} \end{bmatrix} P^{(i)}[(P^{(i)}K^{(i)}P^{(i)})^\dagger]P^{(i)} \begin{bmatrix} K^{(i,\mathcal{F})} & K^{(i)}{G^{(i)}}^T \end{bmatrix}, \end{aligned}$$

which operates to $[\vec{x}_i^{(\mathcal{F})T}\ \ \vec{x}_i^{(0)T}]^T \in \mathbb{R}^{N_Y^{(i)}+N_X^{(i)}} \oplus \mathbb{R}^7$. Here $\dagger$ means the pseudo inverse (the Moore-Penrose inverse) operator. This assumption of regularity is satisfied in the case of the discretized Stokes equation by the stabilized finite element method of penalty type, which is shown in the next section.

Let $\Lambda^{(0)} := \{1, 2, \cdots, 7D\}$ be an index set of unknowns of rigid body movements and pressure constant $(\vec{u}^{(0)}, \vec{p}^{(0)}) \in \mathbb{R}^{6D+D}$, and $\tilde{R}^{(i)}$ be a restriction operator from $\mathbb{R}^{N_Y^{(\mathcal{F})}+N_X^{(\mathcal{F})}} \oplus \mathbb{R}^{7D}$ onto $\mathbb{R}^{N_Y^{(\mathcal{F}),i}+N_X^{(\mathcal{F}),i}} \oplus \mathbb{R}^7$. We finally define Schur complement system to (2.6) by

$$S := \tilde{P}\left(\sum_{1\le i\le D} \tilde{R}^{(i)T} S^{(i)} \tilde{R}^{(i)}\right)\tilde{P}.$$

Here $\tilde{P}$ is an orthogonal projection from $\mathbb{R}^{N_Y^{(\mathcal{F})}+N_X^{(\mathcal{F})}} \oplus \mathbb{R}^{7\,D}$ onto $\vec{W}$ which is defined by

$$\vec{W} := \left\{ \begin{bmatrix} \vec{u}^{(\mathcal{F})} \\ \vec{p}^{(\mathcal{F})} \\ \vec{u}^{(0)} \\ \vec{p}^{(0)} \end{bmatrix} \in \mathbb{R}^{N_Y^{(\mathcal{F})}+N_X^{(\mathcal{F})}+6\,D+D}\,;\ (\vec{u}^{(\mathcal{F})}, \vec{n}_\mu) = 0\ (\mu \in \Lambda_\Gamma)\,, \right.$$
$$(\vec{u}^{(\mathcal{F})}, R_Y^{(\mathcal{F})}\vec{m}_k) + \sum_{1\le i\le D}\sum_{1\le l\le 6} (\vec{\tilde{m}}_l^{(i)}, R_Y^{(\mathcal{F})}\vec{m}_k)[\vec{u}^{(0)}]_{6i+l} = 0\ (1 \le k \le 6)\,,$$
$$\left. (\vec{p}^{(\mathcal{F})}, R_X^{(\mathcal{F})}\vec{m}_0) + \sum_{1\le i\le D} (\vec{\tilde{m}}_0^{(i)}, R_X^{(\mathcal{F})}\vec{m}_0)[\vec{p}^{(0)}]_i = 0 \right\}.$$

Remark 3. Since the Schur complement matrix S is symmetric and regular on $\vec{W}$, a preconditioned conjugate gradient (PCG) method can be employed. We note that S is indefinite and a Balancing Neumann-Neumann preconditioner which is constructed in the similar manner to elasticity problems is also indefinite. In the iterative procedure, PCG generates approximate solutions in Krylov subspaces with initial residual in the space of both velocity and pressure unknowns not within so-called "benign space"[4].

3.3. Regularity of Dirichlet subproblems and direct factorization solver. Since $A^{(i)}$ is defined with inner nodes of $\Omega^{(i)}$, it corresponds to homogeneous Dirichlet boundary data. The stabilized matrix $D^{(i)}$ has the same property of pressure Poisson equation.

Proposition 2. *We have*

$$A^{(i)} \text{ is symmetric and positive definite on } \mathbb{R}^{N_Y^{(i)}}\,,$$
$$D^{(i)} \text{ is symmetric and positive definite on } \mathbb{R}^{N_X^{(i)}}\,.$$

We obtain the regularity of the local stiffness matrix as a corollary of Lemma 1.

Corollary 1. $K^{(i)}$ *is regular on* $V^{(i)} \times Q^{(i)}$.

Advantage of iterative substructuring method in computational costs is that the algorithm allows to use combination of fast direct solver for the subproblems whose size are small, and fast iterative solver with preconditioner for unknowns on the artificial interface. We will show a block-wise LDL$^{\mathrm{T}}$ factorization can be used to solve the subproblem to find $\vec{x}^{(i)} \in V^{(i)} \times Q^{(i)}$ satisfying

$$P^{(i)}K^{(i)}P^{(i)}\vec{x}^{(i)} = P^{(i)}\vec{g}^{(i)}\,. \tag{3.1}$$

The subproblem (3.1) equals to the problem with Lagrange multiplier $\vec{\lambda}$, to find $(\vec{x}^{(i)}, \vec{\lambda}) \in \mathbb{R}^{N_Y^{(i)}+N_X^{(i)}} \times \mathbb{R}^7$ satisfying

$$\begin{bmatrix} K^{(i)} & G^{(i)T} \\ G^{(i)} & 0 \end{bmatrix} \begin{bmatrix} \vec{x}^{(i)} \\ \vec{\lambda} \end{bmatrix} = \begin{bmatrix} \vec{g}^{(i)} \\ 0 \end{bmatrix}. \tag{3.2}$$

To proceed a block-wise LDL$^{\mathrm{T}}$ factorization, we first introduce re-ordering of indices of $\Lambda_Y^{(i)} \oplus \Lambda_X^{(i)}$. Re-ordered index is defined as

$$\hat{\Lambda}^{(i)} := \{\alpha = [\alpha_0,\, \alpha_1], \alpha_0 \in \Lambda_X^{(i)}, 0 \le \alpha_1 \le 3\}\,,$$

where 0-th index of the component is for the pressure. Let $\hat{K}^{(i)}$ be the re-ordered matrix of $K^{(i)}$ with $\hat{\Lambda}^{(i)}$ and

$$\bar{G}^{(i)T} := [\hat{m}_0^{(i)}, \hat{m}_1^{(i)}, \hat{m}_2^{(i)}, \hat{m}_3^{(i)}] \,,$$
$$\hat{G}^{(i)T} := [\hat{m}_0^{(i)}, \hat{m}_1^{(i)}, \hat{m}_2^{(i)}, \hat{m}_3^{(i)}, \hat{m}_4^{(i)}, \hat{m}_5^{(i)}, \hat{m}_6^{(i)}] \,.$$

THEOREM 1. *Re-ordered subproblem to find* $(\vec{\hat{x}}^{(i)}, \vec{\lambda}, \lambda_0) \in \mathbb{R}^{N_Y^{(i)}+N_X^{(i)}} \times \mathbb{R}^8$ *satisfying*

$$\begin{bmatrix} \hat{K}^{(i)} & \hat{G}^{(i)T} & 0 \\ \hat{G}^{(i)} & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \vec{\hat{x}}^{(i)} \\ \vec{\lambda} \\ \lambda_0 \end{bmatrix} = \begin{bmatrix} \vec{\hat{g}}^{(i)} \\ 0 \\ 0 \end{bmatrix} \tag{3.3}$$

can be solved by 4×4 *block-LDL*$^{\mathrm{T}}$ *factorization. Here, the last column and the last row are added to make the matrix size be* $4 \times (N_X^{(i)} + 2)$ *and* λ_0 *is a dummy variable.*

Proof. Let R_m be a restriction from $\mathbb{R}^{N_Y^{(i)}+N_X^{(i)}}$ onto $\mathbb{R}^{4m}$ for $m = 1, 2, \cdots, N_X^{(i)}$. From Lemma 1, we have $R_m \hat{K}^{(i)} R_m^T$ is regular on $\mathbb{R}^{4m}$ that is equivalent to $\hat{K}^{(i)}$ is regular on $\mathrm{Ker} R_m$. For the $(N_X^{(i)} + 1)$-th step of factorization, we deal with a linear system with constraint matrix $\bar{G}^{(i)}$,

$$\begin{bmatrix} \hat{K}^{(i)} & \bar{G}^{(i)T} \\ \bar{G}^{(i)} & 0 \end{bmatrix} .$$

This system can be factorized by 4×4 block because $\hat{K}^{(i)}$ is regular on $\mathrm{Ker}\bar{G}^{(i)}$. At the finial stage, $\hat{K}^{(i)}$ is also regular on $\mathrm{Ker}\hat{G}^{(i)}$. □

4. Balancing Neumann-Neumann preconditioner. The balancing Neumann-Neumann preconditioner [3] is known as an optimal preconditioner for iterative substructuring method for elasticity problems. In the balancing preconditioner, a coarse space is added to local Neumann-Neumann preconditioner for global communication among subdomains. This preconditioner is extended to the Stokes equations with P2/P0-discontinuous pressure element by Pavarino and Widlund [4]. However, in the case of the Stokes equations, supplementary inf-sup condition should be satisfied in the coarse space. In our method using P1/P1 element with stabilizing technique, Schur complement matrix is regular on a coarse space which is generated in the same way of elasticity problems. We can directly apply PCG method to the Schur complement system with Balancing Neumann-Neumann preconditioner which is also indefinite.

4.1. Feasibility of algorithm. We construct a coarse space from rigid body modes and pressure constant with weight of mass matrices and a partition of unity [7].

$$\vec{Z}^{(i)} := \mathrm{span} \left[\left\{ \begin{array}{c} \vec{m}_k^{(\mathcal{F}),i} \\ \left[\{ (\vec{\tilde{m}}_l^{(i)}, \vec{m}_k^{(i)}) \}_{0 \le l \le 6} \right] \end{array} \right\}_{0 \le k \le 6} \right] \;(\, 1 \le i \le D \,) \,,$$
$$\vec{Z} := \left\{ \; \vec{z} \in \mathbb{R}^{N_Y^{(\mathcal{F})} + N_X^{(\mathcal{F})}} \oplus \mathbb{R}^{7\,D} \,;\, \vec{z} = \sum_{1 \le i \le D} \tilde{R}^{(i)T} \tilde{D}^{(i)} \vec{z}^{(i)}, \;\; \vec{z}^{(i)} \in \vec{Z}^{(i)} \; \right\} .$$

Here $\tilde{D}^{(i)} = \begin{bmatrix} D^{(i)} & 0 \\ 0 & I_7 \end{bmatrix}$ and $\{D^{(i)}\}_{1\le i\le D}$ is a partition of unity corresponding to the domain decomposition. A coarse space is defined by $\vec{Y} := \vec{Z} \cap \vec{W}$. Let P_Y be an orthogonal projection from $\mathbb{R}^{N_Y^{(\mathcal{F})}+N_X^{(\mathcal{F})}} \oplus \mathbb{R}^{7\,D}$ onto $\vec{Y}$ and $S_0 := P_Y S P_Y$. We have regularity properties of local Schur matrix $S^{(i)}$ on $\vec{Z}^{(i)}$ and of global Schur complement matrix S_0 on $\vec{Y}$.

LEMMA 2. *Local Schur matrix $S^{(i)}$ is regular on $\vec{Z}^{(i)\perp}$.*

This is obtained from a property of local Neumann problem,

$$\begin{bmatrix} K^{(i)} & K^{(i,\mathcal{F})} \\ K^{(\mathcal{F},i)} & K_i^{(\mathcal{F})} \end{bmatrix} \text{ is regular on span} \left\{ \begin{bmatrix} \vec{m}_k^{(i)}) \\ \vec{m}_k^{(\mathcal{F}),i} \end{bmatrix} \right\}^{\perp} .$$

REMARK 4. Local Neumann problem is expressed as to find $(\vec{x}^{(i)}, \vec{x}^{(\mathcal{F}),i}, \vec{\lambda}) \in \mathbb{R}^{N_Y^{(i)}+N_X^{(i)}} \times \mathbb{R}^{N_Y^{(\mathcal{F}),i}+N_X^{(\mathcal{F}),i}} \times \mathbb{R}^7$ satisfying

$$\begin{bmatrix} K^{(i)} & K^{(i,\mathcal{F})} & G^{(i)T} \\ K^{(\mathcal{F},i)} & K_i^{(\mathcal{F})} & G^{(\mathcal{F}),iT} \\ G^{(i)} & G^{(\mathcal{F}),i} & 0 \end{bmatrix} \begin{bmatrix} \vec{x}^{(i)} \\ \vec{x}^{(\mathcal{F}),i} \\ \vec{\lambda} \end{bmatrix} = \begin{bmatrix} 0 \\ \vec{g}^{(\mathcal{F}),i} \\ 0 \end{bmatrix} . \tag{4.1}$$

We can also employ the 4×4 block-LDL$^{\mathrm{T}}$ factorization as Dirichlet local problem.

THEOREM 2. *S_0 is regular on $\vec{Y}$.*

This follows from the fact that the total Stokes stiffness matrix is regular on $(\vec{V}_1 \times \vec{Q}_1) \oplus (\vec{V}_2 \times \vec{Q}_2) \oplus \cdots \oplus (\vec{V}_D \times \vec{Q}_D) \oplus \vec{Y}$, which is obtained from Lemma 1.

Finally we can define a balancing Neumann-Neumann preconditioner Q_{BNN} which operates to $[\vec{x}^{(\mathcal{F})T}\ \vec{x}^{(0)T}]^T \in \vec{W}$.

$$Q_{BNN} := \tilde{P}(I - S_0^{\dagger} S) \left(\sum_{1\le i\le D} \tilde{R}^{(i)T} \tilde{D}^{(i)} S^{(i)\dagger} \tilde{D}^{(i)} \tilde{R}^{(i)} \right) (I - S\, S_0^{\dagger}) \tilde{P} + S_0^{\dagger} .$$

Here $S^{(i)\dagger}$ is a pseudo-inverse of $S^{(i)}$ on $\vec{Z}^{(i)}$ and $S_0^{\dagger}$ is a pseudo-inverse of S_0 on $\vec{Y}$.

4.2. Numerical results. We consider the Stokes problem in the spherical domain with $R_1 = 0.5$ and $R_2 = 1$. Fig. 4.1 shows the domain and way of domain decomposition into a union of 8 subdomains, where one subdomain is removed to show domain decomposition. We prepared mesh subdivisions with three different mesh sizes, and observed number of iterations for convergence of approximate solution with relative residual less than 10^{-6}. Table 4.1 shows meshes, degrees of freedom, number of iterations of preconditioned CG method to the original linear system (2.6) with incomplete modified Cholesky method (incomplete LDL$^{\mathrm{T}}$-factorization) as the preconditioner, and number of iterations of iterative substructuring method with the balancing Neumann-Neumann preconditioner. While number of iterations of ICCG increases with the rate of $O(1/h)$ as the elasticity problem, number of iterations of iterative substructuring method increases slowly.

5. Concluding remarks. We proposed an iterative substructuring method with a balancing Neumann-Neumann preconditioner for the discretized Stokes equations with penalty type stabilization. Solvability of the Schur complement matrix on the coarse space is ensured by the property of the stiffness matrix of the Stokes equations

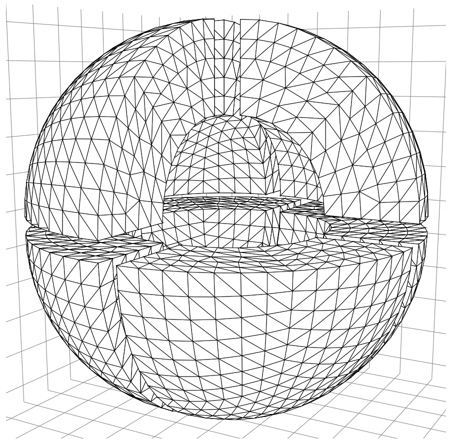

Fig. 4.1. *Finite element mesh and domain decomposition with 8 subdomains*

Table 4.1
Number of iterations of ICCG and iterative substructuring solver

mesh	h	D.O.F.			ICCG	iterative
		velocity		pressure		subtstucturing
a	0.2056	12,512	+	4,692	130	13
b	0.1081	96,134	+	34,126	194	13
c	0.0556	754,922	+	259,962	431	16

with stabilization. Convergence of the iterative solver is verified by a preliminary numerical experiment. Efficient implementation of the coarse space solver and evaluation of the computational cost on parallel computers are future works.

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