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DISCRETE MAXIMUM PRINCIPLE FOR PRISMATIC FINITE ELEMENTS*

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Abstract. The paper deals with a diffusion-reaction problem with homogeneous Dirichlet boundary conditions and presents conditions for the prismatic finite element meshes which guarantee the validity of the corresponding discrete maximum principle (DMP). These conditions are easy to verify and they imply a sufficient and a necessary bound to the maximal angle $\alpha_{\max}^{(T)}$ in the triangular base T of a prism. The sufficient condition is $\alpha_{\max}^{(T)} \leq \arctan\sqrt{7}$ and the necessary condition is $\alpha_{\max}^{(T)} \leq \arctan\sqrt{8}$. If the maximal angle is in between these two values then the other angles in the triangle play a role.

Key words. prismatic finite elements, diffusion-reaction problem, discrete maximum principle

AMS subject classifications. 65N30, 65N50, 35B50, 35K57

1. Introduction. Many physical quantities, for example temperature, concentration, density, etc., are naturally nonnegative. These quantities are usually modeled by second-order partial differential equations and the nonnegativity of their solutions is guaranteed by the maximum principle [5, 10]. A natural question is whether an approximate solution of the original equations satisfies the maximum principle. We speak about the discrete maximum principles (DMP).

Especially in the context of the finite element method, the DMP is not automatically satisfied. Various conditions for the validity of the DMP are studied for several decades, see e.g. [2, 9, 14]. Sufficient conditions for DMP are often formulated as certain geometric limitations on the finite element meshes. For example, maximum angle conditions for simplicial elements, nonnarrowness conditions for rectangular elements, Delaunay triangulations, etc., see e.g. [1, 3, 4, 8, 12, 15].

The DMP on meshes consisting of right triangular prisms was studied very recently in [6] for the elliptic problems and in [7] for the parabolic problems. In this contribution we continue in the analysis from [6] and derive explicit angle conditions for the DMP in the case of prismatic elements.

The paper is organized as follows. Sections 2 and 3 briefly define the model problem, its weak formulation, and its discretization by the finite elements. The main purpose of these sections is to introduce the notation. Section 4 describes the prismatic finite elements and the corresponding shape functions. The global and local (element) matrices are defined in Section 5 and their entries are computed in Section 6 for a line segment, a triangle, and a prism. An interesting tensor product structure of these matrices is presented, too. Finally, Section 7 recalls the main result from [6] and Section 8 presents the sufficient and the necessary angle conditions. Section 9 provides the conclusions.

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2. Model problem. Let $\Omega \subset \mathbb{R}^d$, $d \geq 1$, be a domain with Lipschitz boundary. We consider the following reaction-diffusion boundary value problem

$$-\Delta u + cu = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \tag{2.1}$$

where Δ denotes the Laplacian. To simplify the exposition, we consider the reaction coefficient $c \geq 0$ to be constant. This is a technical simplification only. The analysis of the DMP for (2.1) with nonconstant c can be done equally well, see [6].

In order to disretize problem (2.1) by the finite element method, we introduce its weak formulation: find $u \in V$ such that

$$a(u, v) = (f, v)_\Omega \quad \forall v \in V, \tag{2.2}$$

where $V = H^1_0(\Omega)$ is the Sobolev space $W^{1,2}_0(\Omega)$ of functions with zero traces on the boundary $\partial\Omega$, the right-hand side f is considered in $L^2(\Omega)$, the bilinear form $a : V \times V \mapsto \mathbb{R}$ and the inner product in $L^2(\Omega)$ are defined as

$$a(u, v) = (\nabla u, \nabla v)_\Omega + c(u, v)_\Omega, \quad u, v \in V,$$
$$(u, v)_\Omega = \int_\Omega uv \, \mathrm{d}x,$$

respectively, and ∇ stands for the gradient. Due to the Lax-Milgram lemma a unique weak solution $u \in V$ of (2.2) exists .

3. Finite element discretization. Let $V_h \subset V$ be a finite dimensional subspace. Then the finite element solution $u_h \in V_h$ is defined by requirement

$$a(u_h, v_h) = (f, v_h)_\Omega \quad \forall v_h \in V_h. \tag{3.1}$$

Again the existence and uniqueness of $u_h \in V_h$ is guaranteed by the Lax-Milgram lemma.

Let $\Phi_1, \Phi_2, \ldots, \Phi_N$ form a basis of V_h. Then the finite element solution u_h can be expressed as

$$u_h = \sum_{j=1}^{N} y_j \Phi_j.$$

The coefficients $\mathbf{y} = (y_1, y_2, \ldots, y_N)^\top$ are uniquely determined by the system of linear algebraic equations

$$\mathbf{A}\mathbf{y} = \mathbf{F}, \tag{3.2}$$

where the *finite element matrix* $\mathbf{A} \in \mathbb{R}^{N\times N}$ and the *load vector* $\mathbf{F} \in \mathbb{R}^N$ have entries

$$A_{ij} = a(\Phi_j, \Phi_i) \quad \text{and} \quad F_i = (f, \Phi_i)_\Omega, \quad i, j = 1, 2, \ldots, N. \tag{3.3}$$

4. Prismatic finite elements. In this paper, we concentrate on the discretization based on the right triangular prisms. Therefore, we consider the domain Ω to be three-dimensional, $d = 3$. Furthermore, we assume that it can be partitioned (face-to-face) into right triangular prisms. We denote such a partition by $\mathcal{T}_h$ and call it *prismatic mesh* or *prismatic partition.* Each element of $\mathcal{T}_h$ is a right triangular prism $P = T \times I$, where T is a triangle and I a line segment, see Fig. 4.1. A typical domain

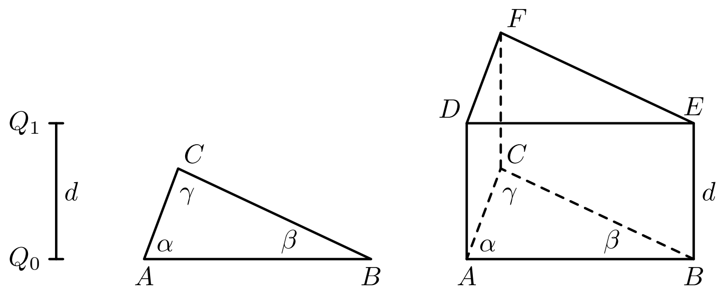

FIG. 4.1. *The notation for a line segment, a triangle, and a prism.*

for which a prismatic partition exists is a cylindrical domain $\Omega = \mathcal{G} \times \mathcal{I}$ where $\mathcal{G} \subset \mathbb{R}^2$ is a polygon and $\mathcal{I} \subset \mathbb{R}$ a line segment. The prismatic partition $\mathcal{T}_h$ of Ω can be then constructed as the Cartesian product of a triangulation of $\mathcal{G}$ and a partition of $\mathcal{I}$. However, we remark that the domain Ω can be much more complicated. In general, it can be a finite union of cylindrical domains.

The finite element space $V_h \subset H_0^1(\Omega)$ associated to $\mathcal{T}_h$ is defined in this case as follows:

$$V_h = \Big\{ \varphi \in H_0^1(\Omega) : \ \varphi(x,y,z)|_P = \sum_{i=1}^{3} \sum_{j=1}^{2} \sigma_{ij} \lambda_i(x,y) \ell_j(z), \text{ where } P \in \mathcal{T}_h,$$
$$P = T \times I, \ \sigma_{ij} \in \mathbb{R}, \ \lambda_i \in \mathbb{P}^1(T), \ \ell_j \in \mathbb{P}^1(I) \Big\},$$

where $\mathbb{P}^1(T)$ and $\mathbb{P}^1(I)$ stand for the spaces of linear functions defined in the triangle T and in the interval I, respectively.

The dimension N of V_h is equal to the number of interior nodes in $\mathcal{T}_h$. The interior nodes $B_1, B_2, \dots, B_N$ in $\mathcal{T}_h$ are those vertices of prisms $P \in T_h$ which do not lie on $\partial\Omega$. Each prismatic finite element basis function Φ_i corresponds to an interior node B_i and it is uniquely given by the following delta property

$$\Phi_i(B_j) = \delta_{ij}, \quad i, j = 1, 2, \dots, N,$$

where δ_{ij} is the Kronecker delta.

Thus, a basis function Φ_i restricted to a prism $P = T \times I$, $P \in \mathcal{T}_h$, either vanishes (if B_i is not a vertex of P) or it is a product of a linear function $\lambda(x,y)$ on a triangle T and a linear function $\ell(z)$ on a line segment I. Let us discuss this structure of basis functions in more detail.

In general, the restriction of a finite element basis function to an element is called the *shape function*. Here, we recall the standard shape functions on a line segment I and on a triangle T, see Fig. 4.2. If I is a line segment with the end points Q_0 and Q_1 then there are two linear shape functions $\ell_0 = \ell_0(z)$ and $\ell_1 = \ell_1(z)$ such that $\ell_i(Q_j) = \delta_{ij}$, $i, j = 0, 1$.

Similarly, in the case of the triangle T, there are three linear shape functions $\lambda_A = \lambda_A(x,y), \dots, \lambda_C = \lambda_C(x,y)$. These functions have the similar delta property like the shape functions ℓ_0 and ℓ_1 and they coincide with the barycentric coordinates on the triangle T.

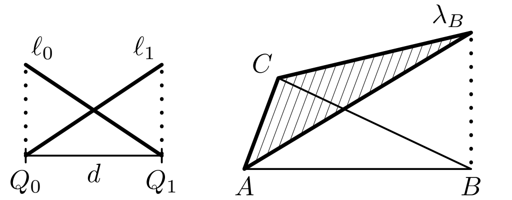

FIG. 4.2. *The shape functions ℓ_0 and ℓ_1 on a line segment (left) and the shape functions λ_B on a triangle (right).*

5. Global and local finite element matrices. The finite element matrix $\mathbf{A}$, see (3.3), can be constructed using contributions from individual elements

$$A_{ij} = a(\Phi_j, \Phi_i) = \sum_{P \in \mathcal{T}_h} a_P(\Phi_j, \Phi_i),$$

where

$$a_P(u, v) = (\nabla u, \nabla v)_P + c(u, v)_P \quad \text{with} \quad (u, v)_P = \int_P uv \, \mathrm{d}x.$$

This can be written in a matrix form as

$$\mathbf{A} = \sum_{P \in \mathcal{T}_h} \bar{\mathbf{A}}^{(P)}, \tag{5.1}$$

where the *local matrix* (or sometimes *element matrix*) $\bar{\mathbf{A}}^{(P)} \in \mathbb{R}^{N \times N}$ has entries $\bar{A}^{(P)}_{ij} = a_P(\Phi_j, \Phi_i)$.

In the case of prismatic elements, the local N-by-N matrix $\bar{\mathbf{A}}^{(P)}$ contains (at most) 6×6 nonzero entries, hence, it can be condensed to a 6-by-6 local matrix $\mathbf{A}^{(P)}$. The particular positions of the nonzero entries in the (rarefied) matrix $\bar{\mathbf{A}}^{(P)}$ is determined by the topology of the mesh and by the enumeration of the nodes. However, it is not a goal of this paper to discuss this issue.

Since the bilinear forms a and a_P are composed of two terms, we split the finite element matrix $\mathbf{A}$ into two contributions

$$\mathbf{A} = \mathbf{S} + c\mathbf{M},$$

where the entries of the *stiffness* and *mass matrices* $\mathbf{S}$ and $\mathbf{M}$ are $S_{ij} = (\nabla\Phi_j, \nabla\Phi_i)_\Omega$ and $M_{ij} = (\Phi_j, \Phi_i)_\Omega$, $i, j = 1, 2, \ldots, N$, respectively. Similarly, we split the local $\mathbf{S}^{(P)}$ and $\mathbf{M}^{(P)}$ matrices

$$\bar{\mathbf{A}}^{(P)} = \bar{\mathbf{S}}^{(P)} + c\bar{\mathbf{M}}^{(P)},$$

where the entries of the local stiffness and mass matrices $\bar{\mathbf{S}}^{(P)}$ and $\bar{\mathbf{M}}^{(P)}$ are given by

$$\bar{S}^{(P)}_{ij} = (\nabla\Phi_j, \nabla\Phi_i)_P \quad \text{and} \quad \bar{M}^{(P)}_{ij} = (\Phi_j, \Phi_i)_P, \quad i, j = 1, 2, \ldots, N.$$

The same applies for the condensed local matrices: $\mathbf{A}^{(P)} = \mathbf{S}^{(P)} + c\mathbf{M}^{(P)}$.

In practice, the global finite element matrix $\mathbf{A}$ is assembled from the (condensed) local matrices $\mathbf{S}^{(P)}$ and $\mathbf{M}^{(P)}$, see e.g. [11]. In the case of prismatic elements, the matrices $\mathbf{S}^{(P)}$ and $\mathbf{M}^{(P)}$ can be explicitly computed from the local matrices for the line segment element and for the triangle.

6. Local matrices for a segment, a triangle, and a prism. First, let us consider the line segment element I with the end points Q_0 and Q_1 and with the corresponding shape functions ℓ_0 and ℓ_1, see Section 4. Its local stiffness and mass matrices $\mathbf{S}^{(I)}$ and $\mathbf{M}^{(I)}$ are defined by the bilinear forms $(u', v')_I$ and $(u, v)_I$, respectively, where the primes denote the derivatives. Entries of these matrices depend solely on the length $d = |I|$ of I and they can be easily computed:

$$\mathbf{S}^{(I)} = \frac{1}{d}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad \mathbf{M}^{(I)} = \frac{d}{6}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Similarly, the local matrices $\mathbf{S}^{(T)}$ and $\mathbf{M}^{(T)}$ for the triangular element T are defined by the bilinear forms $(\nabla u, \nabla v)_T$ and $(u, v)_T$, respectively. Their entries depend on the area $|T|$ of T and on its angles α, β, γ, see Fig. 4.1. A simple calculation reveals that

$$\mathbf{S}^{(T)} = \frac{1}{2}\begin{pmatrix} \cot\beta + \cot\gamma & -\cot\gamma & -\cot\beta \\ -\cot\gamma & \cot\alpha + \cot\gamma & -\cot\alpha \\ -\cot\beta & -\cot\alpha & \cot\alpha + \cot\beta \end{pmatrix}, \ \mathbf{M}^{(T)} = \frac{|T|}{12}\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

Finally, it is an easy exercise to verify that the local stiffness and mass matrices $\mathbf{S}^{(P)}$ and $\mathbf{M}^{(P)}$ for the prism $P = T \times I$, see Fig. 4.1, are given by the tensor (Kronecker) product of the local matrices for the segment and for the triangle

$$\mathbf{S}^{(P)} = \frac{d}{6}\begin{pmatrix} 2\mathbf{S}^{(T)} & \mathbf{S}^{(T)} \\ \mathbf{S}^{(T)} & 2\mathbf{S}^{(T)} \end{pmatrix} + \frac{1}{d}\begin{pmatrix} \mathbf{M}^{(T)} & -\mathbf{M}^{(T)} \\ -\mathbf{M}^{(T)} & \mathbf{M}^{(T)} \end{pmatrix}, \ \mathbf{M}^{(P)} = \frac{d}{6}\begin{pmatrix} 2\mathbf{M}^{(T)} & \mathbf{M}^{(T)} \\ \mathbf{M}^{(T)} & 2\mathbf{M}^{(T)} \end{pmatrix},$$

or briefly

$$\mathbf{S}^{(P)} = \mathbf{M}^{(I)} \otimes \mathbf{S}^{(T)} + \mathbf{S}^{(I)} \otimes \mathbf{M}^{(T)} \quad \text{and} \quad \mathbf{M}^{(P)} = \mathbf{M}^{(I)} \otimes \mathbf{M}^{(T)}. \tag{6.1}$$

This structure enables to infer the explicit formulas for the entries of the local matrices $\mathbf{S}^{(P)}$ and $\mathbf{M}^{(P)}$. Due to the symmetry, there only are the following four nontrivial possibilities for the entries of $\mathbf{S}^{(P)}$ and $\mathbf{M}^{(P)}$. If

$$\varphi_A(x,y,z) = \lambda_A(x,y)\ell_0(z), \qquad \varphi_B(x,y,z) = \lambda_B(x,y)\ell_0(z),$$
$$\varphi_D(x,y,z) = \lambda_A(x,y)\ell_1(z), \qquad \varphi_E(x,y,z) = \lambda_B(x,y)\ell_1(z),$$

then

$$\int_P |\nabla\varphi_A|^2 \,\mathrm{d}P = \frac{d}{6}\left(\cot\beta + \cot\gamma + \frac{|T|}{d^2}\right), \qquad \int_P \varphi_A^2 \,\mathrm{d}P = \frac{d\,|T|}{18},$$

$$\int_P \nabla\varphi_A \cdot \nabla\varphi_B \,\mathrm{d}P = -\frac{d}{12}\left(2\cot\gamma - \frac{|T|}{d^2}\right), \qquad \int_P \varphi_A\varphi_B \,\mathrm{d}P = \frac{d\,|T|}{36},$$

$$\int_P \nabla\varphi_A \cdot \nabla\varphi_D \,\mathrm{d}P = \frac{d}{12}\left(\cot\beta + \cot\gamma - \frac{2|T|}{d^2}\right), \quad \int_P \varphi_A\varphi_D \,\mathrm{d}P = \frac{d\,|T|}{36},$$

$$\int_P \nabla\varphi_A \cdot \nabla\varphi_E \,\mathrm{d}P = -\frac{d}{12}\left(\cot\gamma + \frac{|T|}{d^2}\right), \qquad \int_P \varphi_A\varphi_E \,\mathrm{d}P = \frac{d\,|T|}{72}.$$

REMARK 1. *If $\Omega = \mathcal{G} \times \mathcal{I}$ is a cylindrical domain, then the interesting tensor product structure* (6.1) *exists on the level of global matrices as well*

$$\mathbf{S} = \mathbf{M}^{(\mathcal{I})} \otimes \mathbf{S}^{(\mathcal{G})} + \mathbf{S}^{(\mathcal{I})} \otimes \mathbf{M}^{(\mathcal{G})} \quad \textit{and} \quad \mathbf{M} = \mathbf{M}^{(\mathcal{I})} \otimes \mathbf{M}^{(\mathcal{G})},$$

where

$$\mathbf{S}^{(\mathcal{I})} = \sum_{I \in \mathcal{T}_h^{\mathcal{I}}} \bar{\mathbf{S}}^{(I)}, \quad \mathbf{M}^{(\mathcal{I})} = \sum_{I \in \mathcal{T}_h^{\mathcal{I}}} \bar{\mathbf{M}}^{(I)},$$
$$\mathbf{S}^{(\mathcal{G})} = \sum_{T \in \mathcal{T}_h^{\mathcal{G}}} \bar{\mathbf{S}}^{(T)}, \quad \mathbf{M}^{(\mathcal{G})} = \sum_{T \in \mathcal{T}_h^{\mathcal{G}}} \bar{\mathbf{M}}^{(T)}$$

are the global stiffness and mass matrices on the partition $\mathcal{T}_h^{\mathcal{I}}$ *of the segment* $\mathcal{I}$ *and on the triangulation* $\mathcal{T}_h^{\mathcal{G}}$ *of the polygon* $\mathcal{G}$*.*

However, this simple structure disappears if Ω *is not cylindrical, e.g., if it contains a hole inside.*

7. Discrete maximum principle. The explicit formulas for the entries of the local stiffness and mass matrices on the prismatic element allow for an analysis of the corresponding *discrete maximum principle* (DMP). The DMP mimics the well-known maximum principle for problem (2.1). In this simple setting the maximum principle is equivalent to the well-known conservation of nonnegativity. Problem (2.1) conserves nonnegativity if any nonnegative right-hand side f provides a nonnegative solution u. Therefore, we say that problem (3.1) satisfies the discrete maximum principle if the finite element solution u_h is nonnegative for all nonnegative right-hand sides f.

Unfortunately, the finite element method does not satisfy the DMP in general. The basis functions $\Phi_1, \Phi_2, \dots, \Phi_N$ are nonnegative and therefore the load vector $\mathbf{F}$ is nonnegative (all its entries are nonnegative) as well. Thus, we immediately see that the corresponding discrete solution u_h of (3.1) is nonnegative if and only if the stiffness matrix $\mathbf{A}$ is nonotone, i.e. if $\mathbf{A}^{-1} \geq 0$, see (3.2).

Monotone matrices are difficult to handle, but their subclass, so-called M-matrices [13], are characterized easily enough by nonpositive off-diagonal entries. Hence, we usually try to find conditions for nonpositivity of the off-diagonal entries in the local finite element matrices $\mathbf{A}^{(P)}$. This implies nonpositive off-diagonal entries of the global finite element matrix $\mathbf{A}$, see (5.1), and it further implies monotonicity of $\mathbf{A}$ and consequently the DMP.

This approach has been widely used for various types of finite elements and for various problems. Recently we analyzed the prismatic finite elements, see [6], and we obtained certain sufficient conditions for the prismatic meshes. We present this result as Theorem 7.2, below.

DEFINITION 7.1. *Let* $P = T \times I$ *be a prism and let* $\alpha_{\max}^{(T)} \geq \alpha_{\text{med}}^{(T)} \geq \alpha_{\min}^{(T)} > 0$ *be the maximal, medium, and minimal angle of the triangular base* T *of the prism* P*, respectively. We define the lower and upper bounds for the altitude of the prism* P *as*

$$d_L^{(P)} = \left(\frac{2 \cot \alpha_{\max}^{(T)}}{|T|} - \frac{c}{3} \right)^{-\frac{1}{2}}, \quad d_U^{(P)} = \left(\frac{c}{6} + \frac{\cot \alpha_{\text{med}}^{(T)} + \cot \alpha_{\min}^{(T)}}{2\,|T|} \right)^{-\frac{1}{2}}. \tag{7.1}$$

The lower bound $d_L^{(P)}$ *is well defined only if* $\dfrac{2 \cot \alpha_{\max}^{(T)}}{|T|} - \dfrac{c}{3} > 0$.

Notice that $\alpha_{\text{med}}^{(T)} < \pi/2$ and $\alpha_{\min}^{(T)} \leq \pi/3$ for any triangle. Thus, $d_U^{(P)}$ is always well defined by (7.1). Without loss of generality, we assume that $d_L^{(P)}$ is well defined in what follows. Consequently, we consider $\alpha_{\max}^{(T)} \leq \pi/2$.

THEOREM 7.2. *Let* $\mathcal{T}_h$ *be a prismatic partition of a domain* Ω*. For a prism* $P \in \mathcal{T}_h$*, let values* $d_L^{(P)}$ *and* $d_U^{(P)}$ *be defined by (7.1), and let* $d^{(P)}$ *denote the altitude* *of the prism P. If*

$$d_L^{(P)} \le d^{(P)} \le d_U^{(P)} \quad \text{for all } P \in \mathcal{T}_h, \tag{7.2}$$

then problem (3.1) *satisfies the DMP.*

Proof. See [6, Theorem 2]. □

The crucial condition (7.2) is easy to verify for a given prismatic partition. However, it is not immediately clear how to construct prismatic partitions which satisfy (7.2). Condition (7.2) tells us how to choose the altitudes $d^{(P)}$ of the prisms, but the values $d_L^{(P)}$ and $d_U^{(P)}$ depends on the angles in the triangular base of the prism. Not all combinations of these angles yield $d_L^{(P)} \le d_U^{(P)}$. Therefore, we formulate a sufficient and a necessary conditions in terms of the maximal angle $\alpha_{\max}^{(T)}$ in the following section. These conditions provide a certain aid how to construct suitable prismatic partitions satisfying (7.2).

8. Angle conditions. In [6], we formulated certain limitations to the maximal and minimal angles in the triangular bases of prisms. These limitations are necessary for the DMP in a cylindrical domain. In this section, we generalize this result and in Lemmas 8.1 and 8.2 we provide a sufficient and a necessary angle condition. These conditions are not restricted to the partition of a cylindrical domain and apply in general.

LEMMA 8.1. *Let $0 < \gamma \le \beta \le \alpha$, $\alpha + \beta + \gamma = \pi$ be the angles in a triangle T. If*

$$\alpha \le \arctan \sqrt{7} \approx 69.2952^\circ \tag{8.1}$$

then an altitude $d = |I|$ of a prism $P = T \times I$ exists such that condition (7.2) *with $c = 0$ is satisfied.*

Proof. Let us set $\alpha = \alpha_{\max}^{(T)}$, $\beta = \alpha_{\mathrm{med}}^{(T)}$, $\gamma = \alpha_{\min}^{(T)}$ and write condition (7.2) equivalently as

$$\frac{c}{6}|T| + \frac{\cot\beta + \cot\gamma}{2} \le \frac{|T|}{(d^{(P)})^2} \le 2\cot\alpha - \frac{c}{3}|T|. \tag{8.2}$$

Since we consider $c = 0$, our goal is to show that

$$\cot\beta + \cot\gamma \le 4\cot\alpha. \tag{8.3}$$

Substituting $\beta = \pi - \alpha - \gamma$, applying the standard trigonometric indentity

$$\cot(\pi - \alpha - \gamma) = \frac{1 - \cot\alpha\cot\gamma}{\cot\alpha + \cot\gamma},$$

and using the short-hand notation $\mathcal{A} = \cot\alpha$ and $\mathcal{C} = \cot\gamma$, we can rewrite (8.3) equivalently as

$$4\mathcal{A}^2 + 4\mathcal{A}\mathcal{C} - 1 - \mathcal{C}^2 \ge 0. \tag{8.4}$$

The validity of (8.4) can be shown with the aid of three inequalities. First, we immediately see that the assumption (8.1) is equivalent to

$$\mathcal{A} \ge 1/\sqrt{7}. \tag{8.5}$$

Second, since the smallest angle in a triangle is at most $\pi/3$ we have

$$\mathcal{C} \geq 1/\sqrt{3} > 1/\sqrt{7}. \tag{8.6}$$

Third, the upper bound (8.1) for the largest angle α yields the following lower bound to the smallest angle γ

$$\gamma = \pi - \alpha - \beta \geq \pi - 2\alpha.$$

This is equivalent to

$$\cot\gamma \leq \cot(\pi - 2\alpha) = \frac{1 - \cot^2\alpha}{2\cot\alpha}.$$

Hence, using (8.5), we can estimate

$$\mathcal{C} \leq \frac{1 - \mathcal{A}^2}{2\mathcal{A}} \leq \frac{3}{\sqrt{7}}. \tag{8.7}$$

Now, the validity of (8.4) follows from (8.5), (8.6), and (8.7):

$$4\mathcal{A}^2 + 4\mathcal{A}\mathcal{C} - 1 - \mathcal{C}^2 \geq -\mathcal{C}^2 + \frac{4}{\sqrt{7}}\mathcal{C} - \frac{3}{7} = \left(\mathcal{C} - \frac{1}{\sqrt{7}}\right)\left(\frac{3}{\sqrt{7}} - \mathcal{C}\right) \geq 0.$$

□

LEMMA 8.2. *Let $P = T \times I$ be a prism and let $0 < \gamma \leq \beta \leq \alpha$, $\alpha + \beta + \gamma = \pi$ be the angles in the triangular base T of P. If*

$$\alpha > \arctan\sqrt{8} \approx 70.5288^\circ \tag{8.8}$$

then condition (7.2) *is not satisfied.*

Proof. Since we assume that $d_L^{(P)}$ is well defined, we have $\alpha \leq \pi/2$. Our aim is to prove the following inequality

$$\cot\beta + \cot\gamma > 4\cot\alpha. \tag{8.9}$$

Indeed, if (8.9) holds then condition (7.2) cannot be satisfied for any value of c, $d^{(P)}$, and $|T|$, cf. (8.2). If we set $\beta = \pi - \alpha - \gamma$, then we can express (8.9) equivalently as

$$4\mathcal{A}^2 + 4\mathcal{A}\mathcal{C} - 1 - \mathcal{C}^2 < 0, \tag{8.10}$$

where we again use the short-hand notation $\mathcal{A} = \cot\alpha$ and $\mathcal{C} = \cot\gamma$, cf. (8.4).

To prove (8.10), we first deduce from (8.8) that $\mathcal{A} < 1/\sqrt{8}$, which implies inequality (8.10) as follows

$$4\mathcal{A}^2 + 4\mathcal{A}\mathcal{C} - 1 - \mathcal{C}^2 < -\mathcal{C}^2 + \sqrt{2}\mathcal{C} - \frac{1}{2} = -\left(\mathcal{C} - \frac{\sqrt{2}}{2}\right)^2 \leq 0.$$

□

If the maximal angle in all triangular bases of all prisms in the partition of Ω satisfies (8.1) then there is a chance that the altitudes of all the prisms satisfy (7.2). On the other hand if this maximal angle is greater than $\arctan\sqrt{8}$, see (8.8), then condition (7.2) cannot be satisfied.

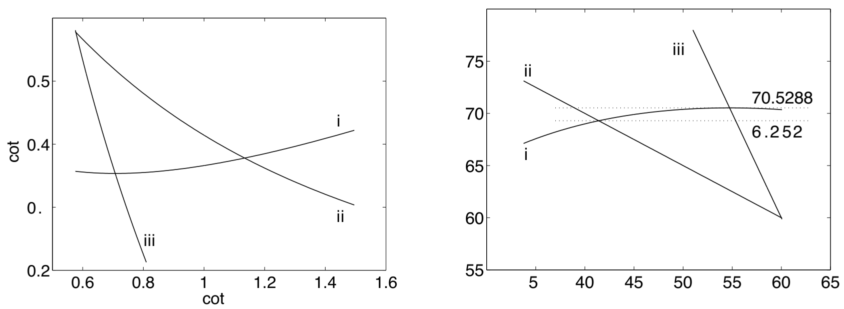

FIG. 8.1. *The graphical representation of inequalities (i)* $d_L^{(P)} \le d_U^{(P)}$*, (ii)* $\beta \le \alpha$*, and (iii)* $\gamma \le \beta$*, where* $c = 0$*, and* $\gamma \le \beta \le \alpha$ *are angles in a triangle. The three curves are given by equalities in (i)–(iii). The symbol (i), (ii), or (iii) lies always on the side of the curve, where the corresponding inequality is satisfied. The area enclosed by these three curves represents triangles, where (i) is satisfied. The values* $\arctan\sqrt{8} \approx 70.5288°$ *and* $\arctan\sqrt{7} \approx 69.2952°$ *are indicated in the right panel, see Lemmas 8.1 and 8.2.*

To illustrate the limitations given by (7.2) in the case $c = 0$, we present Fig. 8.1, where the horizontal axis represents the smallest angle γ of a triangle (right panel) or $\cot\gamma$ (left panel) and the vertical axis represents the largest angle α (right panel) or $\cot\alpha$ (left panel). In these axes, we illustrate conditions (i) $d_L^{(P)} \le d_U^{(P)}$, (ii) $\beta \le \alpha$, and (iii) $\gamma \le \beta$. Each point in these graphs corresponds to a pair of angles γ and α. This pair represents a triangle if it is between curves (ii) and (iii). This triangle satisfies condition (i) if it is above the curve (i) (left panel) or below the curve (i) (right panel). Hence, conditions (i)–(iii) form a curvilinear triangle in both panels and the points of this curvilinear triangle represent the triangular bases of prisms which satisfy (7.2) for $c = 0$ provided their altitudes are suitably chosen.

The result in Lemma 8.1 is limited to the case $c = 0$. However, the bound (8.1) with the strict inequality applies also in the case $c \ne 0$ in the following sense. If $c \ne 0$ then we may ignore this fact for a while and construct a prismatic partition (with the help of (8.1)) such that condition (7.2) holds with strict inequalities for $c = 0$ for all prisms from this prismatic partition. Then a sufficiently fine refinement of this partition exists such that (7.2) holds for the original value of $c \ne 0$ and for all prisms from the refined partition. For more details see [6].

In [7], we analyze the DMP for parabolic problems discretized by the θ-method in time and by the prismatic finite elements in space. The conditions obtained in the elliptic case for $c \ne 0$ are needed in the parabolic case, too. Consequently Lemmas 8.1 and 8.2 apply in the parabolic case as well.

9. Conclusions. We concentrate on an elliptic diffusion-reaction problem discretized by the standard lowest-order finite elements on prismatic meshes. We present explicit formulas for the local (element) mass and stiffness matrices in Section 6 and we show the tensor structure of the local matrices on prisms. These explicit formulas enable to derive the crucial condition (7.2) for the validity of the DMP.

The conditions for the validity of the DMP can be weaken by the well-know mass lumping technique. In this technique, the mass matrix $\mathbf{M}$ is replaced by a diagonal lumped-mass matrix $\widehat{\mathbf{M}}$, which simplifies both the practical computations and the theoretical analysis of various aspects including the DMP. In the case of diffusion-reaction problem, the mass lumping technique provides the DMP under the same conditions as the standard finite element method does for the pure diffusion problem.

However, condition (7.2) is still crucial for both standard and mass-lumped methods and it does not provide any aid in the construction of suitable prismatic partitions for the DMP. In Lemmas 8.1 and 8.2 we provide explicit bounds on the maximal angle in the triangular bases of the prisms from the partition. These bounds can be utilized for the construction of prismatic partitions yielding the DMP for problem (3.1). As mentioned above, these bounds apply for the case $c \neq 0$ and for the parabolic problems as well, provided condition (8.1) holds with strict inequality.

Nevertheless, there is a small gap between the sufficient bound $\arctan\sqrt{7}$ and the necessary bound $\arctan\sqrt{8}$. If the maximal angle lies between these two values then condition (7.2) and hence the DMP can be satisfied, provided the other angles are chosen in a suitable way, see Fig. 8.1.

An interesting generalization and a goal of a future research is to consider prisms with triangular bases being non-parallel. The finite elements based on these generalized prisms would be more flexible and more general geometries could be handled with them. However, definition of finite element spaces and shape functions on generalized prisms is nontrivial as well as the subsequent analysis of properties of such a discretization.

Let us conclude this paper with an interesting coincidence. The angle $\arctan\sqrt{8} \approx 70.5288^\circ$ obtained in Lemma 8.2 is equal to the dihedral angle between two faces of the regular tetrahedron. It is still unknown if this is just a fortuitous coincidence or if there is a deeper relationship between the prismatic and tetrahedral finite elements.

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