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NUMERICAL APPROXIMATION OF NONLINEAR FLUID-STRUCTURE INTERACTION PROBLEMS*

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Abstract. This paper is concerned with the numerical approximation of nonlinear fluid-structure interaction problems, with a detailed description of numerical approximation of turbulent flow by stabilized finite element method. The interaction of flow with flexibly supported airfoil with aileron is numerically analyzed. The motion of the airfoil is described with the aid of system of ordinary differential equations (ODE) for three degrees of freedom coupled with the Reynolds Averaged Navier-Stokes system of equations completed by the $k-\omega$ turbulence model. The flow and structural problems are coupled via strong coupling algorithm. Numerical results showing the comparison of computation carried out by the $k-\omega$ and Spalart-Allmaras methods are presented.

Key words. nonlinear aeroelasticity, finite element method

AMS subject classifications. 74F10, 76M10

1. Introduction. Fluid-structure interactions are important in many technical applications, cf. [14], where usually only simplified linearized problems are used. The solution of nonlinear problems can be important, e.g., in modeling of post-flutter behaviour. In this paper we address the problem of numerical approximation of turbulent flow interactions with an airfoil, whose motion is described with the aid of three degrees of freedom. An extension of previously published analysis of flow interactions with an airfoil with two and three degrees of freedom (DOF) is considered, see [20], [4]. Similar problem was studied in [23], where the motion of airfoil with 3-DOF was analyzed with the aid of Theodorsen's theory. Further, the analytical and numerical analysis of the aeroelastic response of 3-DOF airfoil was also considered in [18] and also in [13], where an active flutter control methods were considered.

This paper focuses on the numerical modeling of the interactions of incompressible turbulent two-dimensional flow with a flexibly supported airfoil with an aileron. The flow is modeled with the aid of the incompressible Reynolds Averaged Navier-Stokes (RANS) equation. For the approximation the stabilized finite element method (FEM) is used, cf. [2], [5], [8], [9], [12], [16], [17], [21]. The turbulent viscosity is modeled by $k-\omega$ turbulence model, cf. [10], [22]. The $k-\omega$ turbulence model is written in ALE form, time discretized and linearized, and numerically approximated by FEM. The detailed description of such a approximation is given. The structure motion is described by system of ordinary differential equations, see [4]. The time-dependent computational domain is taken into account by the Arbitrary Lagrangian-Eulerian method, see [15], [20]. The applicability of the developed method is demonstrated by numerical experiments comparing the results obtained with this method, Spalart-Allmaras turbulence method and NASTRAN calculation.

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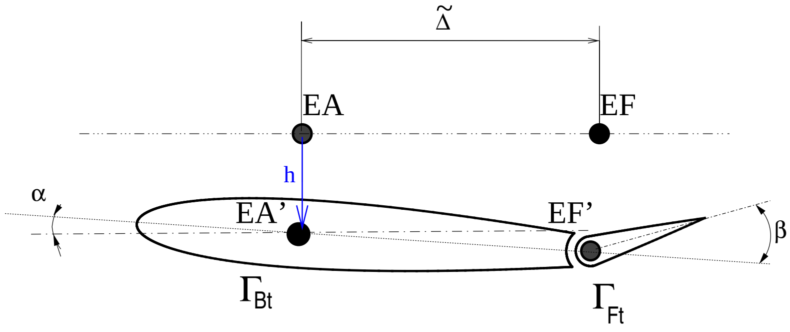


FIG. 2.1. *A sketch of the aeroelastic model of airfoil with aileron in a displaced position given by h, α and β.*

2. Mathematical description.

2.1. Flow model. In what follows we shall be concerned with the approximation of two-dimensional incompressible turbulent flow in time-dependent computational domain $\Omega_t \subset \mathbb{R}^2$, $t \in [0,T]$. In order to treat the time dependence of the domain occupied by fluid, the Arbitrary Lagrangian-Eulerian (ALE) method will be used. We assume that there exists a smooth one-to-one transformation $\mathcal{A}_t$ (*ALE mapping*) of a reference computational domain Ω_0 onto Ω_t for any $t \in [0,T]$. Further, by $\mathbf{w}_D(x,t)$ the *(ALE) domain velocity* and by $D^{\mathcal{A}}/Dt$ the *ALE derivative* shall be denoted, cf. [15], [20], [4]. The flow is modeled by the incompressible RANS equations written in the ALE form:

$$\begin{array}{rcll} \frac{D^{\mathcal{A}}\mathbf{v}}{Dt} - \nabla \cdot (2\nu_{eff}\mathbf{S}(\mathbf{v})) + (\overline{\mathbf{w}} \cdot \nabla)\mathbf{v} + \nabla p &=& 0, & \\ \operatorname{div} \mathbf{v} &=& 0, & \end{array} \quad \text{in } \Omega_t, \tag{2.1}$$

where $\mathbf{v} = (v_1(x,t), v_2(x,t))$ denotes the mean part of the velocity vector, $p = p(x,t)$ denotes the mean part of the kinematic pressure (i.e. pressure divided by the constant fluid density ρ), $\nu_{eff} = \nu + \nu_T$, ν denotes the kinematic viscosity, ν_T is the turbulent viscosity, $\overline{\mathbf{w}} = \mathbf{v} - \mathbf{w}_D$ and $\mathbf{S}(\mathbf{v}) = \frac{1}{2}\left(\nabla\mathbf{v} + \nabla^T\mathbf{v}\right)$. System(2.1) is equipped with the initial condition $\mathbf{v}(x,0) = v^0(x)$, and boundary conditions

$$\begin{array}{lrl} \text{a)} & \mathbf{v}(x,t) = \mathbf{v}_D(x) & \text{for } x \in \Gamma_D, \\ \text{b)} & \mathbf{v}(x,t) = \mathbf{w}_D(x,t) & \text{for } x \in \Gamma_{Wt}, \\ \text{c)} & -p\mathbf{n} + \frac{1}{2}(\mathbf{v}\cdot\mathbf{n})^-\mathbf{v} + \nu\frac{\partial \mathbf{v}}{\partial \mathbf{n}} = 0 & \text{on } \Gamma_O, \end{array} \tag{2.2}$$

considered for $t \in (0,T)$. Here, Γ_D represents the inlet and fixed impermeable walls, Γ_{Wt} is the boundary of the airfoil, Γ_O is the outlet, and $\mathbf{n}$ denotes the unit outward normal to $\partial\Omega_t$.

2.2. Turbulence model. The turbulent viscosity ν_T is determined with the aid of the two-equations turbulence $k - \omega$ model, cf. [22]. This means that the turbulent viscosity ν_T is defined by the relation

$$\nu_T = \frac{k}{\omega} \tag{2.3}$$

where the turbulent kinetic energy $k = k(x,t)$ and the turbulent specific dissipation rate $\omega = \omega(x,t)$ defined for $x \in \overline{\Omega}_t$, $t \in [0,T]$ are solutions of the following initial-boundary value problem (written in the ALE form):

$$\begin{array}{rcl} \frac{D^{\mathcal{A}}k}{Dt} + (\overline{\mathbf{w}} \cdot \nabla)k &=& P_k - \beta^* \omega k + \nabla \cdot (\varepsilon_k \nabla k), \\ \frac{D^{\mathcal{A}}\omega}{Dt} + (\overline{\mathbf{w}} \cdot \nabla)\omega &=& P_\omega - \beta\omega^2 + \nabla \cdot (\varepsilon_\omega \nabla \omega) + C_D. \end{array} \tag{2.4}$$

Here $\varepsilon_k = \nu + \sigma_k \nu_T$, $\varepsilon_\omega = \nu + \sigma_\omega \nu_T$ and σ_k, σ_ω are coefficients given later. The following initial and boundary conditions are considered:

$$\begin{array}{lrclrcll} \text{a)} & k(x,t) &=& 0 & \omega(x,t) &=& \omega_w & \text{for } x \in \Gamma_{Wt}\ t \in (0,T), \\ \text{b)} & k(x,t) &=& k_D & \omega(x,t) &=& \omega_D & \text{for } x \in \Gamma_D\ t \in (0,T), \\ \text{c)} & \frac{\partial k}{\partial n}(x,t) &=& 0 & \frac{\partial \omega}{\partial n}(x,t) &=& 0 & \text{for } x \in \Gamma_O\ t \in (0,T), \end{array} \tag{2.5}$$

where k_D, ω_D and ω_w are prescribed constants used on inlet and walls. The source terms P_k, P_ω and C_D are defined by

$$P_k = \nu_T \mathbf{S}(\mathbf{v}) : \mathbf{S}(\mathbf{v}), \quad P_\omega = \frac{\alpha_\omega \omega}{k} P_k, \quad C_D = \frac{\sigma_D}{\omega} (\nabla k \cdot \nabla \omega)^+.$$

The closure coefficients β, β^*, σ_k, σ_ω, α_ω are chosen according to [10], i.e. the values $\beta = 0.075$, $\beta^* = 0.09$, $\sigma_\omega = 0.5$, $\sigma_k = \frac{2}{3}$, $\kappa = 0.41$, $\sigma_D = 0.5$, $\alpha_\omega = \beta/\beta^* - \sigma_\omega \frac{\kappa^2}{\beta^{*\,1/2}}$ are used.

2.3. Equations of motion. The motion of the airfoil with aileron is described by the system of linear ordinary differential equations (cf. [3])

$$\begin{array}{rcrcrcrcr} m\ddot{h} &+& S_\alpha \ddot{\alpha} &+& S_\beta \ddot{\beta} &+& k_h h &=& -L, \\ S_\alpha \ddot{h} &+& I_\alpha \ddot{\alpha} &+& (\widetilde{\Delta} S_\beta + I_\beta)\ddot{\beta} &+& k_\alpha \alpha &=& M_\alpha, \\ S_\beta \ddot{h} &+& (\widetilde{\Delta} S_\beta + I_\beta)\ddot{\alpha} &+& I_\beta \ddot{\beta} &+& k_\beta \beta &=& M_\beta, \end{array} \tag{2.6}$$

where h is the vertical displacement, α is the angle of rotation of the airfoil around its elastic axis (EA), β is the angle of rotation of the aileron around the aileron axis (EF)(see Fig. 2.1), m is the mass of the airfoil, S_α is the static moment around the axis EA, I_α is the inertia moment around the axis EA, S_β is the static moment of the aileron around the axis EF, I_β is the inertia moment of the aileron around the axis EF, and $\widetilde{\Delta}$ is the distance of EF from EA. In the case of large displacements the nonlinear form of Eqs. (2.6) is considered, cf. [4]. By k_h, k_α and k_β the stiffnesses of springs are denoted. Further, L is the aerodynamic lift force, M_α is the aerodynamic moment acting on the airfoil, and M_β is the aerodynamic moment acting on the aileron. System (2.6) is equipped by initial conditions prescribing the initial values $h(0)$, $\alpha(0)$, $\beta(0)$, $\dot{h}(0)$, $\dot{\alpha}(0)$, $\dot{\beta}(0)$.

2.4. Coupling conditions and coupled problem. The airfoil time dependent boundary Γ_{Wt} is divided into the aileron part Γ_{Ft} and into the front part Γ_{Bt}, see Fig. 2.1. The lift force L and the torsional moments M_α, M_β are defined by

$$L = -l \int_{\Gamma_{Wt}} \sum_{j=1}^{2} \tau_{2j} n_j dS, \quad M_\alpha = l \int_{\Gamma_{Wt}} \sum_{i,j=1}^{2} \tau_{ij} n_j r_i^{\text{ort}} dS, \tag{2.7}$$

and

$$M_\beta = l \int_{\Gamma_{Ft}} \sum_{i,j=1}^{2} \tau_{ij} n_j r_i^{\text{ortEF}} dS, \tag{2.8}$$

where $r_1^{\mathrm{ort}} = -(x_2 - x_{\mathrm{EA2}})$, $r_2^{\mathrm{ort}} = x_1 - x_{\mathrm{EA1}}$, $r_1^{\mathrm{ortEF}} = -(x_2 - x_{\mathrm{EF2}})$, $r_2^{\mathrm{ortEF}} = x_1 - x_{\mathrm{EF1}}$, and where $x_{\mathrm{EA}} = (x_{\mathrm{EA1}}, x_{\mathrm{EA2}})$ and $x_{\mathrm{EF}} = (x_{\mathrm{EF1}}, x_{\mathrm{EF2}})$ denotes the position of the elastic axis EA and EF, respectively. By l the depth of the airfoil section is denoted. Further, the components of the stress tensor are computed by

$$\tau_{ij} = \rho \left[-p\delta_{ij} + \nu_{eff} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \right]. \tag{2.9}$$

The coupled aeroelastic problem consists of system (2.1) with boundary conditions (2.2), system (2.4) with boundary conditions (2.5), relation (2.3) and the initial-value problem for ODE system (2.6) and coupling conditions (2.7), (2.8).

3. Numerical approximation.

3.1. Time discretization. In order to discretize the problem, an equidistant partition $0 = t_0 < t_1 < \cdots < T$, $t_k = k\Delta t$ of the time interval $[0, T]$ with a constant time step Δt is considered. The velocity and pressure are approximated at each time level by $\mathbf{v}(t_n) \approx \mathbf{v}^n$, $p(t_n) \approx p^n$. Similarly k^n, ω^n and ν_T^n are approximations of $k(t_n)$, $\omega(t_n)$ and $\nu_T(t_n)$, respectively. Further $\mathbf{w}_D^n$ is the approximation of the domain velocity $\mathbf{w}_D$ at time t_n, and we set $\overline{\mathbf{w}}^{n+1} = \mathbf{v}^{n+1} - \mathbf{w}_D^{n+1}$. The ALE derivative is approximated by the second-order two-step backward difference formula

$$\frac{D^A \mathbf{v}}{Dt} \approx \frac{3\mathbf{v}^{n+1} - 4\hat{\mathbf{v}}^n + \hat{\mathbf{v}}^{n-1}}{2\Delta t}, \tag{3.1}$$

where we use the notation $\hat{\mathbf{v}}^i = \mathbf{v}^i \circ \mathcal{A}_{t_i} \circ \mathcal{A}_{t_{n+1}}^{-1}$, obtained by the transformation of $\mathbf{v}^n$ and $\mathbf{v}^{n-1}$ to the domain $\Omega := \Omega_{t_{n+1}}$. Similarly, the ALE derivative of functions k and ω are approximated by

$$\frac{D^A k}{Dt} \approx \frac{3k^{n+1} - 4\hat{k}^n + \hat{k}^{n-1}}{2\Delta t}, \quad \frac{D^A \omega}{Dt} \approx \frac{3\omega^{n+1} - 4\hat{\omega}^n + \hat{\omega}^{n-1}}{2\Delta t},$$

where $\hat{k}^i = k^i \circ \mathcal{A}_{t_i} \circ \mathcal{A}_{t_{n+1}}^{-1}$, $\hat{\omega}^i = \omega^i \circ \mathcal{A}_{t_i} \circ \mathcal{A}_{t_{n+1}}^{-1}$ are transformations of functions k^n, k^{n-1}, ω^n, ω^{n-1} onto domain $\Omega_{t_{n+1}}$.

The time discretized problem then reads: Find the functions $\mathbf{v} = \mathbf{v}^{n+1}$, $p := p^{n+1}$ and $k := k^{n+1}$, $\omega := \omega^{n+1}$ defined in the domain $\Omega := \Omega_{t_{n+1}}$ such that

$$\begin{aligned}
\tfrac{3\mathbf{v} - 4\hat{\mathbf{v}}^n + \hat{\mathbf{v}}^{n-1}}{2\tau} + (\overline{\mathbf{w}}^{n+1} \cdot \nabla)\mathbf{v} + \nabla p - \nabla \cdot \left(\nu_{eff} (\nabla \mathbf{v} + \nabla^T \mathbf{v}) \right) &= 0, \\
\nabla \cdot \mathbf{v} &= 0, \\
\tfrac{3k - 4\hat{k}^n + \hat{k}^{n-1}}{2\Delta t} + (\overline{\mathbf{w}}^{n+1} \cdot \nabla)k + \beta^* \omega k - \nabla \cdot (\varepsilon_k \nabla k) &= P_k, \\
\tfrac{3\omega - 4\hat{\omega}^n + \hat{\omega}^{n-1}}{2\Delta t} + (\overline{\mathbf{w}}^{n+1} \cdot \nabla)\omega + \beta \omega^2 - \nabla \cdot (\varepsilon_\omega \nabla \omega) &= P_\omega + C_D,
\end{aligned}$$

where $\nu_{eff} = \nu + \nu_T^{n+1}$, and $\mathbf{v}$, k, ω satisfy boundary conditions (2.2), (2.5).

3.2. Space discretization of RANS equations. In the finite element solution of incompressible RANS equations one has to overcome several obstacles. First, the finite element velocity/pressure pair has to satisfy the Babuška-Breezi condition (cf. [6]) and the dominating convection requires to introduce some additional stabilization, as, e.g. up-winding or streamline-diffusion method (also called SUPG method). In practical computations we assume that the domain Ω is a polygonal approximation of the region occupied by the fluid at time t_{n+1} and $\mathcal{T}_\Delta$ is a regular triangulation in Ω. The fluid velocity and pressure are sought in the finite element space $\mathcal{X}_\Delta$ of continuous piecewise quadratic (vector) functions and the finite element space $\mathcal{Q}_\Delta$ of continuous piecewise linear functions, respectively.

The *fully stabilized problem* reads: Find $U = (\mathbf{v}, p) \in \mathcal{X}_\Delta \times \mathcal{Q}_\Delta$ such that $\mathbf{v}$ satisfies approximately the Dirichlet boundary conditions (2.2) a),b) and the identity

$$a(U,V) + \mathcal{L}(U,V) + \mathcal{P}(U,V) = f(V) + \mathcal{F}(V), \tag{3.2}$$

for all $V = (\mathbf{z}, q) \in \mathcal{X}_\Delta \times \mathcal{Q}_\Delta$, where $a(U,V)$ and $f(V)$ are the Galerkin terms, $\mathcal{L}(U,V)$ and $\mathcal{F}(V)$ are the SUPG/PSPG stabilizing terms, and $\mathcal{P}(U,V)$ is the div-div stabilizing term. These forms are defined in the following way:

$$\begin{aligned}
a(U,V) &= \left(\frac{3\mathbf{v}}{2\Delta t} + \overline{\mathbf{w}}^{n+1}\cdot\nabla\mathbf{v}, \mathbf{z}\right)_\Omega + (\nu_{eff}\nabla\mathbf{v}, \nabla\mathbf{z})_\Omega - (p, \nabla\cdot\mathbf{z})_\Omega + (\nabla\cdot\mathbf{v}, q)_\Omega\,,\\
\mathcal{L}(U,V) &= \sum_{K\in\mathcal{T}_\Delta}\delta_K\Big(\frac{3\mathbf{v}}{2\Delta t} - \nabla\cdot\left(2\nu_{eff}\mathbf{S}(\mathbf{v})\right) + \left(\overline{\mathbf{w}}^{n+1}\cdot\nabla\right)\mathbf{v} + \nabla p, \left(\overline{\mathbf{w}}^{n+1}\cdot\nabla\right)\mathbf{z} + \nabla q\Big)_K,\\
\mathcal{F}(V) &= \sum_{K\in\mathcal{T}_\Delta}\delta_K\Big(\frac{4\hat{\mathbf{v}}^n - \hat{\mathbf{v}}^{n-1}}{2\Delta t}, \left(\overline{\mathbf{w}}^{n+1}\cdot\nabla\right)\mathbf{z} + \nabla q\Big)_K,\\
f(V) &= \left(\frac{4\hat{\mathbf{v}}^n - \hat{\mathbf{v}}^{n-1}}{2\Delta t}, \mathbf{z}\right)_\Omega,\\
\mathcal{P}(U,V) &= \sum_{K\in\mathcal{T}_\Delta}\tau_K(\nabla\cdot\mathbf{v}, \nabla\cdot\mathbf{z})_K.
\end{aligned}$$

The following choice of parameters τ_K, δ_K is used

$$\tau_K = 1, \qquad \delta_K = h_K^2,$$

where h_K denotes the local element size measured in the streamwise direction, see also [5].

3.3. Space discretization of the k-ω turbulence model. The numerical approximation of the time discretized equations of the turbulence model is realized by the application of the SUPG stabilized finite element method, where the nonlinear terms of $k-\omega$ equations (2.4) are linearized. In order to avoid non-physical undershoots/overshoots of the approximations of $k := k^{n+1}$ and $\omega := \omega^{n+1}$, the additional nonlinear crosswind diffusion method is applied, cf. [1], [8], [9]. First, the equations (2.4) are formulated in a weak sense: Find $k, \omega \in H^1(\Omega)$ such that they satisfy boundary conditions (2.5)a),b) and $B(\overline{\mathbf{w}}^{n+1}, \nu_T; \Lambda, \Phi) = L(\Phi)$ for all $\varphi_k, \varphi_\omega \in \mathcal{V}$, $\mathcal{V} = \{\psi \in H^1(\Omega) : \psi = 0 \text{ on } \Gamma_D \cup \Gamma_{Wt}\}$, where the forms

$$\begin{aligned}
B(\overline{\mathbf{w}}, \nu_T; \Lambda, \Phi) = (\varepsilon_k\nabla k, \nabla\varphi_k)_\Omega &+ \left(\frac{3k}{2\Delta t} + (\overline{\mathbf{w}}\cdot\nabla)\,k + 2\beta^*\omega^n k, \varphi_k\right)_\Omega\\
&+ (\varepsilon_\omega\nabla\omega, \nabla\varphi_\omega)_\Omega + \left(\frac{3\omega}{2\Delta t} + (\overline{\mathbf{w}}\cdot\nabla)\,k + 2\beta\omega^n\omega, \varphi_\omega\right)_\Omega,\\
L(\Phi) = \Big(\frac{4\widehat{k}^n - \widehat{k}^{n-1}}{2\Delta t} &+ P_k + \beta^* k^n\omega^n, \varphi_k\Big)_\Omega\\
+\Big(\frac{4\widehat{\omega}^n - \widehat{\omega}^{n-1}}{2\Delta t} &+ \beta(\omega^n)^2 + P_\omega + C_D, \varphi_\omega\Big)_\Omega
\end{aligned}$$

are defined with the use of the linearization of the nonlinear terms of equations (2.4) is used

$$\beta_*\omega k|_{t=t_{n+1}} = 2\beta_*\hat{\omega}^n k^{n+1} - \beta_*\hat{\omega}^n\hat{k}^n,$$

$$\beta\omega^2|_{t=t_{n+1}} = 2\beta\hat{\omega}^n\omega^{n+1} - \beta(\hat{\omega}^n)^2,$$

and the viscous coefficients ε_k, ε_ω, turbulent viscosity ν_T and production terms P_k, P_ω, C_D were taken from the previous time levels, i.e. we set

$$\varepsilon_k|_{t_{n+1}} = \nu + \sigma_k\hat{\nu}_T(t_n), \quad \varepsilon_\omega|_{t_{n+1}} = \nu + \sigma_k\hat{\nu}_T(t_n), \quad \nu_T = \hat{\nu}_T(t_n),$$

and

$$P_k|_{t_{n+1}} = \hat{P}_k(t_n), \quad P_\omega|_{t_{n+1}} = \hat{P}_\omega(t_n), \quad C_D|_{t_{n+1}} = \hat{C}_D(t_n),$$

Further, based on the triangulation $\mathcal{T}_\Delta$ the function spaces $H^1(\Omega)$ and $\mathcal{V}$ are approximated by the finite element subspaces $\mathcal{H}_\Delta$, $\mathcal{V}_\Delta$ of continuous piecewise linear functions, i.e.

$$\mathcal{H}_\Delta = \left\{\varphi \in C(\overline{\Omega}) \,:\, \varphi|_K \in P_1(K)\ \forall K \in \mathcal{T}_\Delta\right\}, \mathcal{V}_\Delta = \mathcal{H}_\Delta \cap \mathcal{V}.$$

To overcome the possible instability of the Galerkin approximations due to the dominating convection, the SUPG stabilization is applied, i.e.

$$\begin{aligned}
B_S(\overline{\mathbf{w}}, \nu_T; \Lambda, \Phi) = &\sum_{K\in\mathcal{T}_\Delta} \delta_K\Big(\frac{3k}{2\Delta t} + \overline{\mathbf{w}}\cdot\nabla k + 2\beta^*\hat{\omega}_n k + \nabla\cdot(\varepsilon_k\,\nabla k)\,, \overline{\mathbf{w}}\cdot\nabla\varphi_k\Big)_K \\
&+ \sum_{K\in\mathcal{T}_\Delta} \widehat{\delta}_K\Big(\frac{3\omega}{2\Delta t} + \overline{\mathbf{w}}\cdot\nabla\omega + 2\beta\hat{\omega}_n\omega + \nabla\cdot(\varepsilon_\omega\,\nabla\omega)\,, \overline{\mathbf{w}}\cdot\nabla\varphi_\omega\Big)_K, \\
L_S(\overline{\mathbf{w}}; \Phi) = &\sum_{K\in\mathcal{T}_\Delta} \delta_K\Big(\frac{4\hat{k}^n - \hat{k}^{n-1}}{2\Delta t} + P_k + \beta^*\hat{k}_n\hat{\omega}_n, \overline{\mathbf{w}}\cdot\nabla\varphi_k\Big)_K \\
&+ \sum_{K\in\mathcal{T}_\Delta} \widehat{\delta_K}\Big(\frac{4\hat{\omega}^n - \hat{\omega}^{n-1}}{2\Delta t} + \beta\hat{\omega}_n^2, P_\omega + C_D, \overline{\mathbf{w}}\cdot\nabla\varphi_k)\Big)_K,
\end{aligned}$$

where the parameters $\delta_K, \widehat{\delta_K}$ are defined by

$$\delta_K^{-1} = \frac{4|\varepsilon_k|_{\infty,K}}{h_K^2} + \frac{2|\overline{\mathbf{w}}|_{\infty,K}}{h_K} + 2\beta^*|\hat{\omega}_n|_{\infty,K},\ \widehat{\delta_K}^{-1} = \frac{4|\varepsilon_\omega|_{\infty,K}}{h_K^2} + \frac{2|\overline{\mathbf{w}}|_{\infty,K}}{h_K} + 2\beta|\hat{\omega}_n|_{\infty,K},$$

and where $|\cdot|_{\infty,K}$ denotes the norm in $L^\infty(K)$.

The use of SUPG stabilization still does not avoid local oscillations near sharp layers, which possibly can lead to negative values of the approximation of the turbulent viscosity ν_T. Therefore the stabilization techniques based on an additional dissipation in *crosswind* direction is applied, cf. [7], [1]. The nonlinear stabilized problem reads: Find $\Lambda = (k, \omega) \in \mathcal{H}_\Delta^2$ such that satisfy approximately boundary conditions (2.5)a),b) and

$$B(\overline{\mathbf{w}}^{n+1}, \nu_T; \Lambda, \Phi) + B_S(\overline{\mathbf{w}}^{n+1}, \nu_T; \Lambda, \Phi) + B_A(\overline{\mathbf{w}}^{n+1}, \nu_T; \Lambda, \Phi) = L(\Phi) + L_S(\overline{\mathbf{w}}^{n+1}; \Phi),$$

holds for all $\Phi = (\varphi_k, \varphi_\omega) \in \mathcal{V}_\Delta^2$, where

$$\begin{aligned}
B_A(\overline{\mathbf{w}}, \nu_T; \Lambda, \Phi) = &\sum_{K\in\mathcal{T}_\Delta}\Big(\alpha_K\nabla k, \nabla\varphi_k\Big)_K + \sum_{K\in\mathcal{T}_\Delta}\Big(\hat{\alpha}_K\nabla\omega, \nabla\varphi_\omega\Big)_K \\
&+ \sum_{K\in\mathcal{T}_\Delta}\int_K \big((\alpha_K - \alpha'_K)^+ - \alpha_K\big)\,\nabla k\cdot\left(\frac{\overline{\mathbf{w}}\otimes\overline{\mathbf{w}}}{\|\overline{\mathbf{w}}\|_K^2}\right)\nabla\varphi_k dx. \\
&+ \sum_{K\in\mathcal{T}_\Delta}\int_K \big((\hat{\alpha}_K - \hat{\alpha}'_K)^+ - \hat{\alpha}_K\big)\,\nabla\omega\cdot\left(\frac{\overline{\mathbf{w}}\otimes\overline{\mathbf{w}}}{\|\overline{\mathbf{w}}\|_K^2}\right)\nabla\varphi_\omega dx.
\end{aligned}$$

Here α'_K and $\hat{\alpha}'_K$ are defined by

$$\alpha'_K = \delta_K \, \|\overline{\mathbf{w}}^{n+1}\|_{0,\infty,K}, \qquad \hat{\alpha}'_K = \hat{\delta}_K \, \|\overline{\mathbf{w}}^{n+1}\|_{0,\infty,K}$$

and α_K, $\hat{\alpha}_K$ are defined in the following way: We define the local element residuals

$$\mathrm{res}_1(k) = \frac{3k - 4\hat{k}^n + \hat{k}^{n-1}}{2\Delta t} + \overline{\mathbf{w}}^{n+1} \cdot \nabla k + 2\beta^* \hat{\omega}_n k - \beta^* \hat{\omega}_n \hat{k}_n - P_k - \nabla \cdot (\varepsilon_k \nabla k)$$

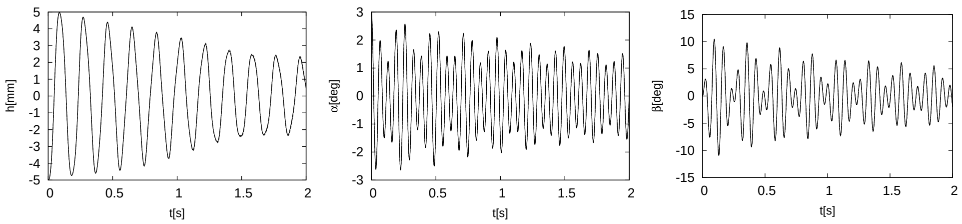
and

$$\mathrm{res}_2(\omega) = \frac{3\omega - 4\hat{\omega}^n + \hat{\omega}^{n-1}}{2\Delta t} + \overline{\mathbf{w}}^{n+1} \cdot \nabla \omega + 2\beta \hat{\omega}_n \omega - \beta^* \hat{\omega}_n^2 - P_\omega - C_D - \nabla \cdot (\varepsilon_\omega \nabla \omega)$$

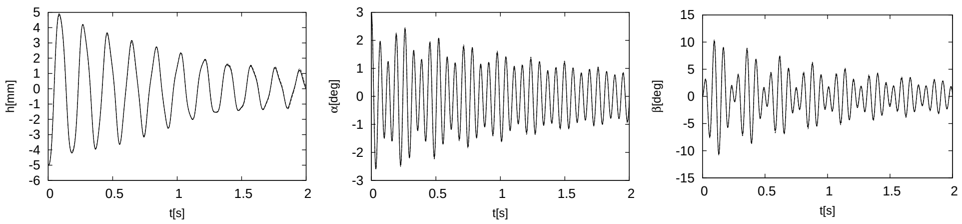
We set

$$\alpha_K = \frac{1}{2} A_K h_K \frac{\|\mathrm{res}_1(k)\|_K}{\|\nabla k\|_K}, \qquad \widehat{\alpha}_K = \frac{1}{2} A_K h_K \frac{\|\mathrm{res}_2(\omega)\|_K}{\|\nabla \omega\|_K}$$

if $\|\nabla k\|_K \neq 0$ and $\|\nabla \omega\|_K \neq 0$. Otherwise we put $\alpha_K = 0$ and $\widehat{\alpha}_K = 0$. Finally, $A_{K,1} = \left(0.7 - \frac{2\varepsilon_k}{\|\mathbf{a}_1\|_K h_K}\right)^+$ and $A_{K,2} = \left(0.7 - \frac{2\varepsilon_\omega}{\|\mathbf{a}_2\|_K h_K}\right)^+$, with $\mathbf{a}_1 = \frac{\mathrm{res}_1(k)}{\|\nabla k\|_K^2} \nabla k$, and $\mathbf{a}_2 = \frac{\mathrm{res}_2(\omega)}{\|\nabla \omega\|_K^2} \nabla \omega$.



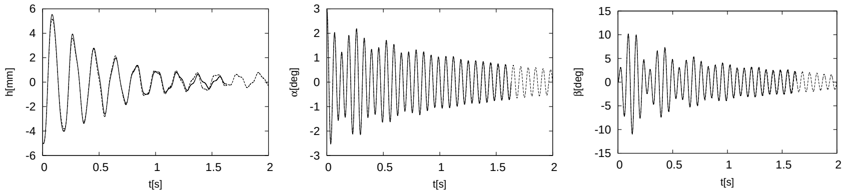
(a) Far field velocity $U_\infty = 2$m/s



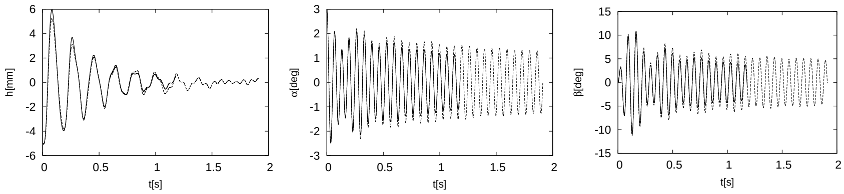
(b) Far field velocity $U_\infty = 4$m/s

FIG. 4.1. *Comparison of aeroelastic response h, α, β for far field velocity $U_\infty = 2\,m/s$ and $U_\infty = 4\,m/s$ computed by $k - \omega$ turbulence model.*

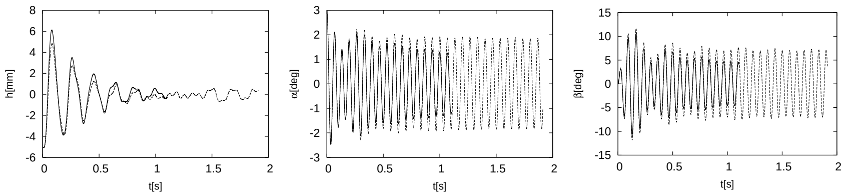
4. Numerical results. Conclusion.. The developed technique was applied for approximation of an aeroelastic problem for the airfoil NACA 0012 with three degrees of freedom. The following choice of parameters was employed: $k_h = 105.109$ N/m, $k_\alpha = 3.69558$ N m/rad, $k_\beta = 0.2$ N m/rad, $m = 0.086622$ kg, $S_\alpha = -0.0007796$ kg/m and $I_\alpha = 0.00048729$ kg m^2. $S_\beta = 0.0$ kg/m and $I_\beta = 3.411037 \times 10^{-5}$ kg m^2, see [11] or [4], where laminar flow model was used. The elastic axis EA is located at 40% of the airfoil and the elastic axis of the aileron EF is located at 80% of the



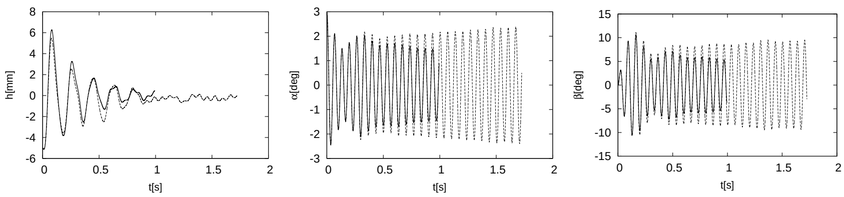
(a) Far field velocity $U_\infty = 8\,\mathrm{m/s}$



(b) Far field velocity $U_\infty = 10\,\mathrm{m/s}$



(c) Far field velocity $U_\infty = 11\,\mathrm{m/s}$



(d) Far field velocity $U_\infty = 12\,\mathrm{m/s}$

FIG. 4.2. *Comparison of aeroelastic response h, α, β for far field velocity $U_\infty = 8\,m/s$, $U_\infty = 10\,m/s$ and $U_\infty = 12\,m/s$ computed by $k-\omega$ turbulence model and compared to results computed by Spalart-Allmaras turbulence model.*

airfoil. The depth of the section is 7.9 cm. The results of Theodorsen's linear theory computed by NASTRAN predicts the aeroelastic instability was predicted for far field velocities higher or equal to $U_\infty = 11.3\,\mathrm{m/s}$, where the instability is caused by coupling of $\alpha - \beta$ motion, cf. [11]. The results were computed for far field flow velocities in the range from $U_\infty = 2\,\mathrm{m/s}$ to $U_\infty = 12\,\mathrm{m/s}$ with the aid of the presented finite element approach and the aeroelastic responses are shown in Figs. 4.1 and 4.2, where the graphs show the angle of rotation α, the angle of rotation of the aileron β and the vertical displacement h of the profile in dependence on time. The initial conditions were specified by $h(0) = -6\,\mathrm{mm}$, $\dot{h}(0) = 0\,\mathrm{mm/s}$, $\alpha(0) = 3^\circ$, $\dot{\alpha}(0) = 0^\circ/\mathrm{s}$. Fig. 4.1 shows damped vibrations for far field velocities bellow the critical velocity. Similarly, Fig. 4.2 shows damped vibrations for velocities lower than 10 m/s, whereas for increasing far field velocities the damping is decreasing, and for velocity $U_\infty =$

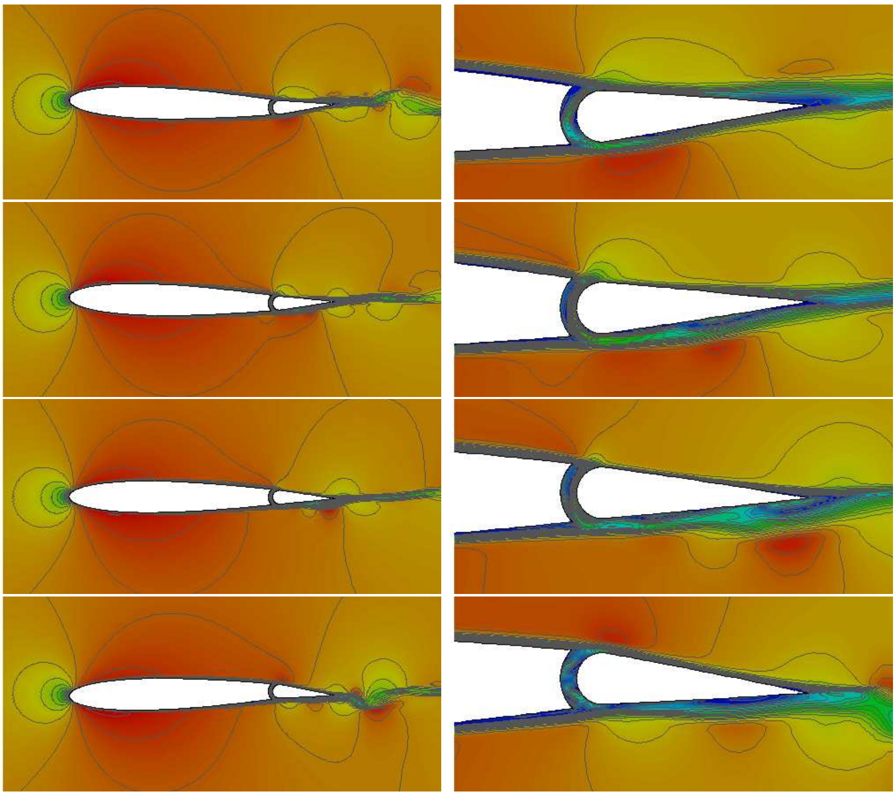


FIG. 4.3. *Flow patterns during the first cycle of aeroelastic response after release of the profile, magnitude of flow velocity for $U_\infty = 8\,m/s$.*

12 m/s undamped vibrations appears. This behaviour is in a good agreement with the linear theory. Furthermore, the comparison of the results computed by $k - \omega$ turbulence model to the numerical results computed with the aid of one equation Spalart-Allmaras turbulence model is shown in Figs. 4.2. Similar behaviour is shown, only small difference of the results computed by two mentioned turbulence models can be seen.

The numerical approximation of the flow field during aeroelastic simulation for far field velocity $U_\infty = 8, \mathrm{m/s}$ is shown in Fig. 4.3, where velocity magnitude is shown around the airfoil and in a detail nearby the aileron. Particularly, the flow through the gap between the airfoil and the aileron can be observed, which influences the aeroelastic behaviour of the system. The numerical results shows that the numerical algorithm for approximation of the turbulent flow interacting with a moving airfoil is applicable and leads to acceptable results for the considered technical application.

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