STOCHASTIC SWIFT-HOHENBERG EQUATION NEAR A CHANGE OF STABILITY

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Abstract. We review recent results on the approximation of stochastic PDEs by amplitude equations. As an example we focus on the Swift-Hohenberg equation.

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AMS subject classifications. 60H10, 60H15, 35R60.

1. Introduction. We consider the stochastic Swift-Hohenberg (SH for short) equation for $u(t,x) \in \mathbb{R}$

$$\partial_t u = -(1 + \partial_x^2)^2 u + \mu_\varepsilon u - u^3 + \sigma_\varepsilon \xi. \tag{1.1}$$

The SH equation was derived in [21] as a toy model for the convective instability in the Rayleigh-Bénard convection. Now it is one of the celebrated models in the theory of pattern formation. We consider small noise given by thermal fluctuations (i.e. additive noise), but also a small possibly noisy distance from bifurcation μ_ε. This allows us to see the effects of noise on the bifurcation. The scalings of μ_ε and σ_ε that we will consider in this paper are motivated by recent experiments. Indeed, when the distance from the change of stability is sufficiently small, then the influence of small noise is detected in various convection experiments (cf. [18, 16, 17]) by formation of pattern below the threshold of instability. The formation of pattern due to thermal noise in convection problems for deterministically stable equations was long conjectured (cf. [21, 10, 12]). The main difficulty of the experiment is to stabilise the control parameters (e.g. temperature in Rayleigh-Bénard convection) to the precision of the noise strength, which is extremely small in the case of thermal fluctuations.

In order to provide a tool to explain these facts rigorously, we rely on the formally well established approximation via amplitude equations, describing the evolution of the dominating modes near bifurcation. On bounded domains the approximation via amplitude equations was first rigorously verified in [8], and later on extended in [4, 5, 6]. There the amplitude equation for the dominant modes is given by an SDE. All these results are mainly limited to long but transient behaviour. It is also possible to approximate the long–time behaviour in terms of the structure of invariant measures for the corresponding Markov semigroup (cf. [1, 2]). The case of unbounded or just very large domains is significantly different. The amplitudes of the dominant modes are subject to a long–range modulation in space, and hence not given by an SDE, but an SPDE instead. The case of large, but still bounded, domains is discussed in [3]. See also [15] for the deterministic

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equation. In both cases the domain is bounded, but it scales with respect to the distance from bifurcation. There is also a very large literature on the rigorous derivation of amplitude equations for deterministic equations in unbounded domains (e.g. [9, 13, 14, 19, 20]). A similar result, however, seems to be out of reach for SPDEs.

The main difference between small and large domains is the existence of a large spectral gap of order $\mathcal{O}(1)$. On bounded domains, a finite number $e = (e_1, \ldots, e_n)$ of modes (or eigenfunctions) change stability at the bifurcation, while all other eigenvalues are negative and bounded away from 0. Formal arguments yield an amplitude equation for the amplitude $\mathcal{A} \in \mathbb{R}^n$ of the dominating modes and to show that the solution u of the SPDE is well approximated by

$$u(t, x) = \varepsilon \mathcal{A}(\varepsilon^2 t) \cdot e(x) + \mathcal{O}(\varepsilon^2) \,,$$

where ε^2 is the typical scale for the distance from bifurcation.

On unbounded or just very large domains this picture changes completely. Even very close to the bifurcation, i.e. for $\varepsilon \ll 1$, a large number of modes is near or already above the threshold of stability. In this case $\mathcal{A}$ is subject to slow modulations in x, reflecting that u is concentrated in Fourier space, which takes into account the large number of weakly (un)stable modes. The solution u is now given by

$$u(t, x) = \varepsilon \mathcal{A}(\varepsilon^2 t, \varepsilon x) \cdot e(x) + \mathcal{O}(\varepsilon^2)$$

and $\mathcal{A}$ fulfils a (stochastic) PDE, which is called the *amplitude or modulation equation*.

2. Multiplicative Noise. For simplicity, we focus on a simple multiplicative noise, which has no spatial dependence. The proofs will be somewhat easier, as in this case we have Itô's formula and Burkholder-Davis-Gundy (BDG for short) type inequalities at our disposal, which we cannot use easily for additive space-time white noise. We follow partly [6].

Multiplicative noise appears naturally in models with noisy control parameters. Consider an equation of the following type

$$\partial_t u = Lu + \mu u - u^3 \,, \tag{2.1}$$

where L is an appropriate differential operator. The question is whether we can see the influence of small noise in the bifurcation parameter μ in the case where we are near or at the bifurcation $\mu = 0$. This is an important issue in many experiments, as μ represents physical quantities like temperature, which are naturally subject to small (random) perturbations. Suppose now that the control parameter $\mu \in \mathbb{R}$ in (2.1) is perturbed by small white noise. To be more precise, let μ be a Gaussian noise with mean and covariance

$$\mathbb{E}\mu(t) = \mu_\varepsilon \in \mathbb{R}, \qquad \mathbb{E}(\mu(t) - \mu_\varepsilon)(\mu(s) - \mu_\varepsilon) = \varepsilon^2 \delta(t - s) \,,$$

respectively. Thus, $\mu = \mu_\varepsilon + \varepsilon \xi$, where $\xi = \partial_t \beta$ is the generalised derivative of a real valued one–dimensional Brownian motion $\beta = \{\beta(t)\}_{t \geq 0}$. Hence, we can rewrite (2.1) as a stochastic PDE

$$\partial_t u = Lu + \mu_\varepsilon u - u^3 + \varepsilon u \partial_t \beta, \tag{2.2}$$

where the Itô interpretation for stochastic integrals has been used.

2.1. Setting. Consider SH in the Hilbert space $X = L^2([-\pi, \pi])$ with scalar product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$.

$$du = [Lu + \varepsilon^2 \nu u - u^3]dt + \varepsilon u d\beta. \tag{2.3}$$

We consider only periodic boundary conditions in order to outline the main ideas in a less technical way. Let $L = -(1 + \partial_x^2)^2$ with periodic boundary conditions on $[-\pi.\pi]$. It is well-known that L generates an analytic semigroup $\{e^{tL}\}_{t \geq 0}$ in X. The kernel of this operator is $\mathcal{N} := N(L) = \text{span}\{\sin, \cos\}$. Denote the projection onto $\mathcal{N}$ by P_c and define $P_s = I - P_c$. Due to the spectral gap of order 1 there are constants $M > 0$ and $\omega > 0$ such that for all $t > 0$

$$\|e^{tL}v\| \leq M\|v\| \quad \text{and} \quad \|P_s e^{tL}v\| \leq Me^{-t\omega}\|v\| \quad \text{for all } t > 0, \ v \in X . \tag{2.4}$$

ASSUMPTION 1. *Let $d\beta$ be the Itô differential with respect to the real-valued standard Brownian motion $\{\beta(t)\}_{t \geq 0}$ adapted to some filtration $\{\mathcal{F}_t\}_{t \geq 0}$ on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$.*

PROPOSITION 2.1 (Strong Nonlinear Stability). *There are constants $C, c > 0$ such that*

$$\langle u, \varepsilon^2 \nu u - u^3 \rangle \leq C\varepsilon^4 - c\|u\|^4 \quad \text{and} \quad \langle u^3 - v^3, u - v \rangle \geq 0 \quad \text{for all} \ \ u, v \in X .$$

We consider mild solutions given by the following definitions.

DEFINITION 2.2. An X-valued stochastic process $\{u(t)\}_{t \geq 0}$ is a mild solution of (2.3) in X, if it is adapted to $\{\mathcal{F}_t\}_{t \geq 0}$ and there is a positive stopping time $\tau_e > 0$ such that $u \in C^0([0, \tau_e), X)$ and

$$u(t) = e^{tL}u(0) + \int_0^t e^{(t-\tau)L}[\varepsilon^2 \nu u - u^3](\tau) \, d\tau + \varepsilon \int_0^t e^{(t-\tau)L}u(\tau) \, d\beta(\tau) \tag{2.5}$$

holds for all $0 < t < \tau_e$. We choose $[0, \tau_e)$ as the maximal interval of existence. This means that either $\tau_e = \infty$ or $\|u(t)\| \to \infty$ for $t \to \tau_e$.

DEFINITION 2.3. We call a mild solution of (2.3) in X a strong solution in X, if

$$\mathbb{E}\int_0^t \|[Lu + \varepsilon^2 \nu u - u^3](\tau)\| \, d\tau < \infty, \qquad \mathbb{E}\int_0^t \|u(\tau)\|^2 \, d\tau < \infty \tag{2.6}$$

for all $t < \tau_e$ and

$$u(t) = u(0) + \int_0^t [Lu + \varepsilon^2 \nu u - u^3](\tau) \, d\tau + \varepsilon \int_0^t u(\tau) \, d\beta(\tau) \tag{2.7}$$

in X for $0 < t < \tau_e$. Again we choose τ_e to be maximal. This means that either $\tau_e = \infty$ or one condition in (2.6) fails to be true at $t = \tau_e$.

Our definition of a strong solution is slightly more restrictive that the one in [11], since we actually require that moments of the solution exist. However, this is mainly for simplicity of presentation. Using standard theory given in [11], it is easy to verify that there is a unique mild solution in X with $\tau_e = \infty$. The existence of strong solutions is standard using a priori estimates and regularisation properties of the equation.

2.2. Attractivity. To prove attractivity we need to verify that there is a time $t_\varepsilon > 0$ such that $u(t_\varepsilon) = \varepsilon A_\varepsilon e^{ix} + c.c. + \mathcal{O}(\varepsilon^2)$, where $A_\varepsilon \in \mathbb{C}$ is of $\mathcal{O}(1)$.

THEOREM 2.4. *Let u be a strong solution of* (2.3) *in X, then for all $p > 0$ and $t_0 > 0$ there is a constant $C > 0$ such that*

$$\sup_{t \geq t_0 \varepsilon^{-2}} \mathbb{E}\|u(t)\|^p \leq C\varepsilon^p \tag{2.8}$$

for all sufficiently small $\varepsilon > 0$ and all strong solutions u of (2.3) *in X independent of $u(0)$. In particular, $\tau_e = \infty$, a.s.. Furthermore, for $q \geq 2$, $\delta > 0$, and $p \in [2, q]$ there is some $C > 0$ such that $\mathbb{E}\|u(0)\|^q \leq \delta \varepsilon^q$ for all $\varepsilon \in (0, 1)$ implies*

$$\sup_{t \geq 0} \mathbb{E}\|u(t)\|^p \leq C\varepsilon^p \quad \textit{for all suff. small } \varepsilon > 0. \tag{2.9}$$

Additionally, for $t_\varepsilon = \frac{2}{\omega} \ln(\varepsilon^{-1})$ and all $p \in [4, q/3]$ there is a constant $C > 0$ such that

$$\sup_{t \geq t_\varepsilon} \mathbb{E}\|P_s u(t)\|^p \leq C\varepsilon^{3p} \quad \textit{for all suff. small } \varepsilon > 0. \tag{2.10}$$

The proof is straightforward, but quite technical. For details see Section 2.3 of [6]. The main tools are standard a priori type estimates. The basic idea is to apply Itô formula to $\|u(t)\|^p$ and to use PROPOSITION 2.1. The key technical obstacle is that we do not know a priori that $\mathbb{E}\|u(t)\|^p$ exists for all times. Therefore, we need to use cut-off techniques by approximating the pth power by a monotonically growing sequence of smooth bounded functions.

2.3. Residual. With THEOREM 2.4 at hand we make the ansatz

$$u(t) = \varepsilon A(\varepsilon^2 t)e^{ix} + c.c. + \mathcal{O}(\varepsilon^2),$$

where $A \in \mathbb{C}$. A formal calculation yields the following amplitude equation:

$$dA = (\nu A - 3A|A|^2)dT + Ad\tilde{\beta}, \quad \text{where} \quad \tilde{\beta}(T) = \varepsilon \beta(\varepsilon^{-2}T). \tag{2.11}$$

As usual we consider the equation in the Itô sense. For $a(t) = \varepsilon A(\varepsilon^2 t)e^{ix} + c.c.$ for a solution A of (2.11), we define the residual

$$\text{Res}(\varepsilon a)(t) = -\varepsilon a(t) + \varepsilon e^{tL}a(0) + \varepsilon^2 \int_0^t e^{(t-\tau)L}a(\tau)\,d\beta(\tau)$$

$$+\varepsilon^3 \int_0^t e^{(t-\tau)L}[\nu a - a^3](\tau)\,d\tau. \tag{2.12}$$

THEOREM 2.5. *For all $p > \frac{4}{3}$, $\delta > 0$, and $T_0 > 0$ there is a constant $C > 0$ such that*

$$P_c \text{Res}(\varepsilon a)(t) = 0 \qquad and \qquad \mathbb{E}\Big(\sup_{t \in [0, T_0 \varepsilon^{-2}]} \|P_s \text{Res}(\varepsilon a)(t)\|^p\Big) \leq C\varepsilon^{3p}$$

for all sufficiently small $\varepsilon > 0$ and all solutions a of (2.11) *with $\mathbb{E}\|a(0)\|^{3p} \leq \delta \varepsilon^{3p}$.*

We can also use higher order corrections (cf. [1]), but then the result is more involved.

Proof. Split Res $= P_c\text{Res} + P_s\text{Res}$. First projecting (2.12) to $\mathcal{N}$, using the slow time-scale $T = \varepsilon^2 t$, and (2.11) immediately yields $P_c\text{Res}(\varepsilon a) = 0$. For $P_s\text{Res}$ we project (2.12) to $P_s X$. and use (2.4), in order to derive

$$\mathbb{E} \sup_{t \in [0, \frac{T_0}{\varepsilon^2}]} \|P_s\text{Res}(\varepsilon a)(\varepsilon^2 t)\|^p \leq C\varepsilon^{3p} \mathbb{E} \sup_{t \in [0, \frac{T_0}{\varepsilon^2}]} \left[\int_0^t e^{-(t-\tau)\omega} \|P_s a^3(\tau)\| \, d\tau \right]^p$$

$$\leq C\varepsilon^{3p}(1 + \mathbb{E} \sup_{t \in [0,T_0]} |A(T)|^{3p}) \leq C\varepsilon^{3p} ,$$

by using standard a priori bounds for A (cf. e.g. [6] for $3p \geq 4$). $\square$

2.4. Approximation. Define R as the error of our approximation by

$$\varepsilon^2 R(t) = u(t) - \varepsilon a(t) . \tag{2.13}$$

We split $R = R_c + R_s$ and treat R_s using the a priori estimates on $P_s u$. This information on $P_s u$ is not necessary, but it simplifies the proofs significantly. Our main result is the following:

THEOREM 2.6. *For $p > 4$, $T_0 > 0$, and $\delta > 0$ there is a constant $C > 0$ such that for all strong solutions u of (2.3) in X with*

$$\mathbb{E}\|u(0)\|^{3p} \leq \delta\varepsilon^{3p} \quad and \quad \mathbb{E}\|P_s u(0)\|^p \leq \delta\varepsilon^{3p} \quad for \ all \ \varepsilon \in (0,1) ,$$

we derive

$$\mathbb{E}\left(\sup_{t \in [0,T_0\varepsilon^{-2}]} \|P_s R(t))\|^p \right) \leq C\varepsilon^p \qquad and \qquad \mathbb{E}\left(\sup_{t \in [0,T_0\varepsilon^{-2}]} \|P_c R(t))\|^p \right) \leq C$$

for all sufficiently small $\varepsilon > 0$, where A solves (2.11) such that $P_c u(0) = \varepsilon A(0)e^{ix} + c.c..$

Proof. First use an improvement of the bound of THEOREM 2.4. To be more precise using BDG it is possible to derive from the mild formulation

$$\mathbb{E}(\sup_{t \in [0,T_0\varepsilon^{-2}]} \|u(t)\|^p) \quad and \quad \mathbb{E}(\sup_{t \in [0,T_0\varepsilon^{-2}]} \|P_s u(t)\|^p) \leq C\varepsilon^{3p} .$$

These a priori estimates on u are only possible because of the very strong nonlinear stability. Now (2.13) implies

$$\mathbb{E}\left(\sup_{t \in [0,T_0\varepsilon^{-2}]} \|P_s R(t)\|^p \right) = \mathbb{E}\left(\sup_{t \in [0,T_0\varepsilon^{-2}]} \|\varepsilon^{-2} P_s u(t)\|^p \right)$$

$$\leq C\varepsilon^p . \tag{2.14}$$

We thus proved the bound on R_s. For the bound on R_c, (2.5) and (2.12) yield

$$R(t) = e^{tL} R(0) + \int_0^t e^{(t-\tau)L} [\varepsilon^2 AR - \varepsilon^{-2}(u^3 - (\varepsilon a)^3)](\tau) \, d\tau$$

$$+ \varepsilon \int_0^t e^{(t-\tau)L} R(\tau) \, d\beta(\tau) + \varepsilon^{-2}\text{Res}(\varepsilon a)(t) . \tag{2.15}$$

Now $\varepsilon^{-2}(u^3 - (\varepsilon a)^3) = 3\varepsilon^2 a^2 R + 3\varepsilon^3 aR^2 + \varepsilon^4 R^3$. For $R_c = P_c R$ we derive using THEOREM 2.5 and $R_c(0) = 0$ (by the definition of $a(0)$)

$$R_c(t) = \varepsilon^2 \int_0^t (\nu R_c - P_c[3a^2 R + 3\varepsilon aR^2 + \varepsilon^2 R^3]) \, \mathrm{d}\tau + \varepsilon \int_0^t R_c(\tau) \, \mathrm{d}\beta(\tau) \, . \quad (2.16)$$

Hence, $dR_c(t) = \varepsilon^2(\nu R_c - 3P_c a^2 R_c - 3\varepsilon aR_c^2 - \varepsilon 2 P_c R_c^3 + V_c)dt + \varepsilon R_c \, \mathrm{d}\beta$, where V_c collects all terms at least of order $\mathcal{O}(1)$. For example V_c contains all terms not depending on R_c like $P_c a^2 R_s$, $\varepsilon P_c aR_s^2$, or $\varepsilon^2 P_c R_s^3$, together with terms like $\varepsilon P_c aR_c R$ or $\varepsilon^2 P_c R_s R^2$, which we can easily bound. To be more precise $\mathbb{E}\|V_c(t)\|^q \leq C$ for $q > \frac{4}{3}$ and all $t \in [0, T_0 \varepsilon^{-2}]$. It is straightforward to verify this $\mathcal{O}(1)$-bound using Hölder's inequality and the following bounds. First $A = \mathcal{O}(1)$ by standard a priori bounds for (2.11). Furthermore, $R_s = \mathcal{O}(\varepsilon)$ from (2.14). Finally $R = \mathcal{O}(\varepsilon^{-1})$ and thus $R_c = \mathcal{O}(\varepsilon^{-1})$ from (2.13), if we use again the bound on A together with the attractivity result of THEOREM 2.4.

Using PROPOSITION 2.1 together with Itô's formula and Young's inequality yields

$$d\|R_c\|^q \leq C\varepsilon^2(\|R_c\|^q + \|V_c\|^q)dt + \varepsilon q\|R_c\|^q \, \mathrm{d}\beta \, . \quad (2.17)$$

Thus, $\qquad \mathbb{E}\|R_c(t)\|^q \leq C\varepsilon^2 \int_0^t \mathbb{E}(\|R_c\|^q + \|V_c\|^q)(\tau) \, \mathrm{d}\tau \, .$

Now Gronwall's inequality and the bound on V_c imply $\sup_{t \in [0, T_0 \varepsilon^{-2}]} \mathbb{E}\|R_c(t)\|^q \leq C$. Going back to (2.17) for $q = p/2$ ($p \geq 4$) yields

$$\sup_{t \in [0, \frac{T_0}{\varepsilon^2}]} \|R_c(t)\|^p$$

$$\leq \sup_{t \in [0, \frac{T_0}{\varepsilon^2}]} \left(C\varepsilon^2 \int_0^t (\|R_c\|^{p/2} + \|V_c\|^{p/2})(\tau) \, \mathrm{d}\tau + C\varepsilon \int_0^t \|R_c(\tau)\|^{p/2} \, \mathrm{d}\beta(\tau) \right)^2$$

$$\leq C\varepsilon^4 \left(\int_0^{\frac{T_0}{\varepsilon^2}} (\|R_c\|^{p/2} + \|V_c\|^{p/2})(\tau) \, \mathrm{d}\tau \right)^2 + C \sup_{t \in [0, \frac{T_0}{\varepsilon^2}]} \left(\varepsilon \int_0^t \|R_c(\tau)\|^{p/2} \, \mathrm{d}\beta(\tau) \right)^2 \, .$$

Using BDG and the first bound on R_c, we finish the proof. □

3. Results for Additive Noise. We present some results from [1] which in turn are based on [8]. Consider SH with additive noise

$$\partial_t u = Lu + \mu\varepsilon^2 u - u^3 + \varepsilon^2 \xi, \quad (3.1)$$

subject to periodic boundary conditions on $[-\pi, \pi]$. The additive noise $\varepsilon^2 \xi$ is motivated by the presence of thermal fluctuations in the medium.

3.1. Assumptions. Concerning ξ we assume the following.

ASSUMPTION 2. *The noise process is given by $\xi = Q\partial_t W$, where W is a standard cylindrical Wiener process in X with the identity as a covariance operator and $Q \in \mathcal{L}(X, X)$ is symmetric. Furthermore, there exists a constant $\tilde{\alpha} < \frac{1}{2}$ such that*

$$\|(1 - L)^{-\tilde{\alpha}} Q\|_{\mathrm{HS}(X)} < \infty \, , \quad (3.2)$$

where $\| \cdot \|_{\mathrm{HS}(X)}$ denotes the Hilbert-Schmidt norm of an operator from X to X.

Straightforward computations, combined with properties of analytic semigroups allow us to check that ASSUMPTION 2 implies the following (see [11, Section 5.4] for the first assertion):

- The stochastic convolution $W_L(t) = \int_0^t e^{L(t-s)} Q \, dW(s)$ is an X-valued process with Hölder continuous sample paths.
- There exist positive constants C and γ such that for every $t > 0$

$$\|P_s e^{Lt} Q\|_{\mathrm{HS}(X)} \leq C(1 + t^{-\gamma}) e^{-\omega t} . \tag{3.3}$$

Let us comment on the relationship between the Wiener process QW and the noise ξ. Let ξ be a generalised Gaussian process such that

$$\mathbb{E}\xi(t, x) = 0 \quad \text{and} \quad \mathbb{E}\xi(t, x)\xi(s, y) = \delta(t - s)q(x, y) ,$$

where δ is the usual Delta distribution and q the spatial correlation-function. If we define the linear operator Q via $Qf(x) = \int_{-\pi}^{\pi} q(x, y)f(y) \, dy$, then up to some technical assumptions it is easy to verify that the generalised derivative $\partial_t QW$ has the same properties as ξ (e.g. [7] and the references therein).

We will show that for a solution

$$u(t) \approx \varepsilon A(\varepsilon^2 t) e^{ix} + c.c. + \varepsilon^2 \psi(t) , \tag{3.4}$$

where $A \in \mathbb{C}$ solves

$$dA = (\nu A - 3A|A|^2)dT + d\beta , \tag{3.5}$$

where $\beta(T)$ is the projection of $\varepsilon QW(\varepsilon^2 T)$ onto e^{ix} and $\psi \in P_s X$ solves

$$d\psi = L\psi \, dt + P_s Q \, dW . \tag{3.6}$$

Note that the Wiener processes β and $P_s QW$ are not necessarily independent. Thus (3.5) and (3.6) are coupled through the noise. Nevertheless it is a key point that the two noise terms live on different time-scales.

3.2. Existence of Solutions. We consider mild solutions of (3.1). The existence of a unique mild solution is given in the following proposition, cf. [1].

PROPOSITION 3.1. *For all (stochastic) initial conditions $u(0) \in X$ equation (3.1) has a unique global mild solution u. This means we have a stochastic process u such that $u : [0, \infty) \to X$ is continuous and fulfils*

$$u(t) = e^{tL}u(0) + \int_0^t e^{(t-\tau)L}[\varepsilon^2 \nu u - u^3](\tau) \, d\tau + \varepsilon^2 W_L(t) \quad \text{for all } t > 0 . \tag{3.7}$$

For the proof of this proposition, note that the existence and uniqueness of local solutions is standard (e.g. [11, Section 7]). The global existence follows from standard a priori estimates for $v = u - \varepsilon^2 W_L$, which is a weak solution of the following PDE with random coefficients

$$\partial_t v = Lv + \varepsilon^2 \nu(v + \varepsilon^2 W_L) - (v + \varepsilon^2 W_L)^3. \tag{3.8}$$

The formal idea is to take the scalar product with v, in order to derive standard a priori estimates for $\|v\|^2$ and hence $\|u\|^2$ (cf. e.g. proof of [1, Theorem 4.1]).

3.3. Attractivity. The attractivity justifies the ansatz used in formal computation. After a comparably short time the solution of (3.1) is of the form of (3.4).

THEOREM 3.2. *For all times $T_\varepsilon = T_0 \varepsilon^{-2} > 0$ and for all $p \geq 1$ there are constants $C_p > 0$ explicitly depending on p such that*

$$\mathbb{E}\|u(t + T_\varepsilon)\|^p \leq C_p \varepsilon^p \quad and \quad \mathbb{E}\|P_s u(t + T_\varepsilon)\|^p \leq C_p \varepsilon^{2p} \tag{3.9}$$

for all $t \geq 0$, all mild solutions u of equation (3.1) independent of the initial condition $u(0)$, and for all $\varepsilon \in (0, 1)$.

    Furthermore, if we already assume that $\mathbb{E}\|u(0)\|^p \leq \tilde{C}_p \varepsilon^p$ for a constant $\tilde{C}_p > 0$, then there is a time $t_\varepsilon = \mathcal{O}(\ln(\varepsilon^{-1}))$ and a constant $C > 0$ such that

$$\mathbb{E}\|u(t)\|^p \leq C \varepsilon^p \quad and \quad \mathbb{E}\|P_s u(t + t_\varepsilon)\|^p \leq C \varepsilon^{2p} \tag{3.10}$$

for all $t \geq 0$, all X-valued mild solutions u, and for all $\varepsilon \in (0, 1)$.

The proof of this theorem is based on a priori estimates and it takes into account the global nonlinear stability of the equation. This was not proved directly in [1], but under our somewhat stronger assumptions the proof of this result is similar to the proof in [1, Lemma 4.3]. The proof of this lemma relies on a priori estimates for $v_{\delta_\varepsilon} = u - \varepsilon^2 W_{L-\delta_\varepsilon}$ with $\delta_\varepsilon = \mathcal{O}(\varepsilon^2)$, which fulfils a random PDE similar to (3.8). The main advantage is that the linear semigroup generated by $L - \delta_\varepsilon$ is exponentially stable, which simplifies the bounds for the stochastic convolution.

3.4. Approximation. For a solution A of (3.5) and ψ of (3.6) we define the approximation $\varepsilon w(t) := \varepsilon A(\varepsilon^2 t) e^{ix} + c.c. + \varepsilon^2 \psi(t)$. The residual of εw is given by

$$\mathrm{Res}(\varepsilon w)(t) = -\varepsilon w(t) + e^{tL} u(0) + \varepsilon^3 \int_0^t e^{(t-\tau)L} [\nu w - w^3](\tau)\, d\tau + \varepsilon^2 W_L(t). \tag{3.11}$$

Now the main idea is to obtain bounds on $P_c \mathrm{Res}(\varepsilon w)$ via the amplitude equation and to bound $P_s \mathrm{Res}(\varepsilon w)$ by using the linear stability of (2.4). As usual, these estimates require good a priori bounds on the approximation εw, but do not require any a priori knowledge on the solution u of the original equation.

    Bounds on the residual easily imply approximation results, since they enable us to establish bounds on the difference between εw_k and u using (3.11) and (3.7).

THEOREM 3.3. *Let u be the mild solution of (3.1) with (random) initial value $u(0)$, which fulfils (3.9). This means there exists a family of positive constants $\{C_p\}_{p \geq 1}$ such that*

$$\mathbb{E}\|u(0)\|^p \leq C_p \varepsilon^p \quad and \quad \mathbb{E}\|P_s u(0)\|^p \leq C_p \varepsilon^{2p} \tag{3.12}$$

Then for all $p \geq 1$, $1 \gg \kappa > 0$ and $T_0 > 0$ there is a constant $C > 0$ such that

$$\mathbb{E}\left(\sup_{t \in [0, T_0 \varepsilon^{-2}]} \|u(t) - \varepsilon w(t)\|^p \right) \leq C \varepsilon^{3p - \kappa} \quad for \ all \ \varepsilon \in (0, 1) .$$

    The proof of this theorem can be found in [1, Sec. 4].

4. Large Domains. In this section we summarise the results of [3]. This paper contains the first rigorous derivation of amplitude equations for SPDEs on large domains, near a change of stability. We choose to study our SPDE on a large but bounded domain of length $\mathcal{O}(\varepsilon^{-1})$, rather than the whole real line in order to avoid the technical difficulties arising for SPDEs on unbounded domains. Nevertheless, we manage to capture the high dimensionality of the space changing stability.

4.1. Setting. Consider the stochastic SH equation on $D_\varepsilon = [-L/\varepsilon, L/\varepsilon]$

$$\partial_t u = -(1 + \partial_x^2)^2 u + \varepsilon^2 \nu u - u^3 + \varepsilon^{\frac{3}{2}} \sigma \xi_\varepsilon, \quad u(t,x) \in \mathbb{R} \quad \text{periodic in } x \in D_\varepsilon , \quad (4.1)$$

with $\nu, \sigma \in [-1, 1]$, and ξ_ε is space–time white noise on the torus. Thus, ξ_ε is a generalised centred Gaussian field such that up to L/ε−periodicity of $\xi_\varepsilon(t, \cdot)$

$$\mathbb{E}\xi_\varepsilon(s,x) = 0, \quad \mathbb{E}\xi_\varepsilon(s,x)\xi_\varepsilon(t,y) = \delta(t-s)\delta(|x-y|) . \quad (4.2)$$

In order to handle the fact that the dominating modes $\mathrm{e}^{\pm ix/\varepsilon}$ are not necessarily $2L$-periodic, we introduce $N_\varepsilon = [\frac{L}{\varepsilon\pi}]$, where $[z] \in \mathbb{Z}$ is the nearest integer of $z \in \mathbb{R}$ with $[\frac{1}{2}] = 1$, $\delta_\varepsilon = \frac{1}{\varepsilon} - \frac{\pi}{L}N_\varepsilon$, and the dominant wave-number $\rho_\varepsilon = N_\varepsilon \frac{\pi\varepsilon}{L}$.
   Our goal is to show that one can approximate solutions u of (4.1) by

$$u(t,x) \approx \varepsilon A(\varepsilon^2 t, \varepsilon x)\mathrm{e}^{ix} + c.c.,$$

where $\Re$ denotes the real part of a complex number and $A = A(T, X)$ is a solution of the following complex stochastic Ginzburg– Landau equation

$$\partial_t A = \Delta_\varepsilon A + \nu A - 3|A|^2 A + \sigma\eta , \quad \Delta_\varepsilon := -4(i\partial_X + \delta_\varepsilon)^2 , \quad (4.3)$$

with L-periodic boundary conditions. Here η is a version of complex space–time white noise. i.e. a generalised Gaussian field with

$$\mathbb{E}\eta(T,X) = 0, \quad \mathbb{E}\eta(T,X)\eta(S,Y) = 0, \quad \mathbb{E}\eta(T,X)\overline{\eta(S,Y)} = \delta(T-S)\delta(X-Y) \quad (4.4)$$

up to L−periodicity of $\eta(T, \cdot)$. Note that all ε−dependent terms in the amplitude equation are uniformly small in L.

REMARK 1. Correlations in space for the noise ξ_ε in (4.2) are possible, but if the correlation decays sufficiently fast (e.g., in L^2), then we end up with (4.4) for (4.3). Thus we encounter regularity problems for spatial dimension larger than 1.

4.2. Results. We consider a class of *admissible* initial conditions given below, which is a natural condition due to THEOREM 4.3. Note that the number of modes near criticality is of order $\mathcal{O}(1/\varepsilon)$. Thus any bound for solutions on the dominant part needs a decay condition in Fourier space.

DEFINITION 4.1. A family of random variables $\{A^\varepsilon\}_{\varepsilon\in(0,1]}$ with $A^\varepsilon \in L^2([-L, L], \mathbb{C})$ is *admissible* if there exists a decomposition $A^\varepsilon = W_0^\varepsilon + A_1^\varepsilon$, a constant $C_0 > 0$, and a family of positive constants $\{C_q\}_{q\geq 1}$ such that
  1. $A_1^\varepsilon \in H^1([-L, L], \mathbb{C})$ almost surely and $\mathbb{E}\|A_1^\varepsilon\|_{H^1}^q \leq C_q$ for every $q \geq 1$.
  2. W_0^ε are centred Gaussian random variables with $|\mathbb{E}\langle e_k, W_0^\varepsilon\rangle\langle e_\ell, W_0^\varepsilon\rangle| \leq C_0 \frac{\delta_{k\ell}}{1+|k|^2}$, for all $k, \ell \in \mathbb{Z}$ ($\delta_{k\ell} = 1$ for $k = \ell$ and 0, otherwise).

DEFINITION 4.2. A family of random variables $\{u^\varepsilon\}_{\varepsilon\in(0,1]}$ with $u^\varepsilon \in L^2(D_\varepsilon, \mathbb{R})$ is admissible if the family $\varepsilon^{-1}u^+\mathrm{e}^{-i\rho_\varepsilon x} \in L^2([-L, L], \mathbb{C})$ is admissible in the sense of Definition 4.1. Here, for $u = \sum_{k\in\mathbb{Z}} u_k \mathrm{e}^{ik\pi x/L}$ we use $u^+ = \frac{1}{2}u_0 + \sum_{k=1}^\infty u_k \mathrm{e}^{ik\pi x/L}$.

It is a quite natural assumption to have admissible initial conditions, as solutions of SH are admissible after some usually large time. In particular the following result is true (cf. Theorem 1.1 or 5.1 of [3]). It balances the smoothing properties of the deterministic PDE, which would lead to exponentially fast decaying Fourier modes, with the roughening

of the noise. The H^1-part in the definition above reflects deterministic smoothing, while the Gaussian part collects the effects of the noise.

THEOREM 4.3 (Attractivity). *Let $\{u^\varepsilon\}_{\varepsilon \in (0,1]}$ be a family given by mild solutions of* (4.1) *with arbitrary initial conditions in $L^2(D)$. Then for fixed $T_0 > 0$ the family $u^\varepsilon(T\varepsilon^{-2})$ is admissible for all $T \geq T_0$.*

Our main result on the approximation is the following (cf. [3, Theorem 1.2 or 4.1]).

THEOREM 4.4 (Approximation). *Let $\{u^\varepsilon\}_{\varepsilon \in (0,1]}$ be a family given by mild solutions of* (4.1) *with an admissible initial condition $u_0^\varepsilon(x) = \varepsilon(A_0(\varepsilon x)e^{ix\rho_\varepsilon} + c.c.)$. Consider the mild solution A to* (4.3) *with $A(0) = A_0$. Then, for every $T_0 > 0$, $\kappa > 0$, and $p \geq 1$, one can find joint realisations of the noises η and ξ_ε such that for all $\varepsilon \in (0,1]$*

$$\left(\mathbb{E} \sup_{t \in [0, \frac{T_0}{\varepsilon^2}]} \sup_{x \in D_\varepsilon} |u^\varepsilon(t,x) - \varepsilon\big(A(t, \varepsilon x)e^{ix\rho_\varepsilon} + c.c.\big)|^p \right)^{1/p} \leq C_{\kappa, p, L, T_0} \, \varepsilon^{3/2 - \kappa} \, . \quad (4.5)$$

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