OSCILLATION CRITERIA FOR HALF-LINEAR PARTIAL DIFFERENTIAL EQUATIONS VIA PICONE'S IDENTITY[*]

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Abstract. A Picone's identity is established for a class of half-linear partial differential equations, and oscillation criteria are obtained by using the Picone's identity. By reducing the oscillation problem for half-linear partial differential equations to a one-dimensional oscillation problem for half-linear ordinary differential equations, we derive various oscillation results.

Key words. Picone's inequality, oscillation criteria, half-linear, partial differential equation

AMS subject classifications. 35B05

1. Introduction. Recently there has been much interest in studying the oscillatory behavior of solutions of half-linear differential equations. There are many papers (or books) dealing with oscillations of half-linear partial differential equations, see, e.g. Bognár and Došlý [2], Došlý [3, 4], Došlý and Mařík [5], Došlý and Řehák [6] Dunninger [7], Kusano, Jaroš and Yoshida [10], Mařík [13, 14] and Yoshida [15]. Picone identity plays an important role in Sturmian comparison theory and oscillation theory of differential equations. We mention the papers [1, 2, 3, 5, 7, 10, 15] which deal with Picone identity for half-linear partial differential equations. In particular, the paper [15] treats the half-linear partial differential equation with first order term

$$\nabla \cdot \left(A(x)|\nabla v|^{\alpha-1}\nabla v \right) + (\alpha + 1)|\nabla v|^{\alpha-1}B(x) \cdot \nabla v + C(x)|v|^{\alpha-1}v = 0. \qquad (*)$$

The purpose of this paper is to establish a Picone identity for the half-linear partial differential equation

$$P_\alpha[v] \equiv \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\left(A_i(x)\right)^2|\nabla_A v|^{\alpha-1}\frac{\partial v}{\partial x_i} \right) + (\alpha+1)|\nabla_A v|^{\alpha-1}B(x) \cdot \nabla_A v \\ + C(x)|v|^{\alpha-1}v = 0 \qquad (1.1)$$

and to derive oscillation results for (1.1) using the Picone identity, where $\alpha > 0$ is a constant and

$$\nabla_A v = \left(A_1(x)\frac{\partial v}{\partial x_1}, ..., A_n(x)\frac{\partial v}{\partial x_n} \right).$$

We note that the half-linear partial differential equation (1.1) is a generalization of $(*)$. In fact, if

$$A_1(x) = A_2(x) = \cdots = A_n(x) = A(x)^{\frac{1}{\alpha+1}} \ (A(x) > 0),$$

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we see that (1.1) reduces to

$$\nabla \cdot \left(A(x)|\nabla v|^{\alpha-1}\nabla v\right) + (\alpha+1)A(x)^{\frac{\alpha}{\alpha+1}}|\nabla v|^{\alpha-1}B(x) \cdot \nabla v$$
$$+ C(x)|v|^{\alpha-1}v = 0.$$

2. Picone identity. In this section we establish a Picone identity for (1.1), and obtain a sufficient condition for every solution v of (1.1) to have a zero on $\overline{G}$, where G is a bounded domain in $\mathbb{R}^n$ with piecewise smooth boundary ∂G.

It is assumed that $A_i(x) \in C(\overline{G}; (0, \infty))$ $(i = 1, 2, ..., n)$, $B(x) \in C(\overline{G}; \mathbb{R}^n)$ and $C(x) \in C(\overline{G}; \mathbb{R})$.

The domain $\mathcal{D}_{P_\alpha}(G)$ of P_α is defined to be the set of all functions v of class $C^1(\overline{G}; \mathbb{R})$ with the property that $\left(A_i(x)\right)^2|\nabla_A v|^{\alpha-1}\frac{\partial v}{\partial x_i} \in C^1(G; \mathbb{R}) \cap C(\overline{G}; \mathbb{R})$ $(i = 1, 2, ..., n)$.

THEOREM 2.1 (Picone identity). *If* $v \in \mathcal{D}_{P_\alpha}(G)$, $v \neq 0$ *in* G, *then the following Picone identity holds for any* $u \in C^1(G; \mathbb{R})$:

$$
\begin{aligned}
&-\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(u\varphi(u)\frac{\left(A_i(x)\right)^2|\nabla_A v|^{\alpha-1}\frac{\partial v}{\partial x_i}}{\varphi(v)}\right) \\
&= -\left|\nabla_A u - u\,B(x)\right|^{\alpha+1} + C(x)|u|^{\alpha+1} \\
&\quad + \left|\nabla_A u - u\,B(x)\right|^{\alpha+1} + \alpha\left|\frac{u}{v}\nabla_A v\right|^{\alpha+1} \\
&\quad - (\alpha+1)\left(\nabla_A u - u\,B(x)\right) \cdot \Phi\left(\frac{u}{v}\nabla_A v\right) \\
&\quad - \frac{u\varphi(u)}{\varphi(v)}P_\alpha[v],
\end{aligned}
\tag{2.1}
$$

where $\varphi(s) = |s|^{\alpha-1}s$ $(s \in \mathbb{R})$ *and* $\Phi(\xi) = |\xi|^{\alpha-1}\xi$ $(\xi \in \mathbb{R}^n)$.

Proof. A direct calculation yields

$$
\begin{aligned}
&-\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(u\varphi(u)\frac{\left(A_i(x)\right)^2|\nabla_A v|^{\alpha-1}\frac{\partial v}{\partial x_i}}{\varphi(v)}\right) \\
&= -\sum_{i=1}^{n} \frac{\partial u}{\partial x_i}\varphi(u)\frac{\left(A_i(x)\right)^2|\nabla_A v|^{\alpha-1}\frac{\partial v}{\partial x_i}}{\varphi(v)} \\
&\quad - \sum_{i=1}^{n} u\varphi'(u)\frac{\partial u}{\partial x_i}\frac{\left(A_i(x)\right)^2|\nabla_A v|^{\alpha-1}\frac{\partial v}{\partial x_i}}{\varphi(v)} \\
&\quad - \sum_{i=1}^{n} u\varphi(u)\left(-\frac{\varphi'(v)}{\varphi(v)^2}\frac{\partial v}{\partial x_i}\right)\left(A_i(x)\right)^2|\nabla_A v|^{\alpha-1}\frac{\partial v}{\partial x_i} \\
&\quad - \sum_{i=1}^{n} u\varphi(u)\frac{\frac{\partial}{\partial x_i}\left(\left(A_i(x)\right)^2|\nabla_A v|^{\alpha-1}\frac{\partial v}{\partial x_i}\right)}{\varphi(v)}.
\end{aligned}
\tag{2.2}
$$

It is easy to see that

$$
\sum_{i=1}^{n} \frac{\partial u}{\partial x_i}\varphi(u)\frac{\left(A_i(x)\right)^2|\nabla_A v|^{\alpha-1}\frac{\partial v}{\partial x_i}}{\varphi(v)} = \varphi\left(\frac{u}{v}\right)\sum_{i=1}^{n}\left(A_i(x)\right)^2|\nabla_A v|^{\alpha-1}\frac{\partial v}{\partial x_i}\frac{\partial u}{\partial x_i}
\tag{2.3}
$$

in view of the fact that $\varphi(u)/\varphi(v) = \varphi(u/v)$. Since $u\varphi'(u) = \alpha\varphi(u)$, it can be shown that

$$
\sum_{i=1}^{n} u\varphi'(u) \frac{\partial u}{\partial x_i} \frac{\left(A_i(x)\right)^2 |\nabla_A v|^{\alpha-1} \frac{\partial v}{\partial x_i}}{\varphi(v)}
$$
$$
= \alpha\varphi\left(\frac{u}{v}\right) \sum_{i=1}^{n} \left(A_i(x)\right)^2 |\nabla_A v|^{\alpha-1} \frac{\partial v}{\partial x_i} \frac{\partial u}{\partial x_i}.
$$

(2.4)

Using the identity $\varphi'(v) = \alpha(\varphi(v)/v)$, we obtain

$$
\sum_{i=1}^{n} u\varphi(u) \left(-\frac{\varphi'(v)}{\varphi(v)^2} \frac{\partial v}{\partial x_i}\right) \left(A_i(x)\right)^2 |\nabla_A v|^{\alpha-1} \frac{\partial v}{\partial x_i}
$$
$$
= -\alpha \frac{u}{v} \varphi\left(\frac{u}{v}\right) \sum_{i=1}^{n} \left(A_i(x)\right)^2 |\nabla_A v|^{\alpha-1} \left(\frac{\partial v}{\partial x_i}\right)^2.
$$

(2.5)

Combining (2.2)–(2.5), we observe that

$$
-\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(u\varphi(u) \frac{\left(A_i(x)\right)^2 |\nabla_A v|^{\alpha-1} \frac{\partial v}{\partial x_i}}{\varphi(v)}\right)
$$
$$
= \alpha \frac{u}{v} \varphi\left(\frac{u}{v}\right) \sum_{i=1}^{n} \left(A_i(x)\right)^2 |\nabla_A v|^{\alpha-1} \left(\frac{\partial v}{\partial x_i}\right)^2
$$
$$
- (\alpha+1)\varphi\left(\frac{u}{v}\right) \sum_{i=1}^{n} \left(A_i(x)\right)^2 |\nabla_A v|^{\alpha-1} \frac{\partial v}{\partial x_i} \frac{\partial u}{\partial x_i}
$$
$$
- \frac{u\varphi(u)}{\varphi(v)} \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\left(A_i(x)\right)^2 |\nabla_A v|^{\alpha-1} \frac{\partial v}{\partial x_i}\right).
$$

(2.6)

It is easily verified that

$$
\alpha \frac{u}{v} \varphi\left(\frac{u}{v}\right) \sum_{i=1}^{n} \left(A_i(x)\right)^2 |\nabla_A v|^{\alpha-1} \left(\frac{\partial v}{\partial x_i}\right)^2
$$
$$
= \alpha \left|\frac{u}{v}\right|^{\alpha+1} |\nabla_A v|^{\alpha-1} \sum_{i=1}^{n} \left(A_i(x)\right)^2 \left(\frac{\partial v}{\partial x_i}\right)^2
$$
$$
= \alpha \left|\frac{u}{v} \nabla_A v\right|^{\alpha+1}.
$$

(2.7)

A simple computation shows that

$$
-(\alpha+1)\varphi\left(\frac{u}{v}\right) \sum_{i=1}^{n} \left(A_i(x)\right)^2 |\nabla_A v|^{\alpha-1} \frac{\partial v}{\partial x_i} \frac{\partial u}{\partial x_i}
$$
$$
= -(\alpha+1) \left|\frac{u}{v} \nabla_A v\right|^{\alpha-1} \sum_{i=1}^{n} \left(A_i(x) \frac{\partial u}{\partial x_i}\right) \left(\frac{u}{v} A_i(x) \frac{\partial v}{\partial x_i}\right)
$$
$$
= -(\alpha+1) \left|\frac{u}{v} \nabla_A v\right|^{\alpha-1} (\nabla_A u) \cdot \left(\frac{u}{v} \nabla_A v\right).
$$

(2.8)

Hence, combining (2.6)–(2.8) yields the following :

$$
-\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(u\varphi(u) \frac{\left(A_i(x)\right)^2 |\nabla_A v|^{\alpha-1} \frac{\partial v}{\partial x_i}}{\varphi(v)} \right)
$$

$$
= \alpha \left| \frac{u}{v} \nabla_A v \right|^{\alpha+1} - (\alpha+1) \left| \frac{u}{v} \nabla_A v \right|^{\alpha-1} (\nabla_A u) \cdot \left(\frac{u}{v} \nabla_A v \right) \tag{2.9}
$$

$$
- \frac{u\varphi(u)}{\varphi(v)} \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\left(A_i(x)\right)^2 |\nabla_A v|^{\alpha-1} \frac{\partial v}{\partial x_i} \right).
$$

We easily obtain

$$
\frac{u\varphi(u)}{\varphi(v)} \left[(\alpha+1)|\nabla_A v|^{\alpha-1} B(x) \cdot \nabla_A v \right]
$$

$$
= (\alpha+1) \left| \frac{u}{v} \nabla_A v \right|^{\alpha-1} uB(x) \cdot \left(\frac{u}{v} \nabla_A v \right). \tag{2.10}
$$

Combining (2.9) and (2.10), we find that

$$
-\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(u\varphi(u) \frac{\left(A_i(x)\right)^2 |\nabla_A v|^{\alpha-1} \frac{\partial v}{\partial x_i}}{\varphi(v)} \right)
$$

$$
= \alpha \left| \frac{u}{v} \nabla_A v \right|^{\alpha+1} - (\alpha+1) \left| \frac{u}{v} \nabla_A v \right|^{\alpha-1} (\nabla_A u - uB(x)) \cdot \left(\frac{u}{v} \nabla_A v \right)
$$

$$
- \frac{u\varphi(u)}{\varphi(v)} \left[\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\left(A_i(x)\right)^2 |\nabla_A v|^{\alpha-1} \frac{\partial v}{\partial x_i} \right) \right. \tag{2.11}
$$

$$
\left. + (\alpha+1)|\nabla_A v|^{\alpha-1} B(x) \cdot \nabla_A v \right].
$$

Since

$$
\frac{u\varphi(u)}{\varphi(v)} C(x)|v|^{\alpha-1}v = C(x)|u|^{\alpha+1},
$$

we conclude that (2.11) is equivalent to the desired Picone identity (2.1). □

THEOREM 2.2. *Assume that there exists a nontrivial function* $u \in C^1(\overline{G};\mathbb{R})$ *such that* $u = 0$ *on* ∂G *and*

$$
M_G[u] \equiv \int_G \left[|\nabla_A u - u\, B(x)|^{\alpha+1} - C(x)|u|^{\alpha+1} \right] \mathrm{d}x \leq 0.
$$

Then every solution $v \in \mathcal{D}_{P_\alpha}(G)$ *of* (1.1) *must vanish at some point of* $\overline{G}$.

Proof. Suppose to the contrary that there exists a solution $v \in \mathcal{D}_P(G)$ of (1.1) satisfying $v \neq 0$ on $\overline{G}$. THEOREM 2.1 implies that the Picone-type inequality (2.1) holds for the nontrivial function u. Integrating (2.1) over G, we obtain

$$
0 = -M_G[u] + \int_G \left[|\nabla_A u - u\, B(x)|^{\alpha+1} + \alpha \left| \frac{u}{v} \nabla_A v \right|^{\alpha+1} \right.
$$

$$
\left. - (\alpha+1)(\nabla_A u - u\, B(x)) \cdot \Phi\left(\frac{u}{v} \nabla_A v \right) \right] \mathrm{d}x
$$

$$
\geq \int_G \left[|\nabla_A u - u\, B(x)|^{\alpha+1} + \alpha \left| \frac{u}{v} \nabla_A v \right|^{\alpha+1} \right. \tag{2.12}
$$

$$
\left. - (\alpha+1)(\nabla_A u - u\, B(x)) \cdot \Phi\left(\frac{u}{v} \nabla_A v \right) \right] \mathrm{d}x.
$$

It is easily seen that

$$\nabla_A u - u B(x) - \frac{u}{v}\nabla_A v = v\nabla_A\left(\frac{u}{v}\right) - uB(x) = v\left[\nabla_A\left(\frac{u}{v}\right) - \frac{u}{v}B(x)\right].$$

If

$$\nabla_A\left(\frac{u}{v}\right) - \frac{u}{v}B(x) \equiv 0 \quad \text{in} \quad G,$$

then we obtain the following

$$\nabla\left(\frac{u}{v}\right) - \frac{u}{v}B_A(x) \equiv 0 \qquad \text{in } G,$$

where

$$B_A(x) = \left(\frac{B_1(x)}{A_1(x)}, \ldots, \frac{B_n(x)}{A_n(x)}\right).$$

It follows from a result of Jaroš, Kusano and Yoshida [9, Lemma] that

$$\frac{u}{v} = C_0 \exp h(x) \qquad \text{on } \overline{G}$$

for some constant C_0 and some continuous function $h(x)$. Since $u = 0$ on ∂G, we obtain $C_0 = 0$, and hence $u \equiv 0$. This contradicts the fact that u is nontrivial, and therefore we find

$$\nabla_A u - u B(x) \not\equiv \frac{u}{v}\nabla_A v \qquad \text{in } G.$$

Hence, it follows from a result of Kusano, Jaroš and Yoshida [10, Lemma 2.1] that

$$\int_G\left[|\nabla_A u - u\,B(x)|^{\alpha+1} + \alpha\left|\frac{u}{v}\nabla_A v\right|^{\alpha+1}\right.$$
$$\left. -(\alpha+1)\left(\nabla_A u - u\,B(x)\right)\cdot\Phi\left(\frac{u}{v}\nabla_A v\right)\right]\mathrm{d}x > 0,$$

which, combined with (2.12), yields a contradiction. The proof is complete.   □

3. Oscillation results. We consider the half-linear partial differential equation

$$P_\alpha[v] \equiv \sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\left(A_i(x)\right)^2|\nabla_A v|^{\alpha-1}\frac{\partial v}{\partial x_i}\right) + (\alpha+1)|\nabla_A v|^{\alpha-1}B(x)\cdot\nabla_A v$$
$$+ C(x)|v|^{\alpha-1}v = 0 \tag{3.1}$$

in an unbounded domain $\Omega \subset \mathbb{R}^n$, where $\alpha > 0$ is a constant, $A_i(x) \in C(\Omega; (0,\infty))$ $(i = 1, 2, \ldots, n)$, $B(x) \in C(\Omega; \mathbb{R})$ and $C(x) \in C(\Omega; \mathbb{R})$.

The domain $\mathcal{D}_{P_\alpha}(\Omega)$ of P_α is defined to be the set of all functions v of class $C^1(\Omega; \mathbb{R})$ with the property that $\left(A_i(x)\right)^2|\nabla_A v|^{\alpha-1}\frac{\partial v}{\partial x_i} \in C^1(\Omega; \mathbb{R})$ $(i = 1, 2, \ldots, n)$.

A solution $v \in \mathcal{D}_{P_\alpha}(\Omega)$ of (3.1) is said to be *oscillatory* in Ω if it has a zero in Ω_r for any $r > 0$, where

$$\Omega_r = \Omega \cap \{x \in \mathbb{R}^n; |x| > r\}.$$

THEOREM 3.1. *Assume that for any $r > 0$ there exists a bounded and piecewise smooth domain G with $\overline{G} \subset \Omega_r$. If there is a nontrivial function $u \in C^1(\overline{G}; \mathbb{R})$ such that $u = 0$ on*

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∂G and $M_G[u] \leq 0$, where M_G is defined in Theorem 2.2, then every solution $v \in \mathcal{D}_{P_\alpha}(\Omega)$ of (3.1) is oscillatory in Ω.

Proof. Let $r > 0$ be an arbitrary number. THEOREM 2.2 implies that every solution $v \in \mathcal{D}_{P_\alpha}(\Omega)$ of (3.1) has a zero on $\overline{G} \subset \Omega_r$, that is, every solution v of (3.1) is oscillatory in Ω. □

LEMMA 3.2. *Let $0 < \alpha < 1$. Then we obtain the inequality*

$$|\nabla u - uW(x)|^{\alpha+1} \leq \frac{|\nabla u|^{\alpha+1}}{1-\alpha} + \frac{|W(x)|^{\alpha+1}}{1-\alpha}|u|^{\alpha+1} \tag{3.2}$$

for any function $u \in C^1(G; \mathbb{R})$ and any n-vector function $W(x) \in C(G; \mathbb{R})$.

Proof. The following inequality holds:

$$(\nabla u) \cdot \Phi(\nabla u) + \alpha \left(\nabla u - uW(x)\right) \cdot \Phi\left(\nabla u - uW(x)\right)$$
$$- (\alpha + 1)(\nabla u) \cdot \Phi\left(\nabla u - uW(x)\right) \geq 0$$

(see, e.g., Kusano, Jaroš and Yoshida [10, Lemma 2.1]). Hence, we have

$$(\nabla u) \cdot \Phi(\nabla u) + \alpha \left(\nabla u - uW(x)\right) \cdot \Phi\left(\nabla u - uW(x)\right)$$
$$-(\alpha + 1)\left(\nabla u - uW(x) + uW(x)\right) \cdot \Phi\left(\nabla u - uW(x)\right) \geq 0,$$

and therefore

$$|\nabla u|^{\alpha+1} + \alpha|\nabla u - uW(x)|^{\alpha+1}$$
$$-(\alpha + 1)\left[|\nabla u - uW(x)|^{\alpha+1} + uW(x) \cdot \Phi\left(\nabla u - uW(x)\right)\right] \geq 0,$$

or

$$|\nabla u|^{\alpha+1} - (\alpha + 1)uW(x) \cdot \Phi\left(\nabla u - uW(x)\right) \geq |\nabla u - uW(x)|^{\alpha+1}. \tag{3.3}$$

Using Schwarz's inequality and Young's inequality, we find that

$$|(\alpha + 1)uW(x) \cdot \Phi\left(\nabla u - uW(x)\right)|$$
$$\leq (\alpha + 1)|uW(x)||\nabla u - uW(x)|^{\alpha}$$
$$\leq (\alpha + 1)\left[\frac{|uW(x)|^{\alpha+1}}{\alpha + 1} + \frac{|\nabla u - uW(x)|^{\alpha+1}}{\frac{\alpha+1}{\alpha}}\right] \tag{3.4}$$
$$= |uW(x)|^{\alpha+1} + \alpha|\nabla u - uW(x)|^{\alpha+1}.$$

Combining (3.3) with (3.4) yields the following

$$|\nabla u - uW(x)|^{\alpha+1} \leq |\nabla u|^{\alpha+1} + |(\alpha + 1)uW(x) \cdot \Phi\left(\nabla u - uW(x)\right)|$$
$$\leq |\nabla u|^{\alpha+1} + |uW(x)|^{\alpha+1} + \alpha|\nabla u - uW(x)|^{\alpha+1},$$

and hence

$$(1 - \alpha)|\nabla u - uW(x)|^{\alpha+1} \leq |\nabla u|^{\alpha+1} + |W(x)|^{\alpha+1}|u|^{\alpha+1},$$

which is equivalent to (3.2). The proof is complete. □

THEOREM 3.3. *Let* $0 < \alpha < 1$. *Assume that for any* $r > 0$ *there exist a bounded and piecewise smooth domain* G *with* $\overline{G} \subset \Omega_r$ *and a nontrivial function* $u \in C^1(\overline{G}; \mathbb{R})$ *such that* $u = 0$ *on* ∂G *and*

$$\int_G \left[\frac{K(x)}{1-\alpha} |\nabla u|^{\alpha+1} - \left\{ C(x) - \frac{K(x)|B_A(x)|^{\alpha+1}}{1-\alpha} \right\} |u|^{\alpha+1} \right] \mathrm{d}x \le 0,$$

where $K(x) = \left(\max_{1 \le i \le n} A_i(x) \right)^{\alpha+1}$ *and*

$$B_A(x) = \left(\frac{B_1(x)}{A_1(x)}, \ldots, \frac{B_n(x)}{A_n(x)} \right).$$

Then every solution $v \in \mathcal{D}_{P_\alpha}(\Omega)$ *of* (3.1) *is oscillatory in* Ω.

Proof. It is easy to see that

$$|\nabla_A u - uB(x)| \le \sqrt{\max_{1 \le i \le n} (A_i(x))^2} \, |\nabla u - uB_A(x)| \le \left(\max_{1 \le i \le n} A_i(x) \right) |\nabla u - uB_A(x)|$$

and hence

$$|\nabla_A u - uB(x)|^{\alpha+1} \le K(x)|\nabla u - uB_A(x)|^{\alpha+1}. \tag{3.5}$$

Combining (3.2) with (3.5), we obtain

$$|\nabla_A u - uB_A(x)|^{\alpha+1} \le \frac{K(x)}{1-\alpha} |\nabla u|^{\alpha+1} + \frac{K(x)|B_A(x)|^{\alpha+1}}{1-\alpha} |u|^{\alpha+1}.$$

Therefore, we observe that

$$\int_G \left[|\nabla_A u - u\,B(x)|^{\alpha+1} - C(x)|u|^{\alpha+1} \right] \mathrm{d}x$$
$$\le \int_G \left[\frac{K(x)}{1-\alpha} |\nabla u|^{\alpha+1} - \left\{ C(x) - \frac{K(x)|B_A(x)|^{\alpha+1}}{1-\alpha} \right\} |u|^{\alpha+1} \right] \mathrm{d}x$$

and consequently, the conclusion follows from THEOREM 3.1. $\qquad\square$

LEMMA 3.4. *Let* $E(x) \in C(G;(0,\infty))$ *satisfy* $E(x) > \alpha$. *Then the inequality*

$$|\nabla u - uW(x)|^{\alpha+1} \le \frac{E(x)}{E(x)-\alpha} |\nabla u|^{\alpha+1} + \frac{|E(x)W(x)|^{\alpha+1}}{E(x)-\alpha} |u|^{\alpha+1} \tag{3.6}$$

holds for any function $u \in C^1(G;\mathbb{R})$ *and any n-vector function* $W(x) \in C(G;\mathbb{R})$.

Proof. Proceeding as in the proof of LEMMA 3.2, we see that the inequality (3.3) holds. Applying Schwarz's inequality and Young's inequality, we have

$$|(\alpha+1)uW(x) \cdot \Phi(\nabla u - uW(x))|$$
$$= \frac{1}{E(x)} (\alpha+1)|uE(x)W(x)||\nabla u - uW(x)|^\alpha \tag{3.7}$$
$$\le \frac{1}{E(x)} \left(|uE(x)W(x)|^{\alpha+1} + \alpha \, |\nabla u - uW(x)|^{\alpha+1} \right).$$

Combining (3.3) with (3.7) yields the following

$$|\nabla u - uW(x)|^{\alpha+1} \le |\nabla u|^{\alpha+1} + \frac{|E(x)W(x)|^{\alpha+1}}{E(x)}|u|^{\alpha+1} + \frac{\alpha}{E(x)}|\nabla u - uW(x)|^{\alpha+1}$$

and therefore

$$\left(1 - \frac{\alpha}{E(x)}\right)|\nabla u - uW(x)|^{\alpha+1} \le |\nabla u|^{\alpha+1} + \frac{|E(x)W(x)|^{\alpha+1}}{E(x)}|u|^{\alpha+1},$$

which is equivalent to (3.6). The proof is complete. □

THEOREM 3.5. *Let $K(x) > \alpha$. Assume that for any $r > 0$ there exist a bounded and piecewise smooth domain G with $\overline{G} \subset \Omega_r$ and a nontrivial function $u \in C^1(\overline{G}; \mathbb{R})$ such that $u = 0$ on ∂G and*

$$\int_G \left[\frac{(K(x))^2}{K(x) - \alpha}|\nabla u|^{\alpha+1} - \left\{ C(x) - (K(x))^{\alpha+2}\frac{|B_A(x)|^{\alpha+1}}{K(x) - \alpha} \right\}|u|^{\alpha+1} \right] dx \le 0.$$

Then every solution $v \in \mathcal{D}_{P_\alpha}(\Omega)$ of (3.1) is oscillatory in Ω.

Proof. We see from (3.5) and (3.6) with $E(x) = K(x)$ that

$$\int_G \left[|\nabla_A u - u\,B(x)|^{\alpha+1} - C(x)|u|^{\alpha+1} \right] dx$$

$$\le \int_G \left[\frac{(K(x))^2}{K(x) - \alpha}|\nabla u|^{\alpha+1} \right.$$

$$\left. - \left\{ C(x) - (K(x))^{\alpha+2}\frac{|B_A(x)|^{\alpha+1}}{K(x) - \alpha} \right\}|u|^{\alpha+1} \right] dx.$$

Hence, the conclusion follows from THEOREM 3.1. The proof is complete. □

Let $\{Q(x)\}_S(r)$ denote the spherical mean of $Q(x)$ over the sphere $S_r = \{x \in \mathbb{R}^n : |x| = r\}$, that is,

$$\{Q(x)\}_S(r) = \frac{1}{\omega_n r^{n-1}} \int_{S_r} Q(x)\,dS = \frac{1}{\omega_n} \int_{S_1} Q(r, \theta)d\omega,$$

where ω_n is the surface area of the unit sphere S_1 and (r, θ) is the hyperspherical coordinates on $\mathbb{R}^n$.

THEOREM 3.6. *Let $0 < \alpha < 1$. If the half-linear ordinary differential equation*

$$\left(r^{n-1} \left\{ \frac{K(x)}{1-\alpha} \right\}_S (r)\, |y'|^{\alpha-1}y' \right)'$$
$$+ r^{n-1} \left\{ C(x) - \frac{K(x)|B_A(x)|^{\alpha+1}}{1-\alpha} \right\}_S (r)\, |y|^{\alpha-1}y = 0 \tag{3.8}$$

is oscillatory, then every solution $v \in \mathcal{D}_{P_\alpha}(\mathbb{R}^n)$ of (3.1) is oscillatory in $\mathbb{R}^n$.

Proof. Let $\{r_k\}$ be the zeros of a nontrivial solution $y(r)$ of (3.8) such that $r_1 < r_2 < \cdots$, $\lim_{k \to \infty} r_k = \infty$. Letting

$$G_k = \{x \in \mathbb{R}^n; r_k < |x| < r_{k+1}\}\ (k = 1, 2, ...)$$

and $u_k(x) = y(|x|)$, we find that

$$M_{G_k}[u_k] \leq \int_{G_k} \left[\frac{K(x)}{1-\alpha} |\nabla u_k|^{\alpha+1} - \left\{ C(x) - \frac{K(x)|B_A(x)|^{\alpha+1}}{1-\alpha} \right\} |u_k|^{\alpha+1} \right] dx$$

$$= \omega_n \int_{r_k}^{r_{k+1}} \left[\left\{ \frac{K(x)}{1-\alpha} \right\}_S (r)|y'(r)|^{\alpha+1} \right.$$

$$\left. - \left\{ C(x) - \frac{K(x)|B_A(x)|^{\alpha+1}}{1-\alpha} \right\}_S (r)|y(r)|^{\alpha+1} \right] r^{n-1} dr$$

$$= -\omega_n \int_{r_k}^{r_{k+1}} \left[\left(r^{n-1} \left\{ \frac{K(x)}{1-\alpha} \right\}_S (r)|y'(r)|^{\alpha-1}y'(r) \right)' \right.$$

$$\left. + r^{n-1} \left\{ C(x) - \frac{K(x)|B_A(x)|^{\alpha+1}}{1-\alpha} \right\}_S (r)|y(r)|^{\alpha-1}y(r) \right] y(r) dr$$

$$= 0.$$

Hence, the conclusion follows from THEOREM 3.3.   □

THEOREM 3.7. *Let $K(x) > \alpha$ in $\mathbb{R}^n$. If the half-linear ordinary differential equation*

$$\left(r^{n-1} \left\{ \frac{(K(x))^2}{K(x) - \alpha} \right\}_S (r) |y'|^{\alpha-1}y' \right)'$$

$$+ r^{n-1} \left\{ C(x) - (K(x))^{\alpha+2} \frac{|B_A(x)|^{\alpha+1}}{K(x) - \alpha} \right\}_S (r) |y|^{\alpha-1}y = 0 \tag{3.9}$$

is oscillatory, then every solution $v \in \mathcal{D}_{P_\alpha}(\mathbb{R}^n)$ of (3.1) is oscillatory in $\mathbb{R}^n$.

Proof. The proof is quite similar to that of Theorem 3.6, and hence will be omitted.   □

Oscillation results for the half-linear ordinary differential equation

$$\left(p(r)|y'|^{\alpha-1}y' \right)' + q(r)|y|^{\alpha-1}y = 0$$

have been derived by numerous authors (see, e.g., Kusano and Naito [11] and Kusano, Naito and Ogata [12]). Various oscillation results for (3.1) can be obtained by combining THEOREMS 3.6 and 3.7 with the results of [11, 12].

The following THEOREMS 3.8 and 3.9 follow by combining THEOREMS 3.3 and 3.5 with the fact that the half-linear ordinary differential equation

$$\left(K_0 r^{n-1} |y'|^{\alpha-1}y' \right)' + C_0 r^{n-1}|y|^{\alpha-1}y = 0$$

is oscillatory for any $n \in \mathbb{N}$, $\alpha > 0$, $K_0 > 0$ and $C_0 > 0$ (see Kusano, Jaroš and Yoshida [10, Example]).

THEOREM 3.8. *Let $0 < \alpha < 1$. If there are positive constants K_0 and C_0 satisfying*

$$\frac{K(x)}{1-\alpha} \leq K_0, \quad C(x) - \frac{K(x)|B_A(x)|^{\alpha+1}}{1-\alpha} \geq C_0,$$

then every solution $v \in \mathcal{D}_{P_\alpha}(\mathbb{R}^n)$ of (3.1) is oscillatory in $\mathbb{R}^n$.

THEOREM 3.9. *Let $K(x) > \alpha$ in $\mathbb{R}^n$. If there are positive constants K_0 and C_0 satisfying*

$$\frac{(K(x))^2}{K(x) - \alpha} \leq K_0, \quad C(x) - (K(x))^{\alpha+2} \frac{|B_A(x)|^{\alpha+1}}{K(x) - \alpha} \geq C_0,$$

then every solution $v \in \mathcal{D}_{P_\alpha}(\mathbb{R}^n)$ of (3.1) *is oscillatory in* $\mathbb{R}^n$.

EXAMPLE. We consider the half-linear partial differential equation

$$\frac{\partial}{\partial x_1}\left(|\nabla_A v|\frac{\partial v}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(4|\nabla_A v|\frac{\partial v}{\partial x_2}\right)$$
$$+ 3|\nabla_A v|\left(3\frac{\partial v}{\partial x_1} + 16\frac{\partial v}{\partial x_2}\right) + \left(\frac{4}{3}\times 40^3 + 1\right)|v|v = 0 \tag{3.10}$$

for $x = (x_1, x_2) \in \mathbb{R}^2$, where

$$\nabla_A v = \left(\frac{\partial v}{\partial x_1}, 2\frac{\partial v}{\partial x_2}\right).$$

Here $n = \alpha = 2$, $A_1(x) = 1$, $A_2(x) = 2$, $K(x) = 8$, $B(x) = (3,8)$, $B_A(x) = (3,4)$, $C(x) = (4/3)\times 40^3 + 1$. Since

$$\frac{\left(K(x)\right)^2}{K(x)-\alpha} = \frac{32}{3}, \quad C(x) - (K(x))^{\alpha+2}\frac{|B_A(x)|^{\alpha+1}}{K(x)-\alpha} = 1,$$

we can take $K_0 = 32/3$ and $C_0 = 1$. It is easy to see that $K(x) > \alpha$, and hence THEOREM 3.9 implies that every solution v of (3.10) is oscillatory in $\mathbb{R}^2$.

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