On the convergence of the central path in semidefinite optimization

M. Halická* E. de Klerk†‡ C. Roos†§

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Abstract

The central path in linear optimization always converges to the analytic center of the optimal set. This result was extended to semidefinite programming in [1]. In this paper we show that this latter result is not correct in the absence of strict complementarity. We provide a counterexample, where the central path converges to a different optimal solution. This unexpected result raises many questions. We also give a rigorous proof that the central path always converges in semidefinite optimization, by using ideas from algebraic geometry.

Key words: Semidefinite optimization, linear optimization, interior point method, central path, analytic center.

1 Introduction

The central path is of fundamental importance in the study of interior point algorithms. The geometric view of the central path is that of an analytic curve which converges to an optimal solution. Most interior point methods 'follow' the central path approximately to reach the optimal set. In this paper we will re-examine the convergence property of the central path in the case of semidefinite optimization (SDO). We will show that the characterization of the limit point of the central path as found in [1] is not correct in the absence of strict complementarity. This negative result raises the question whether the central path always converges. Since there does not seem to be any detailed proof of the convergence property in the literature, we include a rigorous proof in this paper.

We first formulate semidefinite optimization problems in standard form and recall the definition of the central path and some of its properties.

*Faculty of Mathematics, Physics and Informatics, Comenius University, Mlynska dolina, 842 48 Bratislava, Slovakia. E–mail: halicka@fmph.uniba.sk. The research of this author was supported in part by VEGA grant 1/7675/20 of Slovak Scientific Grant Agency.

†Faculty of Information Technology and Systems, Delft University of Technology, P.O. Box 5031, 2600 GA Delft, The Netherlands.

‡Corresponding author. E–mail: E.deKlerk@ITS.TUDelft.nl

§E–mail: C.Roos@ITS.TUDelft.nl

1.1 The central path in semidefinite optimization

By S^n we denote the space of all real symmetric $n \times n$ matrices and for any $M, N \in S^n$ we define

$$M \bullet N = \mathrm{trace}(MN) = \sum_{i,j} m_{ij} n_{ij}.$$

The convex cones of symmetric positive semidefinite matrices and positive definite matrices will be denoted by S_+^n and S_{++}^n, respectively; $X \succeq 0$ and $X \succ 0$ mean that a symmetric matrix X is positive semidefinite and positive definite, respectively.

We will consider the following primal–dual pair of semidefinite programs in the standard form

$$(P): \qquad \min_{X \in S^n} \left\{ C \bullet X \ : \ A^i \bullet X = b_i \ (i = 1, \ldots, m) \ X \succeq 0 \right\},$$

$$(D): \qquad \max_{y \in \mathbb{R}^m, S \in S^n} \left\{ b^T y \ : \ \sum_{i=1}^m A^i y_i + S = C, \ S \succeq 0 \right\},$$

where $A^i \in S^n$ $(i = 1, \ldots, m)$ and $C \in S^n$, $b \in \mathbb{R}^m$. We assume that A^i $(i = 1, \ldots, m)$ are linearly independent. The solutions X and (y, S) will be referred to as feasible solutions as they satisfy the primal and dual constraints respectively.

We assume that both (P) and (D) satisfy the interior point condition, i.e. there exists (X^0, S^0, y^0) such that

$$A^i \bullet X^0 = b_i \ (i = 1, \ldots, m), \ X^0 \succ 0, \quad \text{and} \quad \sum_{i=1}^m A^i y_i^0 + S^0 = C, \ S^0 \succ 0.$$

The primal and dual feasible sets will be denoted by $\mathcal{P}$ and $\mathcal{D}$ respectively, and $\mathcal{P}^*$ and $\mathcal{D}^*$ will denote the respective optimal sets. It is well-known that under our assumptions both $\mathcal{P}^*$ and $\mathcal{D}^*$ are non-empty and bounded. The optimality conditions for (P) and (D) are

$$\begin{array}{rcl} A^i \bullet X &=& b_i, \quad X \succeq 0 \quad (i = 1, \ldots, m) \\ \sum_{i=1}^m A^i y_i + S &=& C, \quad S \succeq 0 \\ XS &=& 0. \end{array} \tag{1}$$

A strictly complementary solution can be defined as an optimal solution pair (X, S) satisfying the rank condition: rank X + rank $S = n$. Contrary to LO, for SDO the existence of the strictly complementary solution is not generally ensured.

We now relax the optimality conditions (1) to

$$\begin{array}{rcl} A^i \bullet X &=& b_i, \quad X \succeq 0 \quad (i = 1, \ldots, m) \\ \sum_{i=1}^m A^i y_i + S &=& C, \quad S \succeq 0 \\ XS &=& \mu I \end{array} \tag{2}$$

where I is the identity matrix and $\mu \geq 0$. It is easy to see that for $\mu = 0$ (2) gives (1) and hence it may have more solutions. On the other hand, it is well–known that for $\mu > 0$ system (2) has a unique solution, denoted by $(X(\mu), S(\mu), y(\mu))$ (see *e.g.* [5]). Similarly as for linear programming, this solution is seen as the parametric representation of an analytic curve (the *central path*) in terms of the parameter $\mu > 0$.*

*For a proof of the analyticity in the LO case see [8]. This proof can be extended to the SDO case.

It was shown that the central path for the SDO case shares many properties with the central path for the LO case. First, the basic property was established that the central path restricted to $0 < \mu \leq \bar{\mu}$ for some $\bar{\mu} > 0$ is bounded and thus it has limit points as $\mu \downarrow 0$ in the optimal set ([9], [4]). Then it was shown that the limit points are in the relative interior of the optimal set ([4], [1]). Finally, it was claimed by Goldfarb and Scheinberg [1] that the central path converges for $\mu \downarrow 0$ to the so-called analytic center of the optimal solution set. Although this result has been widely cited in the recent literature, we will show in this paper that it is not correct in the absence of strict complementarity. Let us mention that the correct proofs of this fact, however only under the assumption of strict complementarity, are given in [3] and [9].

1.2 Analytic center of the optimal solution set

A pair of optimal solutions $(X, S) \in \mathcal{P}^* \times \mathcal{D}^*$ is called a *maximally complementary solution pair* to the pair of problems (P) and (D) if it maximizes rank (X)+rank (S) over all optimal solution pairs. The set of maximally complementary solutions coincides with the relative interior of $(\mathcal{P}^* \times \mathcal{D}^*)$. Another characterization is: $(\bar{X}, \bar{S}) \in \mathcal{P}^* \times \mathcal{D}^*$ is maximally complementary if and only if

$$\mathcal{R}(\hat{X}) \subset \mathcal{R}(\bar{X}) \quad \forall \hat{X} \in \mathcal{P}^*, \quad \mathcal{R}(\hat{S}) \subset \mathcal{R}(\bar{S}) \quad \forall \hat{S} \in \mathcal{D}^*,$$

where $\mathcal{R}$ denotes the range space. For proofs of these characterizations see [4] and [1] and the references therein.

Let $\bar{X}$ and $\bar{S}$ be a pair of maximally complementary optimal solutions. Denote

$$|B| := \text{rank } \bar{X}, \quad \text{and} \quad |N| := \text{rank } \bar{S}$$

Obviously, $|B| + |N| \leq n$. Without loss of generality (applying an orthonormal transformation of problem data, if necessary) we can assume that

$$\bar{X} = \begin{bmatrix} \bar{X}_B & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \bar{S} = \begin{bmatrix} \bar{0} & 0 & 0 \\ 0 & \bar{S}_N & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where $\bar{X}_B \in S_{++}^{|B|}$ and $\bar{S}_N \in S_{++}^{|N|}$. Therefore, each optimal solution pair $(\hat{X}, \hat{S})$ is of the form

$$\hat{X} = \begin{bmatrix} \hat{X}_B & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \hat{S} = \begin{bmatrix} \hat{0} & 0 & 0 \\ 0 & \hat{S}_N & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where $\hat{X}_B \in S_{+}^{|B|}$ and $\hat{S}_N \in S_{+}^{|N|}$, since $\mathcal{R}(\hat{X}) \subset \mathcal{R}(\bar{X})$ and $\mathcal{R}(\hat{S}) \subset \mathcal{R}(\bar{S})$.

In what follows we consider the partition of any $M \in S^n$ corresponding to the above optimal partition so that

$$M = \begin{bmatrix} M_B & M_{BN} & M_{BT} \\ M_{NB} & M_N & M_{NT} \\ M_{TB} & M_{TN} & M_T \end{bmatrix}$$

We denote $\mathcal{I} = \{B, BN, BT, NB, N, NT, TB, TN, T\}$ the index set corresponding to the optimal partition. Referring to the all blocks of M except of M_B we will write M_i $(i \in \mathcal{I} - B)$.

Now, the optimal solutions sets can be characterized as seen below. Here X and S are considered in the block partition just mentioned above.

$$\mathcal{P}^* = \left\{ X \ : \ A_B^i \bullet X_B = b_i \ (\ i = 1, \ldots, n), \ X_B \in S_+^{|B|}, \ X_k = 0 \ (k \in \mathcal{I} - B) \right\},$$

$$\mathcal{D}^* = \left\{ (S, y) \ : \ \sum_{i=1}^m A_N^i y_i + S_N = C_N, \ S_N \in S_+^{|N|}, \ \sum_{i=1}^m A_k^i y_i = C_k, \ S_k = 0 \ (k \in \mathcal{I} - N) \right\}.$$

The analytic centers of these sets are defined as follows: $X^a \in \mathcal{P}^*$ is the analytic center of $\mathcal{P}^*$ if

$$X_B^a = \arg \max_{X_B \in S_{++}^{|B|}} \left\{ \ln \det X_B \ : \ A_B^i \bullet X_B = b_i, \ i = 1, \ldots, m \right\},$$

and $(y^a, S^a) \in \mathcal{D}^*$ is the analytic center of $\mathcal{D}^*$ if

$$(y^a, S_N^a) = \arg \max_{y \in \mathbb{R}^m, S_N \in S_{++}^{|N|}} \left\{ \ln \det S_N \ : \sum_{i=1}^m A_N^i y_i + S_N = C_N, \ \sum_{i=1}^m A_k^i y_i = C_k, \ k \in \ \mathcal{I} - N \right\}.$$

We end this section with two known results about the central path.

Lemma 1.1 ([4]) *Any limit point (X^*, S^*) of the central path is a maximally complementary optimal solution, i.e., it satisfies*

$$X_B^* \succ 0 \quad \textit{and} \quad S_N^* \succ 0.$$

Lemma 1.2 (see *e.g.* [2], Lemma 2.3.2) *For any $\mu > 0$ the central path $X(\mu), S(\mu), y(\mu)$ is an analytic center of the level set of the duality gap*

$$\left\{ (X, S, y) \ : \ A^i \bullet X = b_i \ (i = 1, \ldots, m), \ \sum_{i=1}^m A^i y_i + S = C, \ C \bullet X - b^T y = \mu n, \ X \in S_+^n, \ S \in S_+^n \right\}.$$

The last two lemmas make it plausible that the central path converges to the analytic center of the optimal set, but in the next section we show that this is not true.

2 Counterexamples

Let $n = 4$, $m = 4$, $b = [1\ 0\ 0\ 0]^T$ and

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \ A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \ A_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \ A_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

The primal problem can be simplified to: minimize x_{44} such that

$$X = \begin{bmatrix} 1-x_{22} & x_{12} & x_{13} & x_{14} \\ x_{12} & x_{22} & -\frac{1}{2}x_{44} & -\frac{1}{2}x_{33} \\ x_{13} & -\frac{1}{2}x_{44} & x_{33} & 0 \\ x_{14} & -\frac{1}{2}x_{33} & 0 & x_{44} \end{bmatrix} \succeq 0.$$

The optimal set of (P) is given by all the positive semidefinite matrices of the form

$$X^* = \begin{bmatrix} 1-x_{22} & x_{12} & 0 & 0 \\ x_{12} & x_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Solutions of the form X^* are clearly optimal, since $C \succeq 0$ and therefore $\mathrm{Tr}\,(CX) \geq 0\ \forall X \in \mathcal{P}$. The analytic center of $\mathcal{P}^*$ is obviously given by

$$\begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

However, we will show that the limit point of the primal central path satisfies

$$X(\mu) \rightarrow \begin{bmatrix} 0.4 & 0 & 0 & 0 \\ 0 & 0.6 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ as } \mu \downarrow 0.$$

The dual problem is to maximize y_1 such that

$$S = \begin{bmatrix} -y_1 & 0 & 0 & 0 \\ 0 & -y_1 & -y_3 & -y_2 \\ 0 & -y_3 & -y_2 & -y_4 \\ 0 & -y_2 & -y_4 & 1-y_3 \end{bmatrix} \succeq 0.$$

Thus the dual problem has a unique optimal solution

$$y_i^* = 0\ (i = 1, 2, 3, 4),\ S^* = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

It is also easy to see that strict complementary does *not* hold.

Due to the structure of feasible $S \in \mathcal{D}$, the primal central path has the following structure

$$X(\mu) = \begin{bmatrix} 1 - x_{22}(\mu) & 0 & 0 & 0 \\ 0 & x_{22}(\mu) & -\frac{1}{2}x_{44}(\mu) & -\frac{1}{2}x_{33}(\mu) \\ 0 & -\frac{1}{2}x_{44}(\mu) & x_{33}(\mu) & 0 \\ 0 & -\frac{1}{2}x_{33}(\mu) & 0 & x_{44}(\mu) \end{bmatrix}.$$

By Lemma 1.2, the point on the central path $X(\mu)$ is, for any $\mu > 0$, the analytic center of a level set. The level set is given by the primal feasibility and a level condition which is $x_{44} = x_{44}(\mu) > 0$ in our case. This implies that $X(\mu)$ maximizes

$$\det \begin{bmatrix} 1 - x_{22} & 0 & 0 & 0 \\ 0 & x_{22} & -\frac{1}{2}x_{44}(\mu) & -\frac{1}{2}x_{33} \\ 0 & -\frac{1}{2}x_{44}(\mu) & x_{33} & 0 \\ 0 & -\frac{1}{2}x_{33} & 0 & x_{44}(\mu) \end{bmatrix} \tag{3}$$

under the conditions

$$x_{22} \in (0,1), \quad x_{33} > 0, \quad x_{22}x_{33}x_{44}(\mu) - x_{33}^3 - x_{44}^3(\mu) > 0.$$

Setting the gradient (with respect to x_{22} and x_{33}) of the determinant in (3) to zero, we obtain the two equations:

$$x_{33}(\mu)x_{44}(\mu) - 2x_{22}(\mu)x_{33}(\mu)x_{44}(\mu) + \frac{1}{4}x_{44}(\mu)^3 + \frac{1}{4}x_{33}(\mu)^3 = 0 \tag{4}$$

$$(1 - x_{22}(\mu))\left(x_{22}(\mu)x_{44}(\mu) - \frac{3}{4}x_{33}(\mu)^2\right) = 0. \tag{5}$$

Using $x_{22}(\mu) \in (0,1)$, we deduce from (5) that

$$x_{33}(\mu) = \frac{2}{\sqrt{3}}\sqrt{x_{22}(\mu)x_{44}(\mu)}.$$

Substituting this expression in (4) and simplifying, we obtain:

$$\frac{2}{\sqrt{3}}\sqrt{x_{22}(\mu)} - \frac{10}{3\sqrt{3}}x_{22}(\mu)^{3/2} + \frac{1}{4}x_{44}(\mu)^{3/2} = 0.$$

In the limit where $\mu \downarrow 0$, we have $x_{44}(\mu) \to 0$. Moreover, we can assume that $x_{22}(\mu)$ is positive in the limit, since the limit point of the central path is maximally complementary (Lemma 1.1). Denoting $\lim_{\mu \downarrow 0} x_{22}(\mu) := x_{22}(0) > 0$, we have:

$$\frac{2}{\sqrt{3}}\sqrt{x_{22}(0)} - \frac{10}{3\sqrt{3}}x_{22}(0)^{3/2} = 0,$$

which implies $x_{22}(0) = 0.6$.

An example for the second order cone

The following example show that the central path may already fail to converge to the analytic center of the optimal set in the special case of second order cone optimization.

Consider the problem of minimizing x_{12} subject to

$$\begin{bmatrix} x_{11} & x_{12} & 0 & 0 & 0 \\ x_{12} & x_{22} & 0 & 0 & 0 \\ 0 & 0 & x_{33} & x_{22} & 0 \\ 0 & 0 & x_{22} & x_{12} & 0 \\ 0 & 0 & 0 & 0 & 1-(x_{11}+x_{33}) \end{bmatrix} \succeq 0.$$

Note that this problem is equivalent to a second order cone programming problem. The optimal set is given by all matrices of the above form where $x_{12} = x_{22} = 0$, and the analytic center of the optimal set is given by the optimal solution where $x_{11} = x_{33} = \frac{1}{3}$.

Using exactly the same technique as in the previous example, one can show that the limit point for the central path is $x_{11} = 2/7$, $x_{33} = 3/7$. However, the proof is more technical for this example due to the larger number of variables, and is therefore omitted.

3 Convergence of the central path

Since the central path does not converge to the analytic center in general, it is natural to ask whether it always converges. The convergence property seems to be a 'folklore' result† yet we could not find a rigorous proof in the SDO literature. In [6] the convergence of the central path for the linear complementary problem (LCP) is proven with the help of some results from algebraic geometry. In [5], Kojima *et al.* mention that this proof for LCP can be extended to the monotone semidefinite complementarity problem (which is equivalent to SDO), without giving a formal proof.

For this reason we include a complete proof here, which also uses some ideas from the theory of algebraic sets, but in a different manner as it was done in [6].

Definition 3.1 (Algebraic set) *A subset $V \in R^k$ is called an algebraic set if V is the locus of common zeroes of some collection of polynomial functions on $\mathbb{R}^k$.*

Lemma 3.1 (Curve selection lemma) *Let $V \subset \mathbb{R}^k$ be a real algebraic set, and let $U \subset \mathbb{R}^k$ be an open set defined by finitely many polynomial inequalities:*

$$U = \left\{ x \in \mathbb{R}^k : g_1(x) > 0, ..., g_l(x) > 0 \right\}.$$

If $U \cap V$ contains points arbitrarily close to the origin then there exists an $\epsilon > 0$ and a real analytic curve

$$p : [0, \epsilon) \mapsto \mathbb{R}^k$$

with $p(0) = 0$ and with $p(t) \in U \cap V$ for $t > 0$.

A proof of the Curve selection lemma is given in [7] (Lemma 3.1 on p. 25).

†For example, the convergence property is mentioned in the review paper [10] on p. 74 without supplying references or a proof.

Theorem 3.1 *The central path in semidefinite optimization always converges.*

Proof: Let (X^*, y^*, S^*) be any limit point of the central path of (P) and (D).

With reference to Lemma 3.1, let the real algebraic set V be defined via

$$V = \left\{ (\bar{X}, \bar{S}, \bar{y}, \mu) \;\middle|\; \begin{array}{rcl} \mathrm{Tr}\, A_i \bar{X} & = & 0 \quad (i = 1, \ldots, m) \\ \sum_i (\bar{y}_i) A_i + \bar{S} & = & 0 \\ (\bar{X} + X^*)(\bar{S} + S^*) - \mu I & = & 0 \end{array} \right\}$$

and let the open set U be defined by: $U = (\bar{X}, \bar{S}, \bar{y}, \mu)$ such that all principal minors of $(\bar{X} + X^*)$ and $(\bar{S} + S^*)$ are positive, and $\mu > 0$.

Now $V \cap U$ corresponds to the central path excluding its limit points, in the sense that if $(\bar{X}, \bar{S}, \bar{y}, \mu) \in U \cap V$ then $X(\mu) = (\bar{X} + X^*)$ and $S(\mu) = (\bar{S} + S^*)$, where $X(\mu)$ (resp. $S(\mu)$) denotes the μ-center of (P) (resp. (D)) as before.

Moreover, the zero element is in the closure of $V \cap U$, by construction.

The required result now follows from the curve selection lemma. To see this, note that Lemma 3.1 implies the existence of an $\epsilon > 0$ and an analytic function $p : [0, \epsilon) \mapsto \mathcal{S}^n \times \mathcal{S}^n \times \mathbb{R}^m \times \mathbb{R}$ such that

$$p(t) = \left(\bar{X}(t), \bar{S}(t), \bar{y}(t), \mu(t)\right) \to (0_{n\times n}, 0_{n\times n}, 0_m, 0) \text{ as } t \downarrow 0, \tag{6}$$

and if $t > 0$, $\left(\bar{X}(t), \bar{S}(t), \bar{y}(t), \mu(t)\right) \in U \cap V$, *i.e:*

$$\begin{array}{rcl} \mathrm{Tr}\, A_i \bar{X}(t) & = & 0 \quad (i = 1, \ldots, m) \\ \sum_i \bar{y}_i(t) A_i + \bar{S}(t) & = & 0 \\ (\bar{X}(t) + X^*)(\bar{S}(t) + S^*) - \mu(t) I & = & 0, \end{array} \tag{7}$$

and $\bar{X}(t) \succ 0$, $\bar{S}(t) \succ 0$, $\mu(t) > 0$.

Since the centrality system (2) has a unique solution, the system (7) also has a unique solution given by

$$\bar{X}(t) + X^* = X(\mu(t)), \quad \bar{S}(t) + S^* = S(\mu(t))$$

if $t > 0$. By (6), we therefore have

$$\lim_{t\downarrow 0} X(\mu(t)) = X^*, \quad \lim_{t\downarrow 0} S(\mu(t)) = S^*, \quad \lim_{t\downarrow 0} \mu(t) = 0.$$

Since $\mu(t)$ is a positive function on $(0, \epsilon)$ and $\mu(0) = 0$, and since it is analytic on $[0, \epsilon)$ there exists an interval, say $(0, \epsilon')$ where $\mu'(t) > 0$. Therefore the inverse function $\mu^{-1} : \mu(t) \mapsto t$ exists on the interval $(0, \mu(\epsilon'))$. Moreover, $\mu^{-1}(t) > 0$ for all $t \in (0, \mu(\epsilon'))$ and $\lim_{t\to 0} \mu^{-1}(t) = 0$.

This implies that

$$\lim_{t\downarrow 0} X(t) = \lim_{t\downarrow 0} X(\mu(\mu^{-1}(t))) = \lim_{t\downarrow 0} \bar{X}(\mu^{-1}(t)) + X^* = X^*.$$

Similarly $\lim_{t\downarrow 0} S(t) = S^*$, which completes the proof. $\square$

4 Conclusions and future work

The purpose of this paper was two-fold:

- to show that the central path in SDO may converge to an optimal solution which is not the analytic center of the optimal set (in the absence of strict complementarity);
- to give a rigorous proof that the central path always converges for SDO.

The first result raises some questions:

- Can we give a correct classification of the limit point of the central path?
- For which sub-classes of SDO problems can one guarantee convergence of the central path to the analytic center of the optimal set?

We therefore hope that the observations in this paper may lead to a renewed interest in the limiting behaviour of the central path in semidefinite optimization.

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