Analyticity of the central path at the boundary point in semidefinite programming

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Outline

1. Central path in SDP (overview)
2. Analyticity of central path in LP & LCP (overview)
3. Analyticity of central path at the boundary point in SDP (new result)

1. Central path in SDP

$S^n -$ the space of real symmetric $n \times n$ matrices

$$M \bullet N = \text{trace}(M^T N) = \sum_{i,j} M_{ij} N_{ij}$$

$$X \in S^n : \quad X \succeq 0 \quad \text{positive semidefinite}$$

$$X \succ 0 \quad \text{positive definite}$$

For given C, $A^i \in S^n$, $i = 1, ..., m$, and $b \in R^m$ we consider the primal-dual pair of semidefinite programs

$$\begin{aligned} \min_{X \in S^n} \quad & C \bullet X \\ \text{s.t.} \quad & A^i \bullet X = b^i, \quad i = 1, ..., m \\ & X \succeq 0 \end{aligned} \tag{P}$$

$$\begin{aligned} \max_{y \in \mathbb{R}^m, Z \in S^n} \quad & b^T y \\ \text{s.t.} \quad & \sum_{i=1}^m A^i y^i + Z = C, \\ & Z \succeq 0 \end{aligned} \tag{D}$$

Assumption 1:

A^i, $i = 1, ..., m$, are linearly independent, i.e.,

$$\sum_{i=1}^{m} A^i u^i = 0 \qquad \Rightarrow u^i = 0, \quad i = 1, ..., m.$$

Assumption 2:

Both (P) and (D) have strictly feasible solutions, i.e.,

$$\exists X \in S^n \ : \qquad A^i \bullet X = b^i, \quad i = 1, ..., m, \quad X \succ 0;$$

$$\exists (y, Z) \in R^m \times S^n \ : \qquad \sum_{i=1}^{m} A^i y^i + Z = C, \quad Z \succ 0.$$

Facts:

- (P) and (D) have optimal solutions $\hat{X}$ and $(\hat{y}, \hat{Z})$;

- $\hat{X} \bullet \hat{Z} = 0$;

- $\hat{X}\hat{Z} = \hat{Z}\hat{X} = 0_{n \times n}$.

Centering system for $\mu > 0$:

$$A^i \bullet X = b^i, \quad i = 1, ..., m, \quad X \succ 0, \tag{1}$$

$$\sum_{i=1}^{m} A^i y^i + Z = C, \quad Z \succ 0, \tag{2}$$

$$XZ = \mu I_n, \tag{3}$$

Fact: $\forall\ \mu > 0\ \exists\ !$ solution $(X(\mu), y(\mu), Z(\mu))$ of (1)-(3)

Central path: $\{\ (X(\mu), y(\mu), Z(\mu))\ \}_{\mu>0}$

Central path function: $\mathbb{R}_{++} \to \mathbb{R}^{n\times n} \times \mathbb{R}^m \times \mathbb{R}^{n\times n}$

Fact: The central path restricted to $0 < \mu \leq \bar{\mu}$ is bounded and each of its limit points for $\mu \downarrow 0$ is an (maximally complementary) optimal solution of (P), (D).

Mapping vec : $\mathbb{R}^{r\times s} \to \mathbb{R}^{rs}$

stacks the columns of a matrix in a vector

Notation:

$$A := \begin{bmatrix} (\mathrm{vec}(A^1))^T \\ \vdots \\ (\mathrm{vec}(A^m))^T \end{bmatrix}$$

Centering system rewritten by vec:

$$A\ \mathrm{vec}(X) = b, \quad X \succ 0, \tag{1'}$$

$$A^T y + \mathrm{vec}(Z) = \mathrm{vec}(C), \quad Z \succ 0, \tag{2'}$$

$$XZ = \mu I_n. \tag{3}$$

Jacobian matrix of (1'), (2'), (3)

$$J(X,y,Z) = \begin{pmatrix} A & 0 & 0 \\ 0 & A^T & I_{n^2} \\ Z \otimes I_n & 0 & I_n \otimes X \end{pmatrix}$$

where $\otimes$ is the Kronecker product sign

Notation: $J(\mu) := J(X(\mu), y(\mu), Z(\mu))$, $\mu > 0$.

Fact: $J(\mu)$, $\mu > 0$, is nonsingular.

[Alizadeh, Haeberly & Overton (1998) SIAM J.Opt. 8, 746-768]

Corollary:

The central path (as a function of $\mu > 0$) is analytic.

Proof: Application of the implicit function theorem

What are the properties of the central path as $\mu \downarrow 0$?

Assumption 3:

There exists a strictly complementary optimal solution, i.e., $\exists(\bar{X}, \bar{y}, \bar{Z}) : \quad \bar{X}$ and $(\bar{y}, \bar{Z})$ are optimal solutions of (P) and (D) and $\mathrm{rank}(\bar{X}) + \mathrm{rank}(\bar{Z}) = n$.

Fact: The central path converges to the analytic center of the optimal solution set as $\mu \downarrow 0$.

[Klerk, Roos, Terlaky: Topics in SD&IPM, 18, 215-236]

Thus the central path can be defined even at $\mu = 0$ by

$$X(0) := \lim_{\mu\downarrow 0} X(\mu), \quad y(0) := \lim_{\mu\downarrow 0} y(\mu), \quad Z(0) := \lim_{\mu\downarrow 0} Z(\mu)$$

Fact: $\|X(\mu) - X(0)\| = O(\mu), \quad \|Z(\mu) - Z(0)\| = O(\mu).$

[Luo, Sturm & Zhang: SIAM J. Optim. (1998) 8, 59-81]

$$\boxed{\text{What are the properties of higher-order derivatives of the central path as } \mu \downarrow 0\text{ ?}}$$

Drawback: $J(\mu)$ may be singular at $\mu = 0$

2. Analyticity of central path in LP & LCP

2.1 Linear programming

Fact: The all-order derivatives of the central path have finite limits as $\mu \downarrow 0$

[O. Güler (1994) Math. Program. 65, 347-363]

Fact: The central path is analytic (as a function of μ) even at $\mu = 0$

[M. Halická (1999) Math. Program. 84, 335-355]
[M. Wechs (1998) Optim., Methods and Software 9, 209-243]

• The proofs proceed from the explicit recursive formulas for all-order derivatives at $\mu > 0$ and from the analysis of their limit properties as $\mu \downarrow 0$. They yield formulas for all-order derivatives at $\mu = 0$.

• Both proofs of analyticity are constructive, estimating the radius of convergence of the corresponding Taylor series, but they are rather technical and quite lengthy.

2.2 Linear complementarity problem

Facts :

(a) If there exists a strictly complementary optimal solution, then the central path is analytic (as a function of μ) even at $\mu = 0$

(b) If there is no strictly complementary solution, then the central path is analytic (as a function of $\sqrt{\mu}$) even at $\mu = 0$

[J.Stoer, M. Wechs (1999) Numer. Math. 81, 631-645]
[J.Stoer, M.Wechs (1998) Math. Progr. 83, 403-423]
[J.Stoer, M.Wechs, S.Mizuno (1998), Path. Oper. Res. 23, 832-862]

• The proof of (a) is very simple, based on the implicit function theorem applied to a system adapted from the equations defining the central path

• In addition to (a) the proof of (b) requires also an analysis of convergence rate of non-strictly complementary variables along the central path, where some results by Robinson from the theory of error bounds for system of linear inequalities were used.

Back to the LP case:

The idea by Stoer & Wechs was applied directly to LP:

M. Halická: Two simple proofs of analyticity of the central path in linear programming (to appear in Oper.Res.Let.)

What about the SDP case?

Surprisingly, this idea can be applied even to SDP (under Assumptions 1-3)

3. Analyticity of central path at $\mu = 0$ in SDP

Theorem. *Under Assumptions 1-3 the central path (as a function of $\mu \geq 0$) is analytic even at $\mu = 0$.*

Proof:

1. Decomposition of matrix variables
2. Introduction of tilde variables
3. Adapting the system for the tilde path
4. Nonsingularity of the corresponding Jacobian at $\mu = 0$.

3.1 Decomposition of matrix variables

Without loss of generality (applying an orthonormal transformation to the problem data if necessary) we may assume that each optimal solution pair X^* and Z^* of (P) and (D) is of the form

$$X^* = \begin{bmatrix} X_{11}^* & 0 \\ 0 & 0 \end{bmatrix}, \text{ where } X_{11}^* \in S^r,\ X_{11}^* \succeq 0,$$

$$Z^* = \begin{bmatrix} 0 & 0 \\ 0 & Z_{22}^* \end{bmatrix}, \text{ where } Z_{22}^* \in S^{n-r},\ Z_{22}^* \succeq 0.$$

$$r := \operatorname{rank}(\bar{X}), \quad \bar{X} \text{ from Assumption 3}$$

Note! r is independent of the choice of strictly complementary optimal solution

Decomposition of any $M \in S^n$ with respect to the above partition of the optimal solution matrices

$$M \in S^n \qquad \Rightarrow \qquad M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

The equations of (1)-(3) can now be partitioned

$$A^i_{11} \bullet X_{11}(\mu) + A^i_{12} \bullet X_{12}(\mu) + A^i_{21} \bullet X_{21}(\mu)$$

$$+A^i_{22} \bullet X_{22}(\mu) = b^i, \quad i = 1, ..., n \tag{4}$$

$$\sum_{i=1}^{m} A^i_{11} y^i(\mu) + Z_{11}(\mu) = C_{11}$$
$$\sum_{i=1}^{m} A^i_{12} y^i(\mu) + Z_{12}(\mu) = C_{12}$$
$$\sum_{i=1}^{m} A^i_{21} y^i(\mu) + Z_{21}(\mu) = C_{21}$$
$$\sum_{i=1}^{m} A^i_{22} y^i(\mu) + Z_{22}(\mu) = C_{22} \tag{5}$$

$$X_{11}(\mu)Z_{11}(\mu) + X_{12}(\mu)Z_{21}(\mu) = \mu I_r$$

$$X_{11}(\mu)Z_{12}(\mu) + X_{12}(\mu)Z_{22}(\mu) = 0$$

$$X_{21}(\mu)Z_{11}(\mu) + X_{22}(\mu)Z_{21}(\mu) = 0$$

$$X_{21}(\mu)Z_{12}(\mu) + X_{22}(\mu)Z_{22}(\mu) = \mu I_{n-r} \tag{6}$$

3.2 Introduction of tilde variables

Since the central path converges to a strictly complementary optimal solution, we have

$$
\begin{array}{ll}
X_{11}(\mu) \to X_{11}(0) \succ 0 & Z_{11}(\mu) \to 0 \\
X_{12}(\mu) \to 0 & Z_{12}(\mu) \to 0 \\
X_{21}(\mu) \to 0 & Z_{21}(\mu) \to 0 \\
X_{22}(\mu) \to 0 & Z_{22}(\mu) \to Z_{22}(0) \succ 0
\end{array}
$$

It is easy to prove that

$$
\frac{X_{22}(\mu)}{\mu} \to \tilde{X}_{22}(0) \succ 0 \quad \text{and} \quad \frac{Z_{11}(\mu)}{\mu} \to \tilde{Z}_{11}(0) \succ 0
$$

From the results by Luo, Sturm & Zhang [SIAM J. Optim. (1998) 8, 59-81] it follows that

$$
\begin{array}{ll}
\dfrac{X_{12}(\mu)}{\mu} \to \tilde{X}_{12}(0) & \dfrac{Z_{12}(\mu)}{\mu} \to \tilde{Z}_{12}(0) \\
\dfrac{X_{21}(\mu)}{\mu} \to \tilde{X}_{21}(0) & \dfrac{Z_{21}(\mu)}{\mu} \to \tilde{Z}_{21}(0)
\end{array}
$$

New **tilde variables** will be introduced, corresponding to the variables with the rate of convergence $O(\mu)$.

The **tilde variables** are defined by

$$\tilde{X}_{12} := \frac{X_{12}}{\mu}, \qquad \tilde{X}_{21} := \frac{X_{12}}{\mu}, \qquad \tilde{X}_{22} := \frac{X_{12}}{\mu},$$
$$\tilde{Z}_{11} := \frac{Z_{11}}{\mu}, \qquad \tilde{Z}_{12} := \frac{Z_{12}}{\mu}, \qquad \tilde{Z}_{21} := \frac{Z_{21}}{\mu}$$

Now, instead of the central path functions

$$X_{11}(\mu), \quad X_{12}(\mu), \quad X_{21}(\mu), \quad X_{22}(\mu), \quad y(\mu)$$
$$Z_{11}(\mu), \quad Z_{12}(\mu), \quad Z_{21}(\mu), \quad Z_{22}(\mu)$$

we will consider the **tilde path** functions

$$X_{11}(\mu), \quad \tilde{X}_{12}(\mu), \quad \tilde{X}_{21}(\mu), \quad \tilde{X}_{22}(\mu), \quad y(\mu)$$
$$\tilde{Z}_{11}(\mu), \quad \tilde{Z}_{12}(\mu), \quad \tilde{Z}_{21}(\mu), \quad Z_{22}(\mu)$$

where only $X_{11}(\mu)$, $y(\mu)$ and $Z_{22}(\mu)$ remain unchanged.

Since $\quad X_{12}(\mu) = \mu\tilde{X}_{12}$, etc., $\quad$ to the analyticity of the central path at $\mu = 0$ it suffices to prove analyticity of the tilde path at $\mu = 0$.

From (4), (5) and (6), using the tilde variables, we can obtain equations defining the tilde path

$$A^i_{11} \bullet X_{11}(\mu) + \mu A^i_{12} \bullet \tilde{X}_{12}(\mu) + \mu A^i_{21} \bullet \tilde{X}_{21}(\mu)$$

$$+\mu A^i_{22} \bullet \tilde{X}_{22}(\mu) = b^i, \quad i = 1, ..., m \quad (7)$$

$$\sum_{i=1}^{m} A^i_{11} y^i(\mu) + \mu \tilde{Z}_{11}(\mu) = C_{11}$$
$$\sum_{i=1}^{m} A^i_{12} y^i(\mu) + \mu \tilde{Z}_{12}(\mu) = C_{12}$$
$$\sum_{i=1}^{m} A^i_{21} y^i(\mu) + \mu \tilde{Z}_{21}(\mu) = C_{21}$$
$$\sum_{i=1}^{m} A^i_{22} y^i(\mu) + Z_{22}(\mu) = C_{22} \quad (8)$$

$$\mu X_{11}(\mu)\tilde{Z}_{11}(\mu) + (\mu)^2 \tilde{X}_{12}(\mu)\tilde{Z}_{21}(\mu) = \mu I_r$$
$$\mu X_{11}(\mu)\tilde{Z}_{12}(\mu) + \mu \tilde{X}_{12}(\mu) Z_{22}(\mu) = 0$$
$$(\mu)^2 \tilde{X}_{21}(\mu)\tilde{Z}_{11}(\mu) + (\mu)^2 \tilde{X}_{22}(\mu)\tilde{Z}_{21}(\mu) = 0$$
$$(\mu)^2 \tilde{X}_{21}(\mu)\tilde{Z}_{12}(\mu) + \mu \tilde{X}_{22}(\mu) Z_{22}(\mu) = \mu I_{n-r} \quad (9)$$

Using the vector symbolic we have

$$A_{11}\mathrm{vec}(X_{11}(\mu))+\mu A_{12}\mathrm{vec}(\tilde{X}_{12}(\mu))+\mu A_{21}\mathrm{vec}(\tilde{X}_{21}(\mu))$$
$$+\mu A_{22}\mathrm{vec}(\tilde{X}_{22}(\mu)) = b, \qquad (7')$$

$$\begin{aligned} A_{11}^T y + \mu \ \mathrm{vec}(\tilde{Z}_{11}) &= \mathrm{vec}(C_{11}) \\ A_{12}^T y + \mu \ \mathrm{vec}(\tilde{Z}_{12}) &= \mathrm{vec}(C_{12}) \\ A_{21}^T y + \mu \ \mathrm{vec}(\tilde{Z}_{21}) &= \mathrm{vec}(C_{21}) \\ A_{22}^T y + \mathrm{vec}(Z_{22}) &= \mathrm{vec}(C_{22}) \end{aligned} \qquad (8')$$

and the system (9) is for $\mu > 0$ equivalent to

$$\begin{aligned} X_{11}(\mu)\tilde{Z}_{11}(\mu) + \mu\tilde{X}_{12}(\mu)\tilde{Z}_{21}(\mu) &= I_r \\ X_{11}(\mu)\tilde{Z}_{12}(\mu) + \tilde{X}_{12}(\mu)Z_{22}(\mu) &= 0 \\ \tilde{X}_{21}(\mu)\tilde{Z}_{11}(\mu) + \tilde{X}_{22}(\mu)\tilde{Z}_{21}(\mu) &= 0 \\ \mu\tilde{X}_{21}(\mu)\tilde{Z}_{12}(\mu) + \tilde{X}_{22}(\mu)Z_{22}(\mu) &= I_{n-r} \end{aligned} \qquad (9')$$

where

$$\begin{aligned} &A_{11} \in \mathbb{R}^{m\times r^2}, && \mathrm{vec}(X_{11}) \in \mathbb{R}^{r^2} \\ &A_{12} \in \mathbb{R}^{m\times r(n-r)}, && \mathrm{vec}(\hat{X}_{12}) \in \mathbb{R}^{r(n-r)} \\ &A_{21} \in \mathbb{R}^{m\times (n-r)r}, && \mathrm{vec}(\hat{X}_{21}) \in \mathbb{R}^{(n-r)r} \\ &A_{22} \in \mathbb{R}^{m\times (n-r)(n-r)}, && \mathrm{vec}(\hat{X}_{22}) \in \mathbb{R}^{(n-r)(n-r)}. \end{aligned}$$

3.3 Adapting the system for the tilde path

The system (7'), (8') can be written in the form

$$Pv(\mu) + \mu Q w(\mu) = d \tag{10}$$

where

$$P := \begin{bmatrix} A_{11} & 0 & 0 \\ 0 & A_{11}^T & 0 \\ 0 & A_{12}^T & 0 \\ 0 & A_{21}^T & 0 \\ 0 & A_{22}^T & I \end{bmatrix}, \quad v(\mu) := \begin{bmatrix} \mathrm{vec}(X_{11}(\mu)) \\ y(\mu) \\ \mathrm{vec}(Z_{22}(\mu)) \end{bmatrix},$$

$$Q := \begin{bmatrix} A_{12} & A_{21} & A_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$w(\mu) := \begin{bmatrix} \mathrm{vec}(\tilde{X}_{12}(\mu)) \\ \mathrm{vec}(\tilde{X}_{21}(\mu)) \\ \mathrm{vec}(\tilde{X}_{22}(\mu)) \\ \mathrm{vec}(\tilde{Z}_{11}(\mu)) \\ \mathrm{vec}(\tilde{Z}_{12}(\mu)) \\ \mathrm{vec}(\tilde{Z}_{21}(\mu)) \end{bmatrix}, \quad d := \begin{bmatrix} b \\ \mathrm{vec}(C_{11}) \\ \mathrm{vec}(C_{12}) \\ \mathrm{vec}(C_{21} \\ \mathrm{vec}(C_{22})) \end{bmatrix}.$$

$$Pv(\mu) + \mu Q w(\mu) = d \tag{10}$$

So, P is $l \times k_1$ and Q is $l \times k_2$, where

$$l := m + r^2 + 2r(n-r) + (n-r)^2 = m + n^2$$

$$k_1 := m + r^2 + (n-r)^2$$

$$k_2 := (2r(n-r) + (n-r)^2 + r^2 + 2r(n-r))$$

Let $s := \mathrm{rank}(P) \leq k_1 \leq l$. Then, there exists a nonsingular $l \times l$ matrix W such that

$$W = \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} \text{ and } W_2 P = 0.$$

Here W_1 is $s \times l$ and W_2 is $(l-s) \times l$. Multiplying (10) by W from the left we obtain

$$W_1 P v(\mu) + \mu W_1 Q w(\mu) = W_1 d \tag{11}$$

$$\mu W_2 Q w(\mu) = W_2 d \tag{12}$$

Since $w(\mu)$ has finite limit as $\mu \to 0$, from (12) at $\mu = 0$ we have $W_2 d = 0$. So, instead of (12) we have

$$W_2 Q w(\mu) = 0 \tag{12'}$$

Evidently, for $\mu > 0$ the system (7), (8) and (9) of equations defining the tilde path is equivalent to (11), (12') and (9') and the corresponding Jacobian $\tilde{J}(\mu),\ \mu > 0$ is nonsingular.

$$W_1 P v(\mu) + \mu W_1 Q w(\mu) = W_1 d \tag{11}$$

$$W_2 Q w(\mu) = 0 \tag{12'}$$

$$\begin{aligned} X_{11}(\mu)\tilde{Z}_{11}(\mu) + \mu\tilde{X}_{12}(\mu)\tilde{Z}_{21}(\mu) &= I_r \\ X_{11}(\mu)\tilde{Z}_{12}(\mu) + \tilde{X}_{12}(\mu)Z_{22}(\mu) &= 0 \\ \tilde{X}_{21}(\mu)\tilde{Z}_{11}(\mu) + \tilde{X}_{22}(\mu)\tilde{Z}_{21}(\mu) &= 0 \\ \mu\tilde{X}_{21}(\mu)\tilde{Z}_{12}(\mu) + \tilde{X}_{22}(\mu)Z_{22}(\mu) &= I_{n-r} \end{aligned} \tag{9'}$$

3.4 Nonsingularity of the Jacobian at $\mu = 0$

Lemma. *The Jacobian $\tilde{J}(\mu)$ of the system (11), (12') and (9') extends to be nonsingular at $\mu = 0$.*

Proof. It suffices to prove that

$$\tilde{J}(0)h = 0 \qquad \Longrightarrow \qquad h = 0$$

Let $h = (h^T_{X_{11}}, h^T_{\tilde{X}_{12}}, h^T_{\tilde{X}_{21}}, h^T_{\tilde{X}_{22}}, h^T_y, h^T_{\tilde{Z}_{11}}, h^T_{\tilde{Z}_{12}}, h^T_{\tilde{Z}_{21}}, h^T_{Z_{22}})$

Denote $\quad h^T_v := (h^T_{X_{11}}, h^T_y, h^T_{Z_{22}}) \quad$ and

$$h^T_w := (h^T_{\tilde{X}_{12}}, h^T_{\tilde{X}_{21}}, h^T_{\tilde{X}_{22}}, h^T_{\tilde{Z}_{11}}, h^T_{\tilde{Z}_{12}}, h^T_{\tilde{Z}_{21}}).$$

Then the equations of $\tilde{J}(0)h = 0$ are

$$W_1 P h_v = 0 \tag{13}$$

$$W_2 Q h_w = 0 \tag{14}$$

$$(I \otimes X_{11}) h_{\tilde{Z}_{11}} + (\tilde{Z}_{11} \otimes I) h_{X_{11}} = 0 \tag{15}$$

$$\begin{aligned}(I \otimes X_{11}) h_{\tilde{Z}_{12}} + (\tilde{Z}^T_{12} \otimes I) h_{X_{11}} + (I \otimes \tilde{X}_{12}) h_{Z_{22}} \\ + (Z_{22} \otimes I) h_{\tilde{X}_{12}} = 0\end{aligned} \tag{16}$$

$$\begin{aligned}(I \otimes \tilde{X}_{21}) h_{\tilde{Z}_{11}} + (\tilde{Z}_{11} \otimes I) h_{\tilde{X}_{21}} + (I \otimes \tilde{X}_{22}) h_{\tilde{Z}_{21}} \\ + (\tilde{Z}^T_{21} \otimes I) h_{\tilde{X}_{22}} = 0\end{aligned} \tag{17}$$

$$(I \otimes \tilde{X}_{22}) h_{Z_{22}} + (Z_{22} \otimes I) h_{\tilde{X}_{22}} = 0 \tag{18}$$

where $\otimes$ is the Kronecker product sign.

From $W_1 P h_v = 0$ (13) and from the properties of W ($W_2 P = 0$, nonsingularity) we have

$$P h_v = 0. \tag{19}$$

From $W_2 Q h_w = 0$ (14) and from the properties of W it follows that and there exist $\mathrm{vec}(U_1)$, u_2 and $\mathrm{vec}(U_3)$ such that

$$P \begin{bmatrix} \mathrm{vec}(U_1) \\ u_2 \\ \mathrm{vec}(U_3) \end{bmatrix} + Q h_w = 0 \tag{20}$$

The equations of (19) are

$$\begin{aligned} A_{11}h_{X_{11}} &= 0 \end{aligned} \tag{21}$$

$$\begin{aligned} A_{11}^T h_y &= 0 \\ A_{12}^T h_y &= 0 \\ A_{21}^T h_y &= 0 \\ A_{22}^T h_y + h_{Z_{22}} &= 0 \end{aligned} \tag{22}$$

and the equations of (20) are

$$A_{11}\text{vec}(U_1) + A_{12}h_{\tilde{X}_{12}} + A_{21}h_{\tilde{X}_{21}} + A_{22}h_{\tilde{X}_{22}} = 0 \tag{23}$$

$$\begin{aligned} A_{11}^T u_2 + h_{\tilde{Z}_{11}} &= 0 \\ A_{12}^T u_2 + h_{\tilde{Z}_{12}} &= 0 \\ A_{21}^T u_2 + h_{\tilde{Z}_{21}} &= 0 \\ A_{22}^T u_2 + \text{vec}(U_3) &= 0 \end{aligned} \tag{24}$$

Now from (21) and (24) the orthogonality property yields

$$h_{X_{11}}^T h_{\tilde{Z}_{11}} = 0$$

and from (22) and (23)

$$h_{\tilde{X}_{22}}^T h_{Z_{22}} = 0$$

Since X_{11}, $\tilde{Z}_{11}$, $\tilde{X}_{22}$ and Z_{22} are positive definite we obtain from (15) and (18) that

$$h_{X_{11}} = 0, \; h_{\tilde{Z}_{11}} = 0 \;\text{ and }\; h_{\tilde{X}_{22}} = 0, \; h_{Z_{22}} = 0. \tag{25}$$

Due to Assumption 1 rank$(A) = m$ and so (22) together with $h_{Z_{22}} = 0$ imply that $h_y = 0$.
Now (16) and (17) together with (25) yield

$$h_{\tilde{X}_{12}} = -(Z_{22} \otimes I)^{-1}(I \otimes X_{11})h_{\tilde{Z}_{12}} \tag{26}$$

$$h_{\tilde{X}_{21}} = -(\tilde{Z}_{11} \otimes I)^{-1}(I \otimes \tilde{X}_{22})h_{\tilde{Z}_{21}}. \tag{27}$$

From (23) and (24) together with (25) we obtain

$$h^T_{\tilde{X}_{12}} h_{\tilde{Z}_{12}} + h^T_{\tilde{X}_{21}} h_{\tilde{Z}_{21}} = 0$$

Setting (26) and (27) into the last equation we obtain

$$-h_{\tilde{Z}_{12}}(Z_{22} \otimes I)^{-1}(I \otimes X_{11})h_{\tilde{Z}_{12}}$$

$$-h_{\tilde{Z}_{21}}(\tilde{Z}_{11} \otimes I)^{-1}(I \otimes \tilde{X}_{22})h_{\tilde{Z}_{21}} = 0$$

Since $(Z_{22} \otimes I)^{-1}(I \otimes X_{11})$ and $(\tilde{Z}_{11} \otimes I)^{-1}(I \otimes \tilde{X}_{22})$ are positive definite, we have $h_{\tilde{Z}_{12}} = 0$ and $h_{\tilde{Z}_{21}} = 0$. From (27), (28) it finally follows $h_{\tilde{X}_{12}} = 0$ and $h_{\tilde{X}_{21}} = 0$. $\square$

Appendix: Kronecker products.

The Kronecker product of $M \otimes N$ matrices $M \in \mathbb{R}^{r_1 \times s_1}$ and $N \in \mathbb{R}^{r_2 \times s_2}$ is defined as the $r_1 r_2 \times s_1 s_2$ matrix

$$M \otimes N = \begin{bmatrix} M_{11}N & \dots & M_{1s_1}N \\ \dots & & \dots \\ M_{r_1 1}N & \dots & M_{r_1 s_1}N \end{bmatrix}$$

Facts:

$$\mathrm{vec}(MN) = (I_s \otimes M)\mathrm{vec}(N), \quad M \in \mathbb{R}^{n \times r}, \quad N \in \mathbb{R}^{r \times s}$$

here $(I_s \otimes M) \in \mathbb{R}^{sn \times sr}$ and $\mathrm{vec}(N) \in \mathbb{R}^{sr}$

$$\mathrm{vec}(MN) = (N^T \otimes I_n)\mathrm{vec}(M), \quad M \in \mathbb{R}^{n \times r}, \quad N \in \mathbb{R}^{r \times s}$$

here $(N^T \otimes I_n) \in \mathbb{R}^{sn \times rn}$ and $\mathrm{vec}(M) \in \mathbb{R}^{nr}$.

Open questions

- Use of analyticity or finiteness of derivatives for construction of algorithms in SDP

- Limiting behavior of the central path in SDP if no strictly complementary solution exists