ANALYTICITY
OF THE CENTRAL PATH AT THE BOUNDARY POINT
IN SEMIDEFINITE PROGRAMMING

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Abstract

In this paper we study the limiting behavior of the central path for semidefinite programming. We show that the central path is an analytic function of the barrier parameter even at the limit point, provided that the semidefinite program has a strictly complementary solution. A consequence of this property is that the derivatives - of any order - of the central path have finite limits as the barrier parameter goes to zero.

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1. Introduction

The central path is one of the most important concepts in the study of interior point methods. It is an analytic curve in the interior of the feasible set which tends to an optimal point at the boundary. Most interior point algorithms follow the central path and hence, the central path is an essential concept when designing and analyzing algorithms for some convex programs.

Numerous facts about this curve have been established, among which the limiting behavior of the first-order derivative of the central path seems to be crucial when investigating the local convergence of the usual interior point methods. The results about the asymptotic behavior of the higher-order derivatives and the analyticity of the central path then again give the theoretical background for the construction of higher-order methods.

The limiting behavior of the central path is most explored for linear programs (LP) and we specially refer to the works by Adler and Monteiro [AM] (see also [WBD]) and by Güler [G], where the convergence of the derivatives of the central path is established. Moreover, it was proved independently by Halická [H1] and Wechs [W] (see also [H2]) that the central path, as an analytic function of the barrier parameter $\mu > 0$, can be analytically extended to the boundary at $\mu = 0$.

The features of the central path are also well explored for linear complementarity problems (LCP). Here, however, the analysis of the asymptotic behavior of the central path derivatives is much more complicated and the results depend on whether the LCP has a strictly complementary solution. The asymptotic results regarding the first-order derivatives of the central path have been established by Monteiro and Tsuchiya [MT] and results about an analytical extension of the central path are presented by Stoer and Wechs in [SW1, SW2].

Less information has been obtained about the limiting behavior of the derivatives of the central path for semidefinite programming (SDP) with the only study being conducted by Goldfarb and Sheinberg [GS]. The results of this work show that the first-order derivative of the central path is unbounded if the SDP does not have a strictly complementary solution. On the other hand, the first-order derivative of the central path converges as

$\mu \downarrow 0$, provided that the SDP satisfies both the assumption of primal and dual non-degeneracy and that of strict complementarity. (The latter ensures the existence of a strictly complementary solution.)

In this paper, we study the limiting behavior of the central path for SDP under the assumption of strict complementarity without any non-degeneracy assumption. We prove that the central path, as an analytic function of the barrier parameter $\mu > 0$, can be analytically extended to $\mu = 0$. As a consequence of this property, we obtain that the derivatives of any order of the central path have finite limits as $\mu \downarrow 0$.

The paper is organized as follows. In Section 2 we recall some relevant properties of the central path for semidefinite programs. In Section 3 we prove the main result establishing the analyticity of the central path at the boundary point, and in Section 4 we discuss some open problems in this area.

2. Preliminaries

2.1. Notation and basic assumptions.

The space of real symmetric $n \times n$ matrices is denoted as S^n and the inner product on S^n is defined by

$$M \bullet N = \operatorname{trace}(MN) = \sum_{i,j} M_{ij} N_{ij}.$$

By $M \succeq 0$ ($M \succ 0$), where $M \in S^n$, we mean that M is positive semidefinite (definite).

A semidefinite program is

$$\begin{aligned} \min_{X \in S^n} \quad & C \bullet X \\ \text{s.t.} \quad & A^i \bullet X = b^i, \quad i = 1, ..., m \\ & X \succeq 0, \end{aligned} \tag{P}$$

where $C,\ A^i \in S^n$, $i = 1, ..., m$, and $b \in R^m$. The corresponding dual program is

$$\begin{aligned} \max_{y \in \mathbb{R}^m, Z \in S^n} \quad & b^T y \\ \text{s.t.} \quad & \sum_{i=1}^m A^i y^i + Z = C, \\ & Z \succeq 0. \end{aligned} \tag{D}$$

Throughout this paper the following assumptions are made.

Assumption 1. The matrices A^i, $i = 1, ..., m$, are linearly independent, i.e.,

$$\sum_{i=1}^m A^i u^i = 0 \quad \Rightarrow \quad u^i = 0, \quad i = 1, ..., m.$$

Assumption 2. Both (P) and (D) have strictly feasible solutions, i.e.,

$$\begin{aligned} \exists\ X \in S^n \ : \quad & A^i \bullet X = b^i, \quad i = 1, ..., m, \quad X \succ 0; \\ \exists\ (y, Z) \in R^m \times S^n \ : \quad & \sum_{i=1}^m A^i y^i + Z = C, \quad Z \succ 0. \end{aligned}$$

Assumption 1 is only a technical one, enforcing a one-to-one correspondence between y and Z in the dual feasible pairs (y, Z). On the other hand, Assumption 2 is essential in the development of the theory. Under this assumption both (P) and (D) have optimal solutions $\hat{X}$ and $(\hat{y}, \hat{Z})$, and the duality gap is $\hat{X} \bullet \hat{Z} = 0$. Moreover, the optimal solutions satisfy $\hat{X}\hat{Z} = \hat{Z}\hat{X} = 0$ [A]. Let us mention that the last property enables us to interpret any primal-dual optimal pair $(\hat{X}, \hat{Z})$ as a point from the boundary of the primal-dual feasible set.

2.2. Definition of the central path. The centering system for $\mu > 0$ is defined by the following system

$$A^i \bullet X = b^i, \quad i = 1, ..., m, \quad X \succ 0, \tag{1}$$

$$\sum_{i=1}^m A^i y^i + Z = C, \quad Z \succ 0, \tag{2}$$

$$XZ = \mu I_n, \tag{3}$$

where I_n is the $n \times n$ identity. It is well known that for any $\mu > 0$ there exists a unique solution $(X(\mu), y(\mu), Z(\mu))$ to the system (1)-(3) [VB, KSH]. The set of solutions for $\mu > 0$ is called the central path for (P) and (D). The central path restricted to $0 < \mu \leq \bar{\mu}$ for some $\bar{\mu} > 0$, is bounded and each of its limit points for $\mu \downarrow 0$ is an (maximally complementary) optimal solution of (P), (D) ([KRT, LSZ, GS]).

2.3. The central path as an analytic function of $\mu > 0$. As indicated above the central path can be considered as a function of the parameter μ. That is, the central path, $(X(\mu), y(\mu), Z(\mu))$, maps $(0, \infty)$ to $\mathbb{R}^{n\times n} \times \mathbb{R}^m \times \mathbb{R}^{n\times n}$. In order to analyze the properties of the central path we introduce the mapping, vec : $\mathbb{R}^{r\times s} \to \mathbb{R}^{rs}$, that stacks the columns of a matrix in a vector. We also define

$$A := \begin{bmatrix} (\mathrm{vec}(A^1))^T \\ \vdots \\ (\mathrm{vec}(A^m))^T \end{bmatrix}.$$

Then, the centering system (1)-(3) can be rewritten as

$$A\ \mathrm{vec}(X) = b, \quad X \succ 0, \tag{1'}$$

$$A^T y + \mathrm{vec}(Z) = \mathrm{vec}(C), \quad Z \succ 0, \tag{2'}$$

$$XZ = \mu I_n. \tag{3}$$

We now describe the Jacobian of this system by means of the Kronecker product. For convenience we recall the definition of the Kronecker product $M \otimes N$: if $M \in \mathbb{R}^{k\times l}$ and $N \in \mathbb{R}^{r\times s}$, then $M \otimes N \in \mathbb{R}^{kr\times ls}$ is a $k \times l$ block matrix whose i, j block is $m_{ij}N$. Following are four properties of the Kronecker product that will be useful later [HJ]:

$$(M \otimes N)(K \otimes L) = MK \otimes NL, \qquad (M \otimes N)^{-1} = M^{-1} \otimes N^{-1},$$

$$(M \otimes N)\mathrm{vec}(K) = \mathrm{vec}(NKM^T), \quad \text{and} \quad \text{if } Z \succ 0 \text{ and } X \succ 0, \text{ then } Z \otimes X \succ 0.$$

Due to the third property it is easy to see that the Jacobian matrix of (1'), (2'), (3) is

$$J(X, y, Z) = \begin{pmatrix} A & 0 & 0 \\ 0 & A^T & I_{n^2} \\ Z \otimes I_n & 0 & I_n \otimes X \end{pmatrix}.$$

Let us denote by $J(\mu)$ the Jacobian evaluated along the central path, i.e.,

$$J(\mu) := J(X(\mu), y(\mu), Z(\mu)), \quad \mu > 0.$$

Since $X(\mu) \succ 0$ and $Z(\mu) \succ 0$, based on the above properties of the Kronecker product it can be easily shown that $J(\mu)$ is nonsingular at any $\mu > 0$ (see also [AHO]). A simple corollary is that the central path is an analytic function at any $\mu > 0$. In fact, since the central path is defined by equations (1'), (2') and (3), and the functions in these equations are analytical functions of X, y Z and μ, the analyticity of the central path follows by the analytic version of the implicit function theorem [D].

2.4. Limiting properties of the central path. For $\mu > 0$, the central path possesses nice features, but the question arises whether it keeps these features as $\mu \downarrow 0$. We will see that the answer to this question is "yes", if a strictly complementary solution exists. This condition is assumed in the sequel of the paper. Formally:

Assumption 3. There exists a strictly complementary solution, -i.e., there exists $(\bar{X}, \bar{y}, \bar{Z})$ such that $\bar{X}$ and $(\bar{y}, \bar{Z})$ are optimal solutions of (P) and (D) and $\text{rank}(\bar{X}) + \text{rank}(\bar{Z}) = n$.

Under this assumption it was proven independently by De Klerk, Roos and Terlaky [KRT], and Luo, Sturm and Zhang [LSZ] that the central path converges as $\mu \downarrow 0$ to the analytic center of the solution set. This fact enables us to extend the domain of the central path to $[0, \infty)$ by defining

$$X(0) := \lim_{\mu \downarrow 0} X(\mu), \quad y(0) := \lim_{\mu \downarrow 0} y(\mu), \quad \text{and} \quad Z(0) := \lim_{\mu \downarrow 0} Z(\mu),$$

and study the analyticity of the central path at $\mu = 0$. Notice that if the Jacobian $J(\mu)$ were nonsingular at $\mu = 0$, the analyticity of the central path for $\mu = 0$ would follow from the implicit function theorem. However, this is not possible generally since the Jacobian may be singular at $\mu = 0$.

In fact, $J(0)$ is non-singular only if a non-degeneracy property is fulfilled [AHO], and hence, trivially for this case, the central path is analytic for any $\mu \geq 0$. Let us mention that the same argument was first used by Fiacco and McCormick [FM] for proving analyticity of the central path in a somewhat different setting. See also [GS], where the convergence of the first-order derivative of the central path for non-degenerate SDP's under strict complementarity is proved.

Since the argument of non-singularity does not work for degenerate SDP's, a more careful analysis is needed for this case. For such an analysis the result by Luo, Sturm and Zhang [LSZ],

$$\|X(\mu) - X(0)\| = O(\mu), \quad \text{and} \quad \|Z(\mu) - Z(0)\| = O(\mu), \tag{R}$$

is of crucial importance. Let us mention, however, that (R) does not imply (as was misinterpreted in [GS]) that the derivatives of the central path are bounded (consider for instance $f(\mu) = \mu \sin 1/\mu$)).

2.5. Relevant results for LP and LCP. The difficulty of a vanishing Jacobian arises for both LP and LCP, and this difficulty has been handled by two different procedures. The LP case was addressed in [G,W,H1], where explicit formulas for the derivatives - of any order - were established for $\mu > 0$, and then their limiting properties were analyzed as $\mu \downarrow 0$. These proofs are constructive, and they yield recursive formulas for the high-order derivatives at $\mu = 0$. They eventually estimate the radius of convergence of the corresponding Taylor series. However, these proofs are technical and lengthy.

A different approach was developed by Stoer and Wechs for LCP in [SW1, SW2, SWM]. This approach is based on an application of the implicit function theorem and provides a much simpler proof of the analyticity of the central path. It was refined, and applied directly to LP by Halická [H2]. Surprisingly, the idea of this proof can be modified and applied even to SDP, as can be seen below.

3. Analyticity of the central path

In this section we prove that the central path for SDP is analytic at $\mu = 0$, provided that there is a strictly complementary solution.

Theorem. *Under Assumptions 1-3 the central path (as a function of $\mu \geq 0$) is analytic, even for $\mu = 0$.*

Proof. The proof consists of five steps.

(a) Decomposition of matrix variables. Let $(\bar{X}, \bar{Z})$ be a strictly complementary solution (see Assumption 3) and $r := \text{rank}(\bar{X})$. As shown in [GS], this r is independent of the choice of the strictly complementary solution, and we may assume without loss of generality (applying an orthonormal transformation if necessary) that each optimal solution pair, X^* and Z^*, of (P) and (D) is of the form

$$X^* = \begin{bmatrix} X_{11}^* & 0 \\ 0 & 0 \end{bmatrix}, \text{ where } X_{11}^* \in S^r, \ X_{11}^* \succeq 0,$$

$$Z^* = \begin{bmatrix} 0 & 0 \\ 0 & Z_{22}^* \end{bmatrix}, \text{ where } Z_{22}^* \in S^{n-r}, \ Z_{22}^* \succeq 0.$$

Throughout this proof, the decomposition of any $n \times n$ matrix M is made with respect to the above partition, -i.e.

$$M \in S^n \qquad \Rightarrow \qquad M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix},$$

where M_{11} is $r \times r$, and M_{22} is $(n-r) \times (n-r)$. With this block partitioning, we write equations (1)-(3) as follows,

$$A_{11}^i \bullet X_{11}(\mu) + A_{12}^i \bullet X_{12}(\mu) + A_{21}^i \bullet X_{21}(\mu) + A_{22}^i \bullet X_{22}(\mu) = b^i, \quad i = 1, ..., n, \tag{4}$$

$$\left.\begin{aligned} \sum_{i=1}^m A_{11}^i y^i(\mu) + Z_{11}(\mu) = C_{11}, \\ \sum_{i=1}^m A_{12}^i y^i(\mu) + Z_{12}(\mu) = C_{12}, \\ \sum_{i=1}^m A_{21}^i y^i(\mu) + Z_{21}(\mu) = C_{21}, \\ \sum_{i=1}^m A_{22}^i y^i(\mu) + Z_{22}(\mu) = C_{22}, \end{aligned}\right\} \tag{5}$$

$$\left.\begin{aligned} X_{11}(\mu)Z_{11}(\mu) + X_{12}(\mu)Z_{21}(\mu) = \mu I_r, \\ X_{11}(\mu)Z_{12}(\mu) + X_{12}(\mu)Z_{22}(\mu) = 0, \\ X_{21}(\mu)Z_{11}(\mu) + X_{22}(\mu)Z_{21}(\mu) = 0, \\ X_{21}(\mu)Z_{12}(\mu) + X_{22}(\mu)Z_{22}(\mu) = \mu I_{n-r}. \end{aligned}\right\} \tag{6}$$

Here $A_{12}^i \bullet X_{12}(\mu) := \text{trace}(A_{12}^i X_{12}^T(\mu))$, and $A_{21}^i \bullet X_{21}(\mu) := \text{trace}(A_{21}^i X_{21}^T(\mu))$.

(b) Introduction of tilde variables. As mentioned above, the central path converges to a strictly complementary solution. This means that

$$\begin{aligned}
&X_{11}(\mu) \to X_{11}(0) \succ 0, && Z_{11}(\mu) \to 0,\\
&X_{12}(\mu) \to 0, && Z_{12}(\mu) \to 0,\\
&X_{21}(\mu) \to 0, && Z_{21}(\mu) \to 0,\\
&X_{22}(\mu) \to 0, && Z_{22}(\mu) \to Z_{22}(0) \succ 0.
\end{aligned}$$

Moreover, from the above-mentioned result (R) of Luo, Sturm and Zhang we have that the functions

$$\frac{X_{12}(\mu)}{\mu}, \quad \frac{X_{21}(\mu)}{\mu}, \quad \frac{X_{22}(\mu)}{\mu}, \quad \frac{Z_{11}(\mu)}{\mu}, \quad \frac{Z_{12}(\mu)}{\mu}, \quad \text{and} \quad \frac{Z_{21}(\mu)}{\mu}$$

are bounded as $\mu \downarrow 0$. This result enables us to divide the path variables into two classes. The variables X_{11} and Z_{22} are $O(1)$ and remain unchanged throughout the proof. The other variables are $O(\mu)$; they are replaced by auxiliary variables denoted by the tilde sign, and called tilde variables. The tilde variables are defined by

$$\begin{aligned}
\tilde{X}_{12} &:= \frac{X_{12}}{\mu}, & \tilde{X}_{21} &:= \frac{X_{21}}{\mu}, & \tilde{X}_{22} &:= \frac{X_{22}}{\mu},\\
\tilde{Z}_{11} &:= \frac{Z_{11}}{\mu}, & \tilde{Z}_{12} &:= \frac{Z_{12}}{\mu}, & \tilde{Z}_{21} &:= \frac{Z_{21}}{\mu}.
\end{aligned}$$

Now, instead of the central path being defined by

$$p(\mu) = (\ X_{11}(\mu),\ X_{12}(\mu),\ X_{21}(\mu),\ X_{22}(\mu),\ y(\mu),\ Z_{11}(\mu),\ Z_{12}(\mu),\ Z_{21}(\mu),\ Z_{22}(\mu)\ ),$$

we consider the tilde path defined by

$$\tilde{p}(\mu) = (\ X_{11}(\mu),\ \tilde{X}_{12}(\mu),\ \tilde{X}_{21}(\mu),\ \tilde{X}_{22}(\mu),\ y(\mu),\ \tilde{Z}_{11}(\mu),\ \tilde{Z}_{12}(\mu),\ \tilde{Z}_{21}(\mu),\ Z_{22}(\mu)\ ),$$

where only $X_{11}(\mu)$, $y(\mu)$, and $Z_{22}(\mu)$ remain unchanged. Substituting the tilde variables into (4), (5), and (6), we obtain the following system for the tilde path

$$A^i_{11} \bullet X_{11}(\mu) + \mu A^i_{12} \bullet \tilde{X}_{12}(\mu) + \mu A^i_{21} \bullet \tilde{X}_{21}(\mu) + \mu A^i_{22} \bullet \tilde{X}_{22}(\mu) = b^i, \quad i = 1, ..., m, \quad (7)$$

$$
\left.\begin{aligned}
\sum_{i=1}^{m} A_{11}^{i} y^{i}(\mu) + \mu \tilde{Z}_{11}(\mu) &= C_{11}, \\
\sum_{i=1}^{m} A_{12}^{i} y^{i}(\mu) + \mu \tilde{Z}_{12}(\mu) &= C_{12}, \\
\sum_{i=1}^{m} A_{21}^{i} y^{i}(\mu) + \mu \tilde{Z}_{21}(\mu) &= C_{21}, \\
\sum_{i=1}^{m} A_{22}^{i} y^{i}(\mu) + Z_{22}(\mu) &= C_{22},
\end{aligned}\right\} \tag{8}
$$

$$
\left.\begin{aligned}
\mu X_{11}(\mu)\tilde{Z}_{11}(\mu) + (\mu)^2 \tilde{X}_{12}(\mu)\tilde{Z}_{21}(\mu) &= \mu I_r, \\
\mu X_{11}(\mu)\tilde{Z}_{12}(\mu) + \mu \tilde{X}_{12}(\mu) Z_{22}(\mu) &= 0, \\
(\mu)^2 \tilde{X}_{21}(\mu)\tilde{Z}_{11}(\mu) + (\mu)^2 \tilde{X}_{22}(\mu)\tilde{Z}_{21}(\mu) &= 0, \\
(\mu)^2 \tilde{X}_{21}(\mu)\tilde{Z}_{12}(\mu) + \mu \tilde{X}_{22}(\mu) Z_{22}(\mu) &= \mu I_{n-r}.
\end{aligned}\right\} \tag{9}
$$

From the formulas $\tilde{X}_{12}(\mu) = \frac{X_{12}(\mu)}{\mu}$, etc., for the tilde path it is easy to see that the analyticity of the central path for $\mu > 0$ implies the analyticity of the tilde path for $\mu > 0$. On the other hand, because $X_{12}(\mu) = \mu \tilde{X}_{12}$, etc., the analyticity of the central path at $\mu = 0$ may be established by proving the analyticity of the tilde path at $\mu = 0$. The drawback is that the tilde path is defined only for $\mu > 0$, and the convergence of the tilde path as $\mu \downarrow 0$ has not been established. However, it suffices to prove that the tilde path can be analytically extended to $\mu = 0$, which can be managed without the convergence of the off-diagonal blocks (their boundedness will suffice). All we need is the convergence of $\tilde{X}_{22}$ and $\tilde{Z}_{11}$ to positive definite matrices. This convergence follows from equations in (9). In fact, from the first equation in (9) we have

$$
\tilde{Z}_{11}(\mu) = X_{11}^{-1}(\mu)(I_r - \mu \tilde{X}_{12}(\mu)\tilde{Z}_{21}(\mu)),
$$

and since $\tilde{X}_{12}(\mu)$ and $\tilde{Z}_{21}(\mu)$ are bounded as $\mu \downarrow 0$, and $X_{11}(\mu) \to X_{11}(0) \succ 0$, we have that

$$
\tilde{Z}_{11}(\mu) \to X_{11}^{-1}(0) \succ 0.
$$

Similarly, from the fourth equation in (9) we obtain that

$$
\tilde{X}_{22}(\mu) \to Z_{22}^{-1}(0) \succ 0.
$$

(c) Operations on the system describing the tilde path. We can see that some of the equations in (9) are multiplied by μ or μ^2, and we divide them by μ (μ^2) to obtain an equivalent system for $\mu > 0$. We also define

$$A_{11} := \begin{bmatrix} (\mathrm{vec}(A_{11}^1))^T \\ \vdots \\ (\mathrm{vec}(A_{11}^m))^T \end{bmatrix}, \qquad A_{12} := \begin{bmatrix} (\mathrm{vec}(A_{12}^1))^T \\ \vdots \\ (\mathrm{vec}(A_{12}^m))^T \end{bmatrix}, \quad \text{etc,}$$

and rewrite the equations in (7) and (8) with the vec operator. We obtain

$$A_{11}\mathrm{vec}(X_{11}(\mu)) + \mu A_{12}\mathrm{vec}(\tilde{X}_{12}(\mu)) + \mu A_{21}\mathrm{vec}(\tilde{X}_{21}(\mu)) + \mu A_{22}\mathrm{vec}(\tilde{X}_{22}(\mu)) = b, \tag{7'}$$

$$\left.\begin{aligned} A_{11}^T y + \mu\ \mathrm{vec}(\tilde{Z}_{11}) &= \mathrm{vec}(C_{11}), \\ A_{12}^T y + \mu\ \mathrm{vec}(\tilde{Z}_{12}) &= \mathrm{vec}(C_{12}), \\ A_{21}^T y + \mu\ \mathrm{vec}(\tilde{Z}_{21}) &= \mathrm{vec}(C_{21}), \\ A_{22}^T y + \mathrm{vec}(Z_{22}) &= \mathrm{vec}(C_{22}), \end{aligned}\right\} \tag{8'}$$

$$\left.\begin{aligned} X_{11}(\mu)\tilde{Z}_{11}(\mu) + \mu\tilde{X}_{12}(\mu)\tilde{Z}_{21}(\mu) &= I_r, \\ X_{11}(\mu)\tilde{Z}_{12}(\mu) + \tilde{X}_{12}(\mu)Z_{22}(\mu) &= 0, \\ \tilde{X}_{21}(\mu)\tilde{Z}_{11}(\mu) + \tilde{X}_{22}(\mu)\tilde{Z}_{21}(\mu) &= 0, \\ \mu\tilde{X}_{21}(\mu)\tilde{Z}_{12}(\mu) + \tilde{X}_{22}(\mu)Z_{22}(\mu) &= I_{n-r}, \end{aligned}\right\} \tag{9'}$$

where

$$\begin{aligned} &A_{11} \in \mathbb{R}^{m\times r^2}, && \mathrm{vec}(X_{11}) \in \mathbb{R}^{r^2}, \\ &A_{12} \in \mathbb{R}^{m\times r(n-r)}, && \mathrm{vec}(\tilde{X}_{12}) \in \mathbb{R}^{r(n-r)}, \\ &A_{21} \in \mathbb{R}^{m\times (n-r)r}, && \mathrm{vec}(\tilde{X}_{21}) \in \mathbb{R}^{(n-r)r}, \\ &A_{22} \in \mathbb{R}^{m\times (n-r)(n-r)}, && \mathrm{vec}(\tilde{X}_{22}) \in \mathbb{R}^{(n-r)(n-r)}. \end{aligned}$$

The equations (7') and (8') are rewritten as the following matrix equation,

$$Pv(\mu) + \mu Q\tilde{w}(\mu) = d, \tag{10}$$

where

$$P := \begin{bmatrix} A_{11} & 0 & 0 \\ 0 & A_{11}^T & 0 \\ 0 & A_{12}^T & 0 \\ 0 & A_{21}^T & 0 \\ 0 & A_{22}^T & I \end{bmatrix}, \quad Q := \begin{bmatrix} A_{12} & A_{21} & A_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and

$$v(\mu) := \begin{bmatrix} \mathrm{vec}(X_{11}(\mu)) \\ y(\mu) \\ \mathrm{vec}(Z_{22}(\mu)) \end{bmatrix}, \quad \tilde{w}(\mu) := \begin{bmatrix} \mathrm{vec}(\tilde{X}_{12}(\mu)) \\ \mathrm{vec}(\tilde{X}_{21}(\mu)) \\ \mathrm{vec}(\tilde{X}_{22}(\mu)) \\ \mathrm{vec}(\tilde{Z}_{11}(\mu)) \\ \mathrm{vec}(\tilde{Z}_{12}(\mu)) \\ \mathrm{vec}(\tilde{Z}_{21}(\mu)) \end{bmatrix}, \quad d := \begin{bmatrix} b \\ \mathrm{vec}(C_{11}) \\ \mathrm{vec}(C_{12}) \\ \mathrm{vec}(C_{21}) \\ \mathrm{vec}(C_{22}) \end{bmatrix}.$$

Here P and Q are constant matrices of dimensions $l \times k_1$ and $l \times k_2$ respectively, where

$$\begin{aligned} l &:= m + r^2 + 2r(n-r) + (n-r)^2 = m + n^2, \\ k_1 &:= m + r^2 + (n-r)^2, \\ k_2 &:= (2r(n-r) + (n-r)^2 + r^2 + 2r(n-r)). \end{aligned}$$

The vector d is constant, and the vectors $v(\mu)$ and $\tilde{w}(\mu)$ are bounded as $\mu \downarrow 0$.

Let $s := \mathrm{rank}(P) \leq k_1 \leq l$. Then, there exists a nonsingular $l \times l$ matrix W such that

$$W = \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} \text{ and } W_2 P = 0,$$

where W_1 is $s \times l$ and W_2 is $(l-s) \times l$. Multiplying (10) by W from the left we obtain

$$W_1 P v(\mu) + \mu W_1 Q \tilde{w}(\mu) = W_1 d, \tag{11}$$

$$\mu W_2 Q \tilde{w}(\mu) = W_2 d. \tag{12}$$

Since $\tilde{w}(\mu)$ is bounded as $\mu \downarrow 0$, from (12) at $\mu = 0$ we have $W_2 d = 0$. So, instead of (12) we have

$$W_2 Q \tilde{w}(\mu) = 0. \tag{12'}$$

Because W is nonsingular the system of (7), (8), and (9) defining the tilde path is equivalent to (11), (12'), and (9') for $\mu > 0$. Hence the corresponding Jacobian, $\tilde{J}(\mu)$, is nonsingular for $\mu > 0$. Now, let

$$p^* = (X_{11}^*, \tilde{X}_{12}^*, \tilde{X}_{21}^*, \tilde{X}_{22}^*, y^*, \tilde{Z}_{11}^*, \tilde{Z}_{12}^*, \tilde{Z}_{21}^*, Z_{22}^*)$$

be one of the limit points of the tilde path as $\mu \downarrow 0$ and let $\{\mu_i\}_{i=1}^{\infty} \downarrow 0$ be the corresponding sequence of μ's for which the tilde path values converge to p^*. From the previously described convergence properties of $\tilde{X}_{22}$ and $\tilde{Z}_{11}$ we have

$$X_{11}^* = X_{11}(0) \succ 0, \quad \tilde{X}_{22}^* = Z_{22}^{-1}(0) \succ 0, \quad \tilde{Z}_{11}^* = X_{11}^{-1}(0) \succ 0, \quad Z_{22}^* = Z_{22}(0) \succ 0.$$

The pair $(v^*, \tilde{w}^*)$, where

$$v^* := \begin{bmatrix} \mathrm{vec}(X_{11}^*) \\ y^* \\ \mathrm{vec}(Z_{22}^*) \end{bmatrix}, \quad \text{and} \quad \tilde{w}^* := \begin{bmatrix} \mathrm{vec}(\tilde{X}_{12}^*) \\ \mathrm{vec}(\tilde{X}_{21}^*) \\ \mathrm{vec}(\tilde{X}_{22}^*) \\ \mathrm{vec}(\tilde{Z}_{11}^*) \\ \mathrm{vec}(\tilde{Z}_{12}^*) \\ \mathrm{vec}(\tilde{Z}_{21}^*), \end{bmatrix},$$

solve the system of (11), (12') and (9) at $\mu = 0$. Denote by $\tilde{J}^*$ the Jacobian of the system at $(v^*, \tilde{w}^*)$ and $\mu = 0$. We prove that $\tilde{J}^*$ is nonsingular.

(d) Nonsingularity of the Jacobian at $\mu = 0$. It suffices to prove that if $\tilde{J}^* h = 0$ for some vector h, then $h = 0$. Let $h^T = (h_{X_{11}}^T, h_{\tilde{X}_{12}}^T, h_{\tilde{X}_{21}}^T, h_{\tilde{X}_{22}}^T, h_y^T, h_{\tilde{Z}_{11}}^T, h_{\tilde{Z}_{12}}^T, h_{\tilde{Z}_{21}}^T, h_{Z_{22}}^T)$. Denote $h_v^T := (h_{X_{11}}^T, h_y^T, h_{Z_{22}}^T)$ and $h_w^T := (h_{\tilde{X}_{12}}^T, h_{\tilde{X}_{21}}^T, h_{\tilde{X}_{22}}^T, h_{\tilde{Z}_{11}}^T, h_{\tilde{Z}_{12}}^T, h_{\tilde{Z}_{21}}^T)$.
Then the equations of $\tilde{J}^* h = 0$ are

$$W_1 P h_v = 0, \tag{13}$$

$$W_2 Q h_w = 0, \tag{14}$$

$$(I \otimes X_{11}^*) h_{\tilde{Z}_{11}} + (\tilde{Z}_{11}^* \otimes I) h_{X_{11}} = 0, \tag{15}$$

$$(I \otimes X_{11}^*) h_{\tilde{Z}_{12}} + ((\tilde{Z}_{12}^*)^T \otimes I) h_{X_{11}} + (I \otimes \tilde{X}_{12}^*) h_{Z_{22}} + (Z_{22}^* \otimes I) h_{\tilde{X}_{12}} = 0, \tag{16}$$

$$(I \otimes \tilde{X}_{21}^*) h_{\tilde{Z}_{11}} + (\tilde{Z}_{11}^* \otimes I) h_{\tilde{X}_{21}} + (I \otimes \tilde{X}_{22}^*) h_{\tilde{Z}_{21}} + ((\tilde{Z}_{21}^*)^T \otimes I) h_{\tilde{X}_{22}} = 0, \tag{17}$$

$$(I \otimes \tilde{X}_{22}^*) h_{Z_{22}} + (Z_{22}^* \otimes I) h_{\tilde{X}_{22}} = 0. \tag{18}$$

Since W_2 has been defined so that $W_2 P = 0$, we also have $W_2 P h_v = 0$. From (13) and the nonsingularity of W we now have

$$P h_v = 0. \tag{19}$$

Hence (13) is rewritten as (19). Now, we rewrite (14). For this purpose we note that $W = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix}$ is nonsigular $l \times l$, and W_2 is $(l - s) \times l$ and hence $\mathrm{rank}(W_2) = l - s$. Moreover, since $\mathrm{rank}(P) = s$ and $W_2 P = 0$ we have that

$$W_2 a = 0 \Leftrightarrow \exists\, b : Pb = a$$

which applied to (14) gives that there exist $\mathrm{vec}(U_1)$, u_2 and $\mathrm{vec}(U_3)$ such that

$$P\begin{bmatrix}\mathrm{vec}(U_1)\\ u_2\\ \mathrm{vec}(U_3)\end{bmatrix} + Qh_w = 0. \tag{20}$$

The equations of (19) are

$$A_{11}h_{X_{11}} = 0, \tag{21}$$

$$\left.\begin{aligned} A_{11}^T h_y &= 0,\\ A_{12}^T h_y &= 0,\\ A_{21}^T h_y &= 0,\\ A_{22}^T h_y + h_{Z_{22}} &= 0, \end{aligned}\right\} \tag{22}$$

and the equations of (20) are

$$A_{11}\mathrm{vec}(U_1) + A_{12}h_{\tilde{X}_{12}} + A_{21}h_{\tilde{X}_{21}} + A_{22}h_{\tilde{X}_{22}} = 0, \tag{23}$$

$$\left.\begin{aligned} A_{11}^T u_2 + h_{\tilde{Z}_{11}} &= 0,\\ A_{12}^T u_2 + h_{\tilde{Z}_{12}} &= 0,\\ A_{21}^T u_2 + h_{\tilde{Z}_{21}} &= 0,\\ A_{22}^T u_2 + \mathrm{vec}(U_3) &= 0. \end{aligned}\right\} \tag{24}$$

Now, from (21) and (24) the orthogonality property yields

$$h_{X_{11}}^T h_{\tilde{Z}_{11}} = 0. \tag{25}$$

Similarly, from (22) and (23) we obtain

$$h_{\tilde{X}_{22}}^T h_{Z_{22}} = 0, \tag{26}$$

and from (23) and (24)

$$\mathrm{vec}(U_1)^T h_{\tilde{Z}_{11}} + h_{\tilde{X}_{22}}^T \mathrm{vec}(U_3) + h_{\tilde{X}_{12}}^T h_{\tilde{Z}_{12}} + h_{\tilde{X}_{21}}^T h_{\tilde{Z}_{21}} = 0. \tag{27}$$

From (15) and (18) we obtain

$$h_{\tilde{Z}_{11}} = -(I\otimes X_{11}^*)^{-1}(\tilde{Z}_{11}^*\otimes I)h_{X_{11}}, \tag{28}$$

$$h_{Z_{22}} = -(I\otimes \tilde{X}_{22}^*)^{-1}(\tilde{Z}_{22}^*\otimes I)h_{\tilde{X}_{22}}, \tag{29}$$

which substituted to (25) and (26) yield

$$0 = h_{X_{11}}^T h_{\tilde{Z}_{11}} = -h_{X_{11}}^T (I \otimes X_{11}^*)^{-1} (\tilde{Z}_{11}^* \otimes I) h_{X_{11}}, \tag{30}$$

$$0 = h_{\tilde{X}_{22}}^T h_{Z_{22}} = -h_{\tilde{X}_{22}}^T (I \otimes \tilde{X}_{22}^*)^{-1} (\tilde{Z}_{22}^* \otimes I) h_{\tilde{X}_{22}}. \tag{31}$$

Since X_{11}^*, $\tilde{Z}_{11}^*$ and $\tilde{X}_{22}^*$, Z_{22}^* are positive definite, due to the properties of the Kronecker product we obtain that

$$(I \otimes X_{11}^*)^{-1}(\tilde{Z}_{11}^* \otimes I) = (I \otimes {X_{11}^*}^{-1})(\tilde{Z}_{11}^* \otimes I) = (\tilde{Z}_{11}^* \otimes {X_{11}^*}^{-1}) \succ 0,$$
$$(I \otimes \tilde{X}_{22}^*)^{-1}(\tilde{Z}_{22}^* \otimes I) = (I \otimes {\tilde{X_{22}^*}}^{-1})(\tilde{Z}_{22}^* \otimes I) = (\tilde{Z}_{22}^* \otimes {\tilde{X_{22}^*}}^{-1}) \succ 0.$$

Hence from (30) and (31)

$$h_{X_{11}} = 0 \quad \text{and} \quad h_{\tilde{X}_{22}} = 0 \tag{32}$$

and substituting (32) to (28) and (29) we obtain

$$h_{\tilde{Z}_{11}} = 0 \;\; \text{and} \;\; h_{Z_{22}} = 0. \tag{33}$$

Due to Assumption 1, $\text{rank}(A) = m$. So (22) together with $h_{Z_{22}} = 0$ imply that $h_y = 0$.

From (16) and (17) we obtain

$$h_{\tilde{X}_{12}} = -(Z_{22}^* \otimes I)^{-1}(I \otimes X_{11}^*) h_{\tilde{Z}_{12}}, \tag{34}$$

$$h_{\tilde{X}_{21}} = -(\tilde{Z}_{11}^* \otimes I)^{-1}(I \otimes \tilde{X}_{22}^*) h_{\tilde{Z}_{21}}. \tag{35}$$

Setting $h_{\tilde{Z}_{11}} = 0$ and $h_{\tilde{X}_{22}} = 0$, and the formulas (34) and (35) into (27) we obtain

$$-h_{\tilde{Z}_{12}}^T (Z_{22}^* \otimes I)^{-1}(I \otimes X_{11}^*) h_{\tilde{Z}_{12}} - h_{\tilde{Z}_{21}}^T (\tilde{Z}_{11}^* \otimes I)^{-1}(I \otimes \tilde{X}_{22}^*) h_{\tilde{Z}_{21}} = 0.$$

Since $(Z_{22}^* \otimes I)^{-1}(I \otimes X_{11}^*)$ and $(\tilde{Z}_{11}^* \otimes I)^{-1}(I \otimes \tilde{X}_{22}^*)$ are positive definite, we have $h_{\tilde{Z}_{12}} = 0$ and $h_{\tilde{Z}_{21}} = 0$. From (34) and (35) it finally follows that $h_{\tilde{X}_{12}} = 0$ and $h_{\tilde{X}_{21}} = 0$. Thus, the Jacobian $\tilde{J}^*$ is nonsingular.

(e) Application of the implicit function theorem. Now, we are ready to apply the analytic version of the implicit function theorem to the system (11), (12') and (9') at

$p^* = (v^*, \tilde{w}^*)$ and $\mu = 0$. Let us recall that $\tilde{p}(\mu) = (v(\mu), \tilde{w}(\mu))$ is the solution of the system for $\mu > 0$. The system is defined by analytic functions, p^* solves the system at $\mu = 0$ and the Jacobian $\tilde{J}^*$ of the system at p^* and $\mu = 0$ is nonsingular. Thus, there exist neighborhoods I and U of $\mu = 0$ and p^* respectively, such that for any $\mu \in I$ there exists a unique $\hat{p}(\mu) \in U$ - the solution to the system, where $\hat{p}(0) = p^*$ and $\hat{p}(\mu)$ is analytic. The tilde path $\tilde{p}(\mu)$ has the property that for some i_o and for all $i \geq i_o$: $\mu_i \in I$ and $\tilde{p}(\mu_i) \in U$, and therefore $\hat{p}(\mu_i) = \tilde{p}(\mu_i)$. However, since both the tilde path $\tilde{p}(\mu)$ and $\hat{p}(\mu)$ are analytic for $\mu > 0$ we have $\tilde{p}(\mu) = \hat{p}(\mu)$ for $\mu \in I \cap (0, \infty)$ by the uniqueness of analytic functions. Thus the tilde path $\tilde{p}(\mu)$ is analytically extended to $\mu = 0$ by prescription $\tilde{p}(0) := \hat{p}(0)$ and the theorem is proved. $\Box$

As a simple consequence of the above results we obtain

Corollary. *Under the assumptions 1-3 all derivatives of the central path*

$$p^{(k)}(\mu) = (X^{(k)}(\mu), y^{(k)}(\mu), Z^{(k)}(\mu)), \quad k = 0, 1, ..., \quad \textit{converge for } \mu \downarrow 0 \textit{ and}$$

$$p^{(k)}(0) = (X^{(k)}(0), y^{(k)}(0), Z^{(k)}(0)) = \lim_{\mu \downarrow 0} (X^{(k)}(\mu), y^{(k)}(\mu), Z^{(k)}(\mu)).$$

Moreover, if the orthonormal transformation (described in the proof of Theorem) was applied to the system, then $X_{11}(0) \succ 0$, $Z_{22}(0) \succ 0$, and

$$X_{12}(\mu) = \mu \tilde{X}_{12}, \quad X_{21}(\mu) = \mu \tilde{X}_{21}(\mu), \quad X_{22}(\mu) = \mu \tilde{X}_{22}(\mu),$$
$$Z_{11}(\mu) = \mu \tilde{Z}_{11}(\mu), \quad Z_{12}(\mu) = \mu \tilde{Z}_{12}(\mu), \quad Z_{21}(\mu) = \mu \tilde{Z}_{21}(\mu)$$

where $\tilde{X}_{12}(\mu), \tilde{X}_{21}(\mu), \tilde{X}_{22}(\mu), \tilde{Z}_{11}(\mu), \tilde{Z}_{12}(\mu), \tilde{Z}_{21}(\mu)$ are analytic for $\mu \geq 0$, and $\tilde{X}_{22}(0) = Z_{22}^{-1}(0) \succ 0$, $\tilde{Z}_{11}(0) = X_{11}^{-1}(0) \succ 0$.

4. Concluding remarks

Let us remind that the above results are proven under the assumption of strict complementarity. If no strictly complementary solution exists, the first-order derivative of the central path is not bounded as $\mu \downarrow 0$, which was shown by Goldfarb and Scheinberg [GS]. The situation here is very similar to that for the LCP where the existence of a strictly complementary solution is not guaranteed and the derivatives are unbounded, as $\mu \downarrow 0$.

Nevertheless, it was proven by Stoer and Wechs [SW1, SW2] that if the LCP does not have a strictly complementary solution, then the central path, considered as the function of $\rho = \sqrt{\mu}$, can be analytically extended to $\rho = 0$. The proof of this fact is based on a classification of the variables along the central path. Accordingly, the variables fall into one of the three classes: $O(1)$, $O(\sqrt{\mu})$ or $O(\mu)$. Let us mention that this classification has been established also by Illes *et al.* [IPRT] (see also [S1] and [P]). To obtain such a classification, some classical results by Robinson and Hoffman from the theory of error bounds for polyhedral sets were used both in [SW1, SW2] and [IPRT]. The drawback is that these old results do not hold for conical sets and thus they do not yield such a classification for SDP. However, the area of error bounds for SDP is a subject of very intense research (see [S, LS, NF] and the references cited therein), and new results obtained here open a way to study the analytical properties of the central path for the general SDP. In fact, some examples of SDPs ([SD], [Pa]) show that the variables along the central path for SDP display much more complex behavior than those for LCP, and the classification of the variables merely into the classes $O(1)$, $O(\sqrt{\mu})$ and $O(\mu)$ does not hold for some SDP instances.

Luo, Sturm and Zhang [LSZ] have used the error bound (R) to construct and analyze algorithms with superlinear convergence. (See also [PS], and [KSS] where the superlinear convergence is analyzed for nondegenarate SDP's). A question now arises as to how the results of this paper may be used to construct higher-order methods for SDP. Such a method for LCP, with very good local convergence properties, has been suggested in [SWM]. For this method, however, another property of the central path is used. The central path is actually considered to be a special case of a weighted central path that is analytic not only at the variable $\mu \geq 0$ but also at the weight vector $\omega > 0$. This path can be constructed to pass through any current point $(x, z) > 0$, and the correspondence between the current point and the weight vector is given by $\omega_i = \frac{x_i z_i}{\mu}$. For SDP the situation is more complicated since there is no obvious connection between the primal-dual strictly feasible pair (X, Z) and the matrix or vector of possible weights ([SZ], [MZ], [MP]). Because of this, the question that still remains open is how to define paths for

SDP, so that their analyticity would serve for the construction of algorithms similarly as the analyticity of the weighted paths serves for LCP [SWM].

The results of this paper may have applications to the sensitivity analysis in SDP. In fact, some results for the central path in LP, which follow from the analyticity of the central path, have been independently established and used in the context of the sensitivity theory in LP (see [HC, HSZ]). Moreover, the technique used in this paper is very suitable for the sensitivity analysis, since from the system (11), (12') and (9) we may obtain the analyticity of the solution not only in the parameter μ, but also in some parameters of problem data. Let us mention that a result of this paper (the boundedness of the first-order derivative) has surprisingly served for the sensitivity result in [SZ2].

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