On interior-point trajectories in linear optimization

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Outline

1. Weighted central paths
2. Limiting behavior technique
3. Interior-point trajectories

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1. Weighted central paths

The primal-dual pair of linear optimization problems

$$\text{(P)} \qquad \min\{\ c^T x \mid Ax = b,\ x \geq 0\ \},$$

$$\text{(D)} \qquad \max\{\ b^T y \mid A^T y + s = c,\ s \geq 0\ \},$$

where A is an $m \times n$ matrix, $c, x, s \in \mathbb{R}^n$; $b, y \in \mathbb{R}^m$.

Assumption 1: $\text{rank}(A) = m < n$

Assumption 2: $\exists\, x > 0,\ s > 0\ :\ Ax = b,\ A^T y + s = c.$

For any $\omega > 0$, $\omega \in \mathbb{R}^n$, the μ parameterized system of **ω-centering conditions**

$$\begin{aligned} Ax &= b, \quad x \geq 0, \\ A^T y + s &= c, \quad s \geq 0, \\ Xs &= \mu\omega, \end{aligned} \tag{1}$$

where $X :=\text{diag}(x)$ and $\mu \in \mathbb{R}$, $\mu \geq 0$.

Fact 1: For any $\omega > 0$ and $\mu > 0$ there exists the unique solution $(x_\omega(\mu),\ y_\omega(\mu),\ s_\omega(\mu))$ of (1).

Definition: Let $\omega > 0$ be fixed. The set of points $\{x_\omega(\mu), y_\omega(\mu), s_\omega(\mu)\}_{\mu>0}$ is called as the (ω- weighted) central path.

The usual central path is when $\omega = (1, ..., 1)^T$.

Fact 2: The ω- central path restricted to $0 < \mu < \bar{\mu}$ for some $\bar{\mu} > 0$ is bounded and each of its limit points as $\mu \downarrow 0$ is a strictly complementary solution to (P)-(D).

Fact 3: Any of the limit points of the ω- central path as $\mu \downarrow 0$ is the ω-(weighted) analytic center of the optimal set.

Fact 4: The ω- central path converges as $\mu \downarrow 0$.

Fact 5: The ω- central path is a smooth curve in the interior of the primal-dual feasible set.

Fact 6: The first-order derivatives of the ω-central path have finite limits as $\mu \downarrow 0$. [Adler, Monteiro (1991)]

Fact 7: The all-order derivatives of the ω-central path have finite limits as $\mu \downarrow 0$. [Güler (1994)]

Fact 8: The ω-central path is analytic (as a function of μ) at $\mu \geq 0$. [Wechs (1998), Halická (1999)]

$$\begin{array}{lll}
\text{Fact 2} \Rightarrow & \boxed{\text{the technique}} & \Rightarrow \text{Fact 8} \Rightarrow \text{Facts 4, 6, 7} \\
& \Downarrow \qquad\quad \Downarrow & \\
& \text{Fact 5} \quad \text{Fact 3} \Rightarrow \text{Fact 4} &
\end{array}$$

2. Limiting behavior technique

The technique first used in LCP:

[J.Stoer, M. Wechs (1999) Numer. Math. 81, 631-645]

[J.Stoer, M.Wechs (1998) Math. Progr. 83, 403-423]

[J.Stoer, M.Wechs, S.Mizuno (1998), Path. Oper. Res. 23, 832-862]

The technique described in:

M. Halicka (2001): Two simple proofs for analyticity of the central path in linear programming, Oper.Res.Let. 28 9-19

Theorem 1. *For any $\omega \in \mathbb{R}^n$, $\omega > 0$, the ω- weighted central path (as a function of μ) is analytic at any $\mu > 0$ and can be analytically extended to $\mu = 0$.*

Outline of the technique:

1. Optimal partition of variables
2. Introduction of new variables
3. Operations on the centering system
4. Characterization of limit points
5. Nonsingularity of the Jacobian
6. Application of the implicit function theorem

Agreements:

- ω fixed throughout the proof
- $(x(0), s(0))$ - an arbitrarily chosen limit point of $(x(\mu), s(\mu))$ as $\mu \downarrow 0$.

2.1 Optimal partition notation

Let (B, N) be the optimal partition for (P)-(D).

Denote by $|B|$ and $|N|$ the cardinality of B and N.

We assume that

$$x \in \mathbb{R}^n \;\; \Rightarrow \;\; x = \begin{pmatrix} x_B \\ x_N \end{pmatrix}, \text{ where } x_B \in \mathbb{R}^{|B|}, x_N \in \mathbb{R}^{|N|}$$

$$A = \begin{pmatrix} A_B & A_N \end{pmatrix}$$

In this notation the equations of (1) can be rewritten to

$$\begin{aligned} A_B x_B + A_N x_N &= b, \\ A_B^T y + s_B &= c_B, \\ A_N^T y + s_N &= c_N, \\ X_B s_B &= \mu \omega_B, \\ X_N s_N &= \mu \omega_N. \end{aligned} \tag{4}$$

The central path is of the form $(x_B(\mu), x_N(\mu), s_B(\mu), s_N(\mu))$.

By Fact 2, any limit point is a strictly complementary solution and since (B, N) is the optimal partition, we have

$$\begin{aligned} x_B(0) > 0, &\qquad x_N(0) = 0, \\ s_B(0) = 0, &\qquad s_N(0) > 0. \end{aligned} \tag{5}$$

"Small" variables x_N and s_B are replaced by auxiliary ones.

2.2 Introduction of "tilde" variables

From the perturbed complementarity:

$$X_B^{-1}\omega_B = \frac{s_B}{\mu} \quad \text{and} \quad S_N^{-1}\omega_N = \frac{x_N}{\mu}$$

We define

$$\begin{aligned} \tilde{s}_B &:= \frac{s_B}{\mu} \quad &\Rightarrow \quad s_B = \mu\tilde{s}_B, \\ \tilde{x}_N &:= \frac{x_N}{\mu} \quad &\Rightarrow \quad x_N = \mu\tilde{x}_N. \end{aligned} \tag{6}$$

The equations (4) of the centering system can be rewritten

$$A_B x_B + \mu A_N \tilde{x}_N = b, \tag{7a}$$

$$A_B^T y + \mu\tilde{s}_B = c_B, \tag{7b}$$

$$A_N^T y + s_N = c_N, \tag{7c}$$

$$X_B\tilde{s}_B = \omega_B, \tag{7d}$$

$$\tilde{X}_N s_N = \omega_N. \tag{7e}$$

Instead of the central path $(x_B(\mu), x_N(\mu), s_B(\mu), s_N(\mu))$, we study the **tilde path**: $(x_B(\mu), \tilde{x}_N(\mu), \tilde{s}_B(\mu), s_N(\mu))$.

We have that:

- $(x_B(\mu), \tilde{x}_N(\mu), \tilde{s}_B(\mu), s_N(\mu))$ is the unique positive solution to (7) at any $\mu > 0$;
- The tilde path has finite limit points as $\mu \to 0$, and they solve (7) at $\mu = 0$.
- Also the tilde components of the limit points are positive, i.e., $\tilde{s}_B(0) > 0$ and $\tilde{x}_N(0) > 0$.

The Jacobian of (7) is

$$J(x_B, y, s_N, \tilde{x}_N, \tilde{s}_B; \mu) = \begin{pmatrix} A_B & 0 & 0 & \mu A_N & 0 \\ 0 & A_B^T & 0 & 0 & \mu I_B \\ 0 & A_N^T & I_N & 0 & 0 \\ \tilde{S}_B & 0 & 0 & 0 & X_B \\ 0 & 0 & \tilde{X}_N & S_N & 0 \end{pmatrix}$$

The Jacobian along the tilde path:
$J(\mu) := J(x_B(\mu), y(\mu),\ s_N(\mu), \tilde{x}_N(\mu), \tilde{s}_B(\mu); \mu)$ for $\mu \geq 0$.

Note that A_B is $m \times |B|$

Nonsingularity of $J(\mu)$:

For $\mu > 0$: $J(\mu)$ is nonsingular (easy to prove, seen later)

For $\mu = 0$: $J(0)$ is nonsingular

$\Leftrightarrow P := \begin{pmatrix} A_B & 0 \\ 0 & A_B^T \end{pmatrix}$ is nonsingular

$\Leftrightarrow |B| = m$ and $\operatorname{rank}(A_B) = m$

$\Leftrightarrow$ Both (P) and (D) have unique solution

The case of $|B| \neq m$ or $\operatorname{rank}(A_B) \neq m$ is treated in what follows.

2.3 Operations on the system for the tilde path

Denote

$$P := \begin{pmatrix} A_B & 0 \\ 0 & A_B^T \end{pmatrix}, \quad Q := \begin{pmatrix} A_N & 0 \\ 0 & I_B \end{pmatrix} \text{ and } q := \begin{pmatrix} b \\ c_B \end{pmatrix}.$$

Then (7a) and (7b) can be rewritten

$$P \begin{pmatrix} x_B \\ y \end{pmatrix} + \mu Q \begin{pmatrix} \tilde{x}_N \\ \tilde{s}_B \end{pmatrix} = q. \tag{8}$$

Let $\operatorname{rank}(P) =: r < m + |B|$. Then there exists a nonsingular $(m + |B|) \times (m + |B|)$ matrix W such that

$$W = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} \text{ and } W_2 P = 0.$$

Here W_2 is an $(m+|B|-r) \times (m+|B|)$ matrix. Multiplying (8) by W on the left we obtain

$$W_1 P \begin{pmatrix} x_B \\ y \end{pmatrix} + \mu W_1 Q \begin{pmatrix} \tilde{x}_N \\ \tilde{s}_B \end{pmatrix} = W_1 q, \tag{10a}$$

$$\mu W_2 Q \begin{pmatrix} \tilde{x}_N \\ \tilde{s}_B \end{pmatrix} = W_2 q. \tag{10b}$$

From (10b) at $\mu = 0$ we get $W_2 q = 0$. Thus instead of (10b) we have

$$W_2 Q \begin{pmatrix} \tilde{x}_N \\ \tilde{s}_B \end{pmatrix} = 0. \tag{11}$$

For any $\mu \geq 0$, the tilde path is a solution to **the tilde system**:

$$W_1 P \begin{pmatrix} x_B \\ y \end{pmatrix} + \mu W_1 Q \begin{pmatrix} \tilde{x}_N \\ \tilde{s}_B \end{pmatrix} = W_1 q, \tag{10a}$$

$$W_2 Q \begin{pmatrix} \tilde{x}_N \\ \tilde{s}_B \end{pmatrix} = 0 \tag{11}$$

$$A_N^T y + s_N = c_N, \tag{7c}$$

$$X_B \tilde{s}_B = \omega_B, \tag{7d}$$

$$\tilde{X}_N s_N = \omega_N. \tag{7e}$$

Lemma 1:

$$W_1 P \binom{x_B}{y} = W_1 q \quad \Leftrightarrow \quad P \binom{x_B}{y} = q$$

Proof: Since $W_2 P = 0$ and $W_2 q = 0$ we have

$$\begin{pmatrix} W_1 \\ W_2 \end{pmatrix} P \begin{pmatrix} x_B \\ y \end{pmatrix} = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} q.$$

Now the lemma follows by nonsingularity of $\binom{W_1}{W_2}$. $\square$

Lemma 2:

$$W_2 Q \begin{pmatrix} \tilde{x}_N \\ \tilde{s}_B \end{pmatrix} = 0$$

if and only if there exist u_1 and u_2 such that

$$P \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + Q \begin{pmatrix} \tilde{x}_N \\ \tilde{s}_B \end{pmatrix} = 0. \tag{13b}$$

Proof: $W = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix}$ is nonsigular $(m + |B|) \times (m + |B|)$; W_2 is $(m + |B| - r) \times (m + |B|)$;

$$\left.\begin{array}{r} \text{rank } W_2 = (m + |B| - r) \\ \text{rank } P = r \\ W_2 P = 0 \end{array}\right\} \quad W_2 a = 0 \Leftrightarrow \exists\, b : Pb = a$$

Lemma follows when setting $a = Q \begin{pmatrix} \tilde{x}_N \\ \tilde{s}_B \end{pmatrix}$. $\square$

2.4 Characterization of limit points

Lemma 3. *(x_B, y, s_N) is an solution to the tilde system at $\mu = 0 \iff$ it is the ω-weighted analytic center of M where*

$$M = \left\{ (x_B, y, s_N) \;:\; \begin{array}{ll} W_1 P \binom{x_B}{y} = W_1 q\,, & x_B \geq 0, \\ A_N^T y + s_N = c_N\,, & s_N \geq 0 \end{array} \right\}.$$

Proof: (x_B, s_N, y) is the ω-weighted analytic center of M

$\iff$ (x_B, s_N, y) is the (unique) solution to

$$\max \left\{ \sum_{i \in B} \omega_i \ln x_i + \sum_{i \in N} \omega_i \ln s_i \;:\; \begin{array}{l} W_1 P \begin{pmatrix} x_B \\ y \end{pmatrix} = W_1 q, \\ A_N^T y + s_N = c_N \end{array} \right\}$$

$\iff$ (x_B, s_N, y) is the solution to

$$\begin{aligned} W_1 P \begin{pmatrix} x_B \\ y \end{pmatrix} &= W_1 q, \\ P \binom{u_1}{u_2} + Q \begin{pmatrix} S_N^{-1} \omega_N \\ X_B^{-1} \omega_B \end{pmatrix} &= 0 \quad \text{for some} \quad u_1, u_2 \\ A_N^T y + s_N &= c_N \end{aligned}$$

$\iff$ (x_B, s_N, y) is the solution to

$$\begin{aligned} W_1 P \begin{pmatrix} x_B \\ y \end{pmatrix} &= W_1 q, \\ W_2 Q \begin{pmatrix} \tilde{x}_N \\ \tilde{s}_B \end{pmatrix} &= 0, \quad \text{where} \quad \begin{array}{l} \tilde{x}_N = S_N^{-1} \omega_N \\ \tilde{s}_B = X_B^{-1} \omega_B \end{array} \\ A_N^T y + s_N &= c_N \end{aligned}$$

$\square$

Convergence of the tilde and central paths:

Let $(x_B(0), y(0), s_N(0))$ be an arbitrarily chosen limit point of $(x_B(t), y(t), s_N(t))$ as $\mu \to 0$.

Then $(x_B(0), y(0), s_N(0))$ solves the tilde system at $\mu = 0$.

Lemma 3 $\Rightarrow$ $(x_B(0), y(0), s_N(0))$ is the ω- center of M.
Note: By Lemma 1, M is the set of optimal solutions.

$$\Downarrow$$

Any limit point of $x_B(t), y(t), s_N(t)$ is the ω- center of M.

The ω- analytic center is defined uniquely.

$$\Downarrow$$

$(x_B(t), y(t), s_N(t))$ has the unique limit point as $\mu \to 0$ and hence it converges.

The convergence of $\tilde{x}_N(t)$ and $\tilde{s}_B(t)$ follows from

$$\tilde{x}_N = S_N^{-1}\omega_N \quad \text{and} \quad \tilde{s}_B = X_B^{-1}\omega_B.$$

The convergence of $x_N(t)$ and $s_B(t)$ follows from

$$x_N = \mu\tilde{x}_N \quad \text{and} \quad s_B = \mu\tilde{s}_B.$$

Notations:

- $(x_B(0), y(0), s_N(0), \tilde{x}_N(0), \tilde{s}_B(0))$ the (unique) limit point
- $\tilde{J}(\mu)-$ the Jacobian of the tilde equations with respect to $(x_B, y, s_N, \tilde{x}_N, \tilde{s}_B)$, evaluated along the tilde path

2.5 Nonsingularity of the Jacobian $\tilde{J}(\mu)$ at $\mu \geq 0$

For any $\mu \geq 0$ we prove that $\quad \tilde{J}(\mu) \cdot dh = 0 \;\Rightarrow\; dh = 0.$

Let $\quad dh = (dx_B, dy, ds_N, d\tilde{x}_N, d\tilde{s}_B), \quad$ then
the equations of $\quad \tilde{J}(\mu) \cdot dh = 0 \quad$ are

$$W_1 P \begin{pmatrix} dx_B \\ dy \end{pmatrix} + \mu W_1 Q \begin{pmatrix} d\tilde{x}_N \\ d\tilde{s}_B \end{pmatrix} = 0, \tag{12a}$$

$$W_2 Q \begin{pmatrix} d\tilde{x}_N \\ d\tilde{s}_B \end{pmatrix} = 0, \tag{12b}$$

$$A_N^T dy + ds_N = 0, \tag{12c}$$

$$\tilde{S}_B(\mu) dx_B + X_B(\mu) d\tilde{s}_B = 0, \tag{12d}$$

$$\tilde{X}_N(\mu) ds_N + S_N(\mu) d\tilde{x}_N = 0. \tag{12e}$$

Unpacking of (12a): Using $W_2 P = 0$, (12b) and nonsingularity of W:

$$P \begin{pmatrix} dx_B \\ dy \end{pmatrix} + \mu Q \begin{pmatrix} d\tilde{x}_N \\ d\tilde{s}_B \end{pmatrix} = 0. \tag{13a}$$

Unpacking of (12b): By Lemma 2 there exist u_1 and u_2 such that

$$P \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + Q \begin{pmatrix} d\tilde{x}_N \\ d\tilde{s}_B \end{pmatrix} = 0. \tag{13b}$$

Moreover, we define $u_3 := -A_N^T u_2$ which yields

$$A_N^T u_2 + u_3 = 0. \tag{14}$$

The equations of (13a) and (12c) are

$$
\begin{aligned}
A_B dx_B + \mu A_N d\tilde{x}_N =&0, && (15a)\\
A_B^T dy + \mu d\tilde{s}_B =&0, && (15b)\\
A_N^T dy + ds_N =&0
\end{aligned}
$$

The equations of (13b) and (14) are:

$$
\begin{aligned}
A_B u_1 + A_N d\tilde{x}_N =&0, && (16a)\\
A_B^T u_2 + d\tilde{s}_B =&0, && (16b)\\
A_N^T u_2 + u_3 =&0
\end{aligned}
$$

Orthogonality properties:

(15a) and (15b): $\mu dx_B^T d\tilde{s}_B + \mu d\tilde{x}_N^T ds_N = 0$ (17a)
(15a) and (16b): $dx_B^T d\tilde{s}_B + \mu d\tilde{x}_N u_3 = 0$ (17b)
(15b) and (16a): $\mu d\tilde{s}_B u_1 + d\tilde{x}_N^T ds_N = 0$ (17c)

For $\mu > 0$: $dx_B^T d\tilde{s}_B + d\tilde{x}_N^T ds_N = 0$ (18a)
For $\mu = 0$: $dx_B^T d\tilde{s}_B = 0$ (18b)
$d\tilde{x}_N^T ds_N = 0$ (18c)

Finally we use (12d) and (12e)

$$
\begin{aligned}
\tilde{S}_B(\mu) dx_B + X_B(\mu) d\tilde{s}_B =&0 \Rightarrow d\tilde{s}_B = -X_B(\mu)^{-1}\tilde{S}_B(\mu) dx_B\\
\tilde{X}_N(\mu) ds_N + S_N(\mu) d\tilde{x}_N =&0 \Rightarrow ds_N = -\tilde{X}_N(\mu)^{-1} S_N(\mu) d\tilde{x}_N.
\end{aligned}
$$

Substituting into (18a) yields for $\mu > 0$:

$$dx_B^T X_B(\mu)^{-1} \tilde{S}_B(\mu) dx_B + d\tilde{x}_N^T \tilde{X}_N(\mu)^{-1} S_N(\mu) d\tilde{x}_N = 0$$

and substituting into (18b) and (18c) yields for $\mu = 0$

$$\begin{aligned} dx_B^T X_B(0)^{-1} \tilde{S}_B(0) dx_B &= 0 \\ d\tilde{x}_N^T \tilde{X}_N(0)^{-1} S_N(0) d\tilde{x}_N &= 0 \end{aligned}$$

Since

$$X_B(\mu)^{-1} \tilde{S}_B(\mu) \ \text{ and } \ \tilde{X}_N(\mu)^{-1} S_N(\mu)$$

are positive definite for any $\mu \geq 0$, we obtain that

$$dx_B = 0 \text{ and } d\tilde{x}_N = 0$$

for both $\mu > 0$ and $\mu = 0$. Consequently,

$$d\tilde{s}_B = 0 \text{ and } ds_N = 0.$$

Hence

$$\left.\begin{aligned} A_B^T dy + \mu d\tilde{s}_B =& 0 \\ A_N^T dy + ds_N =& 0 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} A_B^T dy =& 0 \\ A_N^T dy =& 0 \end{aligned}\right\} \Rightarrow A^T dy = 0,$$

and Assumption 1 (the full row rank of A) yields $dy = 0$.

Thus $dh = 0$ and $\tilde{J}(\mu)$ is nonsingular at any $\mu \geq 0$.

2.6 Application of the implicit function theorem

Denote $p = (x_B^T, \tilde{x}_N^T, y^T, \tilde{s}_B^T, s_N^T)^T \in \mathbb{R}^{2n+m}$ and

the tilde path $p(\mu) = (x_B^T(\mu), \tilde{x}_N^T(\mu), y^T(\mu), \tilde{s}_B^T(\mu), s_N^T(\mu))^T$.

Then the system defining the tilde path

$$W_1 P \begin{pmatrix} x_B \\ y \end{pmatrix} + \mu W_1 Q \begin{pmatrix} \tilde{x}_N \\ \tilde{s}_B \end{pmatrix} - W_1 q = 0, \tag{10a}$$

$$W_2 Q \begin{pmatrix} \tilde{x}_N \\ \tilde{s}_B \end{pmatrix} = 0 \tag{11}$$

$$A_N^T y + s_N - c_N = 0, \tag{7c}$$

$$X_B \tilde{s}_B - \omega_B = 0, \tag{7d}$$

$$\tilde{X}_N s_N - \omega_N = 0. \tag{7e}$$

can be rewritten to

$$F(p, \mu) = 0 \tag{19}$$

where $F(p, \mu) : \mathbb{R}^{2n+m} \times \mathbb{R} \mapsto \mathbb{R}^{2n+m}$ is analytic. Obviously

$$F(p(\mu), \mu) = 0, \quad \det \frac{\partial F(p(\mu), \mu)}{\partial p} \neq 0, \quad \mu \geq 0$$

Note: For any $\mu > 0$ the tilde path $p(\mu)$ is the unique positive solution to (19).

Let $\mu_o \geq 0$ and $p_o := p(\mu_o)$. Note $p_o > 0$.

Application of the analytic version of the implicit function theorem at (p_o, μ_o):

$\exists\ O(\mu_o)$ and $O(p_o)$: $\forall\ \mu \in O(\mu_o)\ \exists\ !\ \hat{p}(\mu) \in O(p_o)$:

(a) $F(\hat{p}(\mu), \mu) = 0\ \forall \mu \in O(\mu_o)$

(b) $\hat{p}(\mu_o) = p_o$, and

(c) $\hat{p}(\mu)$ is analytic on $O(\mu_o)$ (Note: $\hat{p}(\mu) = \hat{p}_{(\mu_o, p_o)}(\mu)$)

Since $\hat{p}(\mu_o) = p_o > 0$ and $\hat{p}(\mu)$ is continuous on $O(\mu_o)$ we have that
$\exists O_1(\mu_o) \subset O(\mu_o)$: $\hat{p}(\mu) > 0$ on $O_1(\mu_o)$

$p(\mu)$, $\hat{p}(\mu)$ are positive solutions to (19) on $O_1(\mu_o) \cap \mathbb{R}_{++}$.
By the uniqueness of positive solutions to (19) on $\mathbb{R}_{++}$:

$$p(\mu) = \hat{p}_{(\mu_o, p_o)}(\mu) \quad \forall \mu \in O_1(\mu_o) \cap \mathbb{R}_{++}. \tag{20}$$

(20) at any $\mu_o > 0 \Rightarrow p(\mu)$ is analytic on $(0, \infty)$.
(20) at $\mu_o = 0$ and (b) give

$$p(\mu) = \hat{p}_{(\mu_o, p_o)}(\mu) \quad \forall \mu \in O_1(\mu_o) \cap \mathbb{R}_{+}$$

and hence $p(\mu)$ is analytic at $\mu = 0$.

Since $x_N(\mu) = \mu \tilde{x}_N(\mu)$, $s_B(\mu) = \mu \tilde{s}_B(\mu)$, the analyticity of $x_N(\mu)$ and $s_B(\mu)$ is evident.

3. Interior-point trajectories

3.1 Dependence of the centering solution on the weight-vector

For any $\omega > 0$, $\omega \in \mathbb{R}^n$, the ω - centering system:

$$\begin{aligned} Ax = b, &\quad x \geq 0, \\ A^T y + s = c, &\quad s \geq 0, \\ Xs = \mu\omega, & \end{aligned} \tag{1}$$

where $\mu \in \mathbb{R}$, $\mu \geq 0$.

Fact 1: For any $\omega > 0$ and $\mu > 0$ there exists the unique solution $(x_\omega(\mu),\, y_\omega(\mu),\, s_\omega(\mu))$ of (1).

Fact 2: The ω- weighted central path restricted to $0 < \mu < \bar{\mu}$ for some $\bar{\mu} > 0$ is bounded and each of its limit points as $\mu \downarrow 0$ is a strictly complementary solution to (P)-(D).

Dependence on ω:

The unique solution to (1) associated with $\omega > 0$ and $\mu > 0$: $(x(\omega, \mu), s(\omega, \mu))$ - **the (ω, μ)-center of (1)**.

The (ω, μ)-centering function:

$$p(\omega, \mu) = (x(\omega, \mu), y(\omega, \mu), s(\omega, \mu)) : \mathbb{R}^n_{++} \times \mathbb{R}_{++} \to \mathbb{R}^{2n+m}$$

To the application of the limit behavior technique we need an equivalent of Fact 2.

Fact 2': The (ω, μ)- centering function $p(\omega, \mu)$ restricted to $0 < \mu < \mu_1$ and $0 < \omega < \omega_1$ for some $\mu_1 > 0$ and $\omega_1 > 0$ is bounded and each of its limit points as $\mu \downarrow 0$ and $\omega \to \omega_0 > 0$ is a strictly complementary solution to (P)-(D).

Theorem 2. *The (ω, μ)-centering function, $p(\omega, \mu)$, is analytic on its definition domain and can be analytically extended to $\mathbb{R}^n_{++} \times \mathbb{R}_+$. Moreover, for any $\omega_o > 0$,*

$$p(\omega_o, 0) = \lim_{\substack{\mu \to 0 \\ \omega \to \omega_o}} p(\omega, \mu)$$

is the ω_o weighted analytic center of the optimal set.

Proof: The same as for the fixed ω.

It uses the analytic dependence of (1) on ω and the assumption that $\omega \to \omega_o > 0$.

The limiting properties of derivatives of $p(\omega, \mu)$ are useful to the analysis and construction of polynomial-time algorithms which are also locally superlinearly or quadratically convergent.

3.2 Re-parameterization of the centering system

Consider the following t-parameterized system

$$\begin{aligned} Ax &= b, \quad x \geq 0, \\ A^T y + s &= c, \quad s \geq 0, \\ Xs &= \mu(t)\omega(t), \end{aligned} \tag{1_t}$$

where $t \in [0, \bar{t}\,]$ and $\mu(t) : [0, \bar{t}\,] \to \mathbb{R}$ and $\omega(t) : [0, \bar{t}\,] \to \mathbb{R}^n$, and satisfy

Assumption 3:

(a) $\omega(t), \mu(t) \in C^r[0, \bar{t}\,]$, where $r \geq 1$ or $r = \infty$ or $r = \Omega$ for analyticity.

(b) For each $t \in (0, \bar{t}\,] : \mu(t) > 0$, and $\mu(0) = 0$.

(c) For each $t \in [0, \bar{t}\,] : \omega(t) > 0$, $\omega(0) =: \omega_o$.

Fact 1_t: For any $t \in (0, \bar{t}\,]$ there exists the unique solution $(x(t),\, y(t),\, s(t))$ of (1_t). It is called **the t-center for (1_t)**.

The t- centering function for (1_t):

$$p(t) = (x(t), y(t), s(t)) : (0, \bar{t}\,] \to \mathbb{R}^{2n+m}$$

Fact 2_t: The t- centering function, $p(t)$, is bounded and each of its limit points as $t \to 0$ is a strictly complementary solution to (P)-(D).

Theorem 3. *Let $p(t)$ be the t-centering function for (1_t) where $\omega(t)$ and $\mu(t)$ satisfy Assumption 3. Then $p(t) \in C^r(0, \bar{t}\,]$ and can be extended as a C^r-function to $[0, \bar{t}\,]$. Moreover, $p(0)$ is the ω_o-analytic center of the optimal set.*

Proof: The same as for the ω-central path. Any ω and μ is replaced by $\omega(t)$ and $\mu(t)$.

- We introduce the optimal partition notation.

- We introduce the tilde variables and the tilde function

$$\tilde{x}_N(t) := \frac{x_N(t)}{\mu(t)} = S_N^{-1}(t)\omega_N(t)$$

$$\tilde{s}_B(t) := \frac{s_B(t)}{\mu(t)} = X_B^{-1}(t)\omega_B(t)$$

which is bounded and positive at any limit point as $t \to 0$.

- We derive a new system for the tilde function. We use that $\tilde{x}_N(t)$ and $\tilde{s}_B(t)$ are bounded as $t \to 0$.

- We prove the convergence of the t-centering function to the ω_o-weighted analytic center of the optimal set.

- We prove the nonsingularity of the Jacobian.

- We apply the C^r-version of the implicit function theorem.

Corollary. $\tilde{x}_N(t),\ \tilde{s}_B(t) \in C^r[0, \bar{t}\,]$.

3.3 Trajectories for potential reduction methods

Reha H. Tűtűncű: Continuous Trajectories for Primal-Dual Potential-Reduction Methods (2001)

[http://www.optimization-online.org/DB_HTML/2001/09/372.html]

Implicit description of trajectories $\bar{x}(\tau), \bar{y}(\tau), \bar{s}(\tau), \tau \in (l, \infty)$ associated with the algorithm of Kojima, Mizuno, and Yoshise:

$$\begin{aligned} A\bar{x}(\tau) = b, \quad \bar{x}(\tau) > 0, \\ A^T\bar{y}(\tau) + \bar{s}(\tau) = c, \quad \bar{s}(\tau) > 0, \\ \bar{X}(\tau)\bar{s}(\tau) = w(\tau). \end{aligned} \tag{21}$$

Here $$w(\tau) = e^{-\tau}\mathbf{p} + h(\tau)\mathbf{e}$$ where

$$\mathbf{p} = X^0 s^0, \quad \mathbf{e} = (1, ...1)^T \qquad \text{and}$$

$$h(\tau) = \frac{\mathbf{e}^T\mathbf{p}}{n} e^{-\tau}\left(e^{\frac{n\tau}{\rho}} - 1\right), \qquad \rho > n$$

$$\lim_{\tau\to\infty} h(\tau) = 0, \quad \lim_{\tau\to\infty} \frac{w(\tau)}{h(\tau)} = \mathbf{e}$$

ρ - the parameter of the potential-reduction function.

How to choose $\mu(t)$ and $\omega(t)$ in order $\mu(t)\omega(t) = w(\frac{1}{t})$, and $\mu(t)$ and $\omega(t)$ would satisfy Assumption 3?

First choice : We have

$$\lim_{\tau \to \infty} h(\tau) = 0, \quad \lim_{\tau \to \infty} \frac{w(\tau)}{h(\tau)} = \mathbf{e}$$

which offer the following choice:

$$\mu(t) := h\left(\frac{1}{t}\right) = \frac{\mathbf{e}^T \mathbf{p}}{n} e^{-\frac{1}{t}} \left(e^{\frac{n}{\rho t}} - 1\right)$$
$$\mu(0) := \lim_{t \to 0} \mu(t) = 0$$

and

$$\omega(t) := \frac{w\left(\frac{1}{t}\right)}{h\left(\frac{1}{t}\right)} = \frac{\mathbf{p}}{\frac{\mathbf{e}^T \mathbf{p}}{n}\left(e^{\frac{n}{\rho t}} - 1\right)} + \mathbf{e}$$
$$\omega(0) := \lim_{t \to 0} \omega(t) = \mathbf{e}$$

Now, it is easy to see that $\mu(t)$ and $\omega(t)$ satisfy our assumptions on $[0, \bar{t}\ ]$ where $\bar{t} = \frac{1}{l}$, they are $C^{\infty}[0, \bar{t}\ ]$ and $\omega_0 = \mathbf{e}$.

Second choice : We define

$$\mu(t) := e^{-(1-\frac{n}{\rho})\frac{1}{t}}$$
$$\mu(0) := \lim_{t\to 0} \mu(t) = 0$$

$$\omega(t) := \frac{w\left(\frac{1}{t}\right)}{\mu(t)} = e^{-\frac{n}{\rho t}}\mathbf{p} + \frac{\mathbf{e}^T\mathbf{p}}{n}\left(1 - e^{-\frac{n}{\rho t}}\right)\mathbf{e}$$
$$\omega(0) := \lim_{t\to 0} \omega(t) = \frac{\mathbf{e}^T\mathbf{p}}{n}\mathbf{e}$$

Obviously, $\mu(t), \omega(t) \in C^\infty[0, \bar{t}\ ]$ and satisfy Assumption 3 with $r = \infty$.

By Theorem 3 we obtain that $(x(t), y(t), s(t)) \in C^\infty[0, \bar{t}\ ]$. By Corollary we obtain that the derivatives of any order of $\tilde{x}_N(t)$ and $\tilde{s}_B(t)$, where

$$\tilde{x}_N(t) = \frac{x_N(t)}{\mu(t)} = e^{(1-\frac{n}{\rho})\frac{1}{t}} x_N(t)$$
$$\tilde{s}_B(t) = \frac{s_B(t)}{\mu(t)} = e^{(1-\frac{n}{\rho})\frac{1}{t}} s_B(t),$$

are bounded as $t \to 0$.

In the paper by Tűtűncű the following property is used for an asymptotical analysis.

Property. *For any $\rho > n$, the functions*

$$e^{(1-\frac{n}{\rho})\tau}\dot{\bar{x}}_N(\tau) \quad \textit{and} \quad e^{(1-\frac{n}{\rho})\tau}\dot{\bar{s}}_B(\tau)$$

are bounded as $\tau \to \infty$.

Proof: Since $t = \frac{1}{\tau}$, the following relations hold between the solution $x(t)$ of (1_t) and the solution $\bar{x}(\tau)$ of (21):

$$\bar{x}_N(\tau) = x_N\left(\frac{1}{\tau}\right) \quad \text{and} \quad \dot{\bar{x}}_N(\tau) = -\frac{1}{\tau^2}\dot{x}_N\left(\frac{1}{\tau}\right).$$

We have obtained above that

$$\dot{\tilde{x}}_N(t) = e^{(1-\frac{n}{\rho})\frac{1}{t}}\dot{x}_N(t) + \left(1-\frac{n}{\rho}\right)\left(-\frac{1}{t^2}\right)e^{(1-\frac{n}{\rho})\frac{1}{t}}x_N(t)$$

is bounded as $t \to 0$ and hence also $t^2\dot{\tilde{x}}_N(t)$ is bounded as $t \to 0$. Since

$$t^2\dot{\tilde{x}}_N(t) = e^{(1-\frac{n}{\rho})\frac{1}{t}}t^2\dot{x}_N(t) - \left(1-\frac{n}{\rho}\right)\tilde{x}_N(t)$$

and $\tilde{x}_N(t)$ is bounded, we obtain that

$$e^{(1-\frac{n}{\rho})\frac{1}{t}}t^2\dot{x}_N(t) = -e^{(1-\frac{n}{\rho})\tau}\dot{\bar{x}}_N(\tau)$$

is bounded as $\tau \to \infty$. $\square$