Limiting behavior of the central path in semidefinite optimization

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Abstract

It was recently shown in [4] that, unlike in linear optimization, the central path in semidefinite optimization (SDO) does not converge to the analytic center of the optimal set in general. In this paper we analyze the limiting behavior of the central path to explain this unexpected phenomenon. This is done by deriving a new necessary and sufficient condition for strict complementarity. We subsequently show that, in the absence of strict complementarity, the central path converges to the analytic center of a certain subset of the optimal set. We further derive sufficient conditions under which this subset coincides with the optimal set, *i.e.* sufficient conditions for the convergence of the central path to the analytic center of the optimal set. Finally, we show that convex quadratically constraint quadratic optimization problems, when rewritten as an SDO problems, satisfy these sufficient conditions. Several examples are given to illustrate the possible convergence behavior.

Key words: Semidefinite optimization, linear optimization, interior point method, central path, analytic center.

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1 Introduction

We first formulate semidefinite programs and recall the definition of the central path and some of its properties.

By S^n we denote the space of all real symmetric $n \times n$ matrices and for any $M, N \in S^n$ we define

$$M \bullet N = \operatorname{trace}(MN) = \sum_{i,j} m_{ij} n_{ij}.$$

The convex cones of symmetric positive semidefinite matrices and positive definite matrices will be denoted by S_+^n and S_{++}^n, respectively; $X \succeq 0$ and $X \succ 0$ mean that a symmetric matrix X is positive semidefinite and positive definite, respectively.

We will consider the following primal–dual pair of semidefinite programs in the standard form

$$(P): \qquad \min_{X \in S^n} \left\{ C \bullet X \ : \ A^i \bullet X = b_i \ (i = 1, \ldots, m), \ X \succeq 0 \right\},$$

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$$(D): \qquad \max_{y\in\mathbb{R}^m, Z\in S^n} \left\{ b^T y \; : \; \sum_{i=1}^m A^i y_i + Z = C, \; Z \succeq 0 \right\},$$

where $A^i \in S^n$ $(i = 1, \ldots, m)$ and $C \in S^n$, $b \in \mathbb{R}^m$. The solutions X and (y, Z) will be referred to as feasible solutions as they satisfy the primal and dual constraints respectively. The primal and dual feasible sets will be denoted by $\mathcal{P}$ and $\mathcal{D}$ respectively, and $\mathcal{P}^*$ and $\mathcal{D}^*$ will denote the respective optimal sets. It is easy to see that for the primal-dual pairs of feasible solution an orthogonality property holds:

$$X^1, X^2 \in \mathcal{P}, \; Z^1, Z^2 \in \mathcal{D} \quad \Rightarrow \quad (X^1 - X^2) \bullet (Z^1 - Z^2) = 0. \tag{1}$$

We make the following two standard assumptions throughout the paper.

Assumption 1.1 A^i $(i = 1, \ldots, m)$ *are linearly independent.*

Assumption 1.2 *There exists* (X^0, Z^0, y^0) *such that*

$$A^i \bullet X^0 = b_i \; (i = 1, \ldots, m), \; X^0 \succ 0, \quad \textit{and} \quad \sum_{i=1}^m A^i y_i^0 + Z^0 = C, \; Z^0 \succ 0.$$

While Assumption 1.1 is only a technical one, and it ensures the one-to-one correspondence between y and Z in $\mathcal{D}$, Assumption 1.2 is important for the theory of interior point methods and the duality results in SDO. In fact, it is well-known that under our assumptions both $\mathcal{P}^*$ and $\mathcal{D}^*$ are non-empty and bounded, and the necessary and sufficient optimality conditions for (P) and (D) are

$$\left.\begin{array}{rcl} A^i \bullet X & = & b_i, \quad X \succeq 0 \quad (i = 1, \ldots, m) \\ \sum_{i=1}^m A^i y_i + Z & = & C, \quad Z \succeq 0 \\ XZ & = & 0. \end{array}\right\} \tag{2}$$

The last equation in (2) can be interpreted as a complementarity condition. A strictly complementary solution is defined as an optimal solution pair (X, Z) satisfying the rank condition: $\mathrm{rank}(X) + \mathrm{rank}(Z) = n$. Contrary to the the case for linear optimization (LO), the existence of a strictly complementary solution is not generally ensured in semidefinite optimization.

In order to define the central path, the optimality conditions (2) are relaxed to

$$\left.\begin{array}{rcl} A^i \bullet X & = & b_i, \quad X \succeq 0 \quad (i = 1, \ldots, m) \\ \sum_{i=1}^m A^i y_i + Z & = & C, \quad Z \succeq 0 \\ XZ & = & \mu I, \end{array}\right\} \tag{3}$$

where I is the identity matrix and $\mu \geq 0$ is a parameter. We refer to the third equality in (3) as the perturbed complementarity condition. It is easy to see that for $\mu = 0$, (3) becomes (2), and hence may have multiple solutions. On the other hand, it is well–known that for $\mu > 0$, the system (3) has a unique solution, denoted by $(X(\mu), Z(\mu), y(\mu))$. Similarly as for linear programming, this solution is seen as the parametric representation of an analytic curve (the *central path*) in terms of the parameter μ.

It has been shown that the central path for SDO shares many properties with the central path for LO. First, the basic property was established that the central path — when restricted to $0 < \mu \leq \bar{\mu}$ for some $\bar{\mu} > 0$ — is bounded and thus has limit points in the optimal set as $\mu \downarrow 0$ ([15], [8]). Then it was shown that the limit points are maximally complementary optimal solutions ([8], [3]). Finally, it was claimed by Goldfarb and Scheinberg [3] that the central path converges for $\mu \downarrow 0$ to the so-called analytic center of the optimal solution set. However, as it was shown in [4], this fact is not generally true for semidefinite programs without strictly complementary solutions. The correct proofs of this fact under the assumption of strict complementarity, are given in [15] and [7]. In [4] (see also [1]) a proof of the convergence of the central path is provided which, however, does not yield any characterization of the limit point.

In this paper we give a new characterization of strict complementarity. This allows us to show that, in the absence of strict complementarity, the central path converges to the analytic center of a certain subset of the optimal set. We derive conditions under which this subset coincides with the optimal set, and hence the central path converges to the analytic center of the optimal set. Finally, we show that the convex quadratically constraint quadratic program, when rewritten as an SDO program, satisfies the sufficient conditions.

2 Preliminaries

A pair of optimal solutions $(X, Z) \in \mathcal{P}^* \times \mathcal{D}^*$ is called a *maximally complementary solution pair* to the pair of problems (P) and (D) if it maximizes rank (X)+rank (Z) over all optimal solution pairs. The set of maximally complementary solutions coincides with the relative interior of $(\mathcal{P}^* \times \mathcal{D}^*)$. Another characterization is: $(\bar{X}, \bar{Z}) \in \mathcal{P}^* \times \mathcal{D}^*$ is maximally complementary if and only if

$$\mathcal{R}(\hat{X}) \subseteq \mathcal{R}(\bar{X}) \quad \forall \hat{X} \in \mathcal{P}^*, \quad \mathcal{R}(\hat{Z}) \subseteq \mathcal{R}(\bar{Z}) \quad \forall \hat{Z} \in \mathcal{D}^*,$$

where $\mathcal{R}$ denotes the range space. For proofs of these characterizations see [8] and [3] and the references therein. In [8] and [3] it was shown that any limit point of the central path as $\mu \downarrow 0$ is a maximally complementary solution. The limit point is in fact unique: a relatively simple proof of the convergence of the central path is given in [4].* These two results may be summarized as follows.

Theorem 2.1 *The central path $(X(\mu), Z(\mu), y(\mu))$ converges as $\mu \downarrow 0$, and the limit point (X^*, Z^*, y^*) is a maximally complementary optimal solution.*

This theorem forms the basis for our subsequent analysis. Denote

$$|B| := \text{rank } X^*, \quad \text{and} \quad |N| := \text{rank } Z^*.$$

Obviously, $|B| + |N| \leq n$. Since X^* and Z^* commute, they can be simultaneously diagonalized. Hence without loss of generality (applying an orthonormal transformation of problem data, if necessary) we can assume that

$$X^* = \begin{bmatrix} X_B^* & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad Z^* = \begin{bmatrix} 0 & 0 & 0 \\ 0 & Z_N^* & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where $X_B^* \in S_{++}^{|B|}$ and $Z_N^* \in S_{++}^{|N|}$ are diagonal. Moreover, each optimal solution pair $(\hat{X}, \hat{Z})$ is of the form

$$\hat{X} = \begin{bmatrix} \hat{X}_B & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \hat{Z} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \hat{Z}_N & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

*The convergence property was known earlier (see *e.g.* [16], p. 74), but it is difficult to pinpoint the exact origin of this result. In [10] the convergence of the central path for the linear complementarity problem (LCP) is proven with the help of some results from algebraic geometry. In [9], Kojima *et al.* mention that this proof for LCP can be extended to the monotone semidefinite complementarity problem (which is equivalent to SDO), without giving a formal proof. A more general convergence result was shown in [1], where convergence is proven for a class of convex semidefinite optimization problems that includes SDO.

where $\hat{X}_B \in S_+^{|B|}$ and $\hat{Z}_N \in S_+^{|N|}$, since $\mathcal{R}(\hat{X}) \subseteq \mathcal{R}(X^*)$ and $\mathcal{R}(\hat{Z}) \subseteq \mathcal{R}(Z^*)$.

In this notation, the condition of existence of an strictly complementary solution is equivalent to $|B| + |N| = n$, and non-existence with $|B| + |N| < n$.

In what follows we consider the partition of any $n \times n$ matrix M corresponding to the above optimal partition so that

$$M = \begin{bmatrix} M_B & M_{BN} & M_{BT} \\ M_{NB} & M_N & M_{NT} \\ M_{TB} & M_{TN} & M_T \end{bmatrix}. \tag{4}$$

We denote by $\mathcal{I} = \{B, BN, BT, NB, N, NT, TB, TN, T\}$ the index set corresponding to the optimal partition. If we refer to the all blocks of M except of M_B, we will write M_i $(i \in \mathcal{I} - B)$. Using this notation, the first two equations in (3) can be rewritten as

$$\begin{aligned} \textstyle\sum_{j \in \mathcal{I}} A_j^i \bullet X_j(\mu) &= b_i \ (\ i = 1, \ldots, n), \\ \textstyle\sum_{i=1}^m A_j^i y_i(\mu) + Z_j(\mu) &= C_j \ (j \in \mathcal{I}). \end{aligned} \tag{5}$$

Due to the optimality conditions (2), the optimal sets can be characterized by using the optimal partition:

$$\mathcal{P}^* = \left\{ X \ : \ A_B^i \bullet X_B = b_i \ (\ i = 1, \ldots, n), \ X_B \in S_+^{|B|}, \ X_k = 0 \ (k \in \mathcal{I} - B) \right\}, \tag{6}$$

$$\mathcal{D}^* = \left\{ (Z, y) \ : \ \sum_{i=1}^m A_N^i y_i + Z_N = C_N, \ Z_N \in S_+^{|N|}, \ \sum_{i=1}^m A_k^i y_i = C_k, \ Z_k = 0 \ (k \in \mathcal{I} - N) \right\}. \tag{7}$$

This description of the optimal sets allows to derive a number of important properties. First of all, since both $\mathcal{P}^*$ and $\mathcal{D}^*$ are non-empty, it is easy to see

$$b \in \mathcal{R}(A_B) \quad \text{and} \quad C_B \in \mathcal{R}(A_B^T). \tag{8}$$

Here $\mathcal{R}(A_B)$ is the range of the linear operator A_B (A_B associates $X_B \in S_+^{|B|}$ with $b \in \mathbb{R}^m$ where $b_i = A_B^i \bullet X_B$ ($i = 1, \ldots, n$)), and $\mathcal{R}(A_B^T)$ is the range of its adjoint linear operator A_B^T (A_B^T associates $y \in \mathbb{R}^m$ with $C_B \in S_+^{|B|}$ where $C_B = \sum_{i=1}^m A_B^i y_i$).

The relative interiors of $\mathcal{P}^*$ and $\mathcal{D}^*$ are respectively given by

$$\text{ri } (\mathcal{P}^*) = \{X \in \mathcal{P}^* \ : \ X_B \in S_{++}^{|B|}\} \quad \text{and} \quad \text{ri } (\mathcal{D}^*) = \{(Z, y) \in \mathcal{D}^* \ : \ Z_N \in S_{++}^{|B|}\}.$$

Due to the construction of the optimal partition it is easy to see that ri $(\mathcal{P}^*) \neq \emptyset$ and ri $(\mathcal{D}^*) \neq \emptyset$.

The analytic centers of these sets are defined as follows: $X^a \in \mathcal{P}^*$ is the analytic center of $\mathcal{P}^*$ if

$$X_B^a = \arg \max_{X_B \in S_{++}^{|B|}} \left\{ \ln \det X_B \ : \ A_B^i \bullet X_B = b_i, \ i = 1, \ldots, m \right\}, \tag{9}$$

and $(y^a, S^a) \in \mathcal{D}^*$ is the analytic center of $\mathcal{D}^*$ if

$$(y^a, S_N^a) = \arg \max_{y \in \mathbb{R}^m, S_N \in S_{++}^{|N|}} \left\{ \ln \det S_N \ : \sum_{i=1}^m A_N^i y_i + S_N = C_N, \ \sum_{i=1}^m A_k^i y_i = C_k, \ k \in \ \mathcal{I} - N \right\}. \tag{10}$$

The analytic centers of $\mathcal{P}^*$ and $\mathcal{D}^*$ can be characterized as is seen in the next lemma.

Lemma 2.1

(a) Let $X^a \in \mathcal{P}^*$. Then the following claims are equivalent:

(i) X^a is the analytic center of $\mathcal{P}^*$,

(ii) $\exists\, u^i (i = 1, \ldots, m) \quad : \quad (X_B^a)^{-1} - \sum_{i=1}^m u^i A_B^i = 0$,

(iii) $(X_B^a)^{-1} \in \mathcal{R}(A_B^T)$.

(b) Let $(Z^a, y^a) \in \mathcal{D}^*$. Then the following claims are equivalent

(i) (Z^a, y^a) is the analytic center of $\mathcal{D}^*$,

(ii) $\exists\, U_k (k \in \mathcal{I} - N) \quad : \quad A_N^i \bullet (Z_N^a)^{-1} + \sum_{k \in \mathcal{I}-N} A_k^i \bullet U_k = 0$

(iii) $A_N (Z_N^a)^{-1} \in \mathcal{R}(A_k,\ k \in \mathcal{I} - N)$. Here $A_N (Z_N^a)^{-1}$ is an m-vector with components $(A_N^i \bullet (Z_N^a)^{-1})$ $(i = 1, \ldots, m)$ and $\mathcal{R}(A_k,\ k \in \mathcal{I} - N)$ is the range of the linear operator which each eight-tuple of matrices $(X_k,\ k \in \mathcal{I} - N)$ associates with an m- vector the components of which are $\sum_{j \in \mathcal{I}-N} A_j^i \bullet X_j$.

Proof: The proof is straightforward by using the optimality conditions of problems (9) and (10). $\Box$

Finally, the above characterization of the optimal set allows us to identify the necessary and sufficient conditions for the uniqueness of the optimal solutions. In fact, the primal optimal solution is uniquely defined if and only if the matrices A_B^i $(i = 1, \ldots, m)$ span the space $S^{|B|}$. Similarly, the dual optimal solution is uniquely defined if and only if, the matrices

$$\begin{bmatrix} A_B^i & A_{BN}^i & A_{BT}^i \\ A_{NB}^i & 0 & A_{NT}^i \\ A_{TB}^i & A_{TN}^i & A_T^i \end{bmatrix}, \quad i = 1, \ldots, m$$

are linearly independent.† These conditions coincide with the concepts of weak dual and weak primal nondegeneracy, respectively, as introduced by Yildirim in [17].

Obviously, if weak primal (dual) nondegeneracy holds, then the dual (primal) central path converges to the unique solution which is the analytic center of $\mathcal{D}^*$ $(\mathcal{P}^*)$. In what follows we are interested in the case when weak primal or dual nondegeneracy does not hold.

3 Inversion along the central path

In this section we identify a property of the central path which holds if and only if a strictly complementary solution exists. In other words, we give an alternative characterization of strict complementarity. Since the possible absence of strict complementarity is the difference between the LO and SDO cases, the absence of this property is the reason why the standard convergence proofs for LO cannot be extended to the SDO. We start with an analogue with LO where the perturbed complementarity condition implies

$$\frac{z_B(\mu)}{\mu} = x_B^{-1}(\mu) \quad \text{and} \quad \frac{x_N(\mu)}{\mu} = z_N^{-1}(\mu) \text{ at any } \mu > 0, \tag{11}$$

and (11) holds even at the limit as $\mu \downarrow 0$. Here we have used the standard LO notation (see *e.g.* [14]). This property is the key to the proofs of convergence of the central path to the analytic center in LO. Unfortunately, an analogue of (11) does not hold for SDO, as we will now show. From the perturbed complementarity we only obtain

$$\tilde{Z}_B(\mu) = \left(X^{-1}(\mu)\right)_B \quad \text{and} \quad \tilde{X}_N(\mu) = \left(Z^{-1}(\mu)\right)_N \text{ at any } \mu > 0, \tag{12}$$

†This observation was made thanks to an e-mail discussion with E.A. Yildirim.

where we have introduced the following notation

$$\tilde{Z}_B(\mu) := \frac{Z_B(\mu)}{\mu} \quad \text{and} \quad \tilde{X}_N(\mu) := \frac{X_N(\mu)}{\mu}. \tag{13}$$

Nevertheless, in order to prove the convergence of the central path to the analytic center the limit analogue of (11), *i.e.*

$$\lim_{\mu \to 0} \tilde{Z}_B(\mu) = (X_B^*)^{-1} \quad \text{and} \quad \lim_{\mu \to 0} \tilde{X}_N(\mu) = (Z_N^*)^{-1}, \tag{14}$$

would suffice. To see this we observe

$$\sum_{i=1}^{m} A_B^i y_i^* = C_B, \tag{15}$$

and by (5) for any $\mu > 0$ one has

$$\sum_{i=1}^{m} A_B^i y_i(\mu) + Z_B(\mu) = C_B. \tag{16}$$

Subtracting (15) from (16) and then dividing the resulting equation by μ yields

$$\sum_{i=1}^{m} A_B^i \frac{y_i(\mu) - y_i^*}{\mu} + \frac{Z_B(\mu)}{\mu} = 0.$$

From this equation it follows that

$$\tilde{Z}_B(\mu) \in \mathcal{R}(A_B^T) \quad \forall \mu > 0.$$

If $\tilde{Z}_B(\mu)$ converges, then

$$\lim_{\mu \to 0} \tilde{Z}_B(\mu) \in \mathcal{R}(A_B^T), \tag{17}$$

since the space $\mathcal{R}(A_B^T)$ is closed. Moreover, if (14) holds, then

$$(X_B^*)^{-1} \in \mathcal{R}(A_B^T),$$

which implies that X^* is the analytic center by Lemma 2.1. Thus we have shown that the central path converges to the analytic center of the optimal set if (14) holds.

The next lemma shows that (14) holds if and only if a strictly complementarity solution exists. In other words, condition (14) is an alternative characterization of strict complementarity.

Lemma 3.1 *Both $\tilde{Z}_B(\mu)$ and $\tilde{X}_N(\mu)$ are bounded as $\mu \downarrow 0$ and there exists a sequence $\{\mu_t\} \to_{t\to\infty} 0$ along which both $\tilde{Z}_B(\mu_t)$ and $\tilde{X}_N(\mu_t)$ converge to positive definite matrices. Moreover,*
(a) if $|B| + |N| = n$, then

$$\lim_{t \to \infty} \tilde{Z}_B(\mu_t) = (X_B^*)^{-1} \quad \textit{and} \quad \lim_{t \to \infty} \tilde{X}_N(\mu_t) = (Z_N^*)^{-1}; \tag{18}$$

(b) if $|B| + |N| < n$, then at least one of the formulas in (18) does not hold, i.e.

$$\lim_{t \to \infty} \tilde{Z}_B(\mu_t) \neq (X_B^*)^{-1} \quad \textit{and/or} \quad \lim_{t \to \infty} \tilde{X}_N(\mu_t) \neq (Z_N^*)^{-1}. \tag{19}$$

Proof: Since $X^*, X(\mu) \in \mathcal{P}$ and $Z^*, Z(\mu) \in \mathcal{D}$ we obtain from the orthogonality property (1) that for any $\mu > 0$

$$X^* \bullet Z(\mu) + Z^* \bullet X(\mu) = X(\mu) \bullet Z(\mu) + X^* \bullet Z^*.$$

Substituting $X(\mu) \bullet Z(\mu) = \mu n$ and $X^* \bullet Z^* = 0$ and then dividing the resulting equation by $\mu > 0$ we get

$$X^* \bullet \frac{Z(\mu)}{\mu} + Z^* \bullet \frac{X(\mu)}{\mu} = n.$$

Now we use the fact that $X_k^* = 0$ for all $k \in \mathcal{I} - B$ and $Z_k^* = 0$ for all $k \in \mathcal{I} - N$. Hence the last equality gives

$$X_B^* \bullet \tilde{Z}_B(\mu) + Z_N^* \bullet \tilde{X}_N(\mu) = n. \tag{20}$$

Since X_B^* and Z_N^* are positive definite and also $Z_B(\mu)$ and $X_N(\mu)$ are positive definite for any $\mu > 0$, the equation (20), which holds for each $\mu > 0$, implies that $\tilde{Z}_B(\mu)$ and $\tilde{X}_N(\mu)$ remain bounded as $\mu \downarrow 0$. That means that there exists a sequence $\{\mu_t\}$ such that $\lim_{t\to\infty} \mu_t = 0$ and both $\tilde{Z}_B(\mu_t)$ and $\tilde{X}_N(\mu_t)$ converge as $t \to \infty$.

Now, we introduce a spectral decomposition of $X(\mu)$ and $Z(\mu)$ along a subsequence of $\{\mu_t\}$. Since $X(\mu)$ and $Z(\mu)$ commute, there exists an $n \times n$ matrix $Q(\mu)$ such that $Q(\mu)Q(\mu)^T = I$, and

$$X(\mu) = Q(\mu)\Lambda(\mu)Q^T(\mu), \quad Z(\mu) = Q(\mu)\Omega(\mu)Q^T(\mu), \tag{21}$$

where $\Lambda(\mu)$ and $\Omega(\mu)$ are diagonal matrices with the eigenvalues of $X(\mu)$ and $Z(\mu)$ on the diagonal. Since $X(\mu) \to X^*$ and $Z(\mu) \to Z^*$ where both X^* and Z^* are diagonal, the spectral decomposition of $X(\mu)$ and $Z(\mu)$ can be chosen in such a way that

$$Q(\mu_t) \to I_n, \quad \Lambda(\mu_t) \to X^* \quad \text{and} \quad \Omega(\mu_t) \to Z^* \tag{22}$$

for some subsequence of $\{\mu_t\}$ where for the simplicity of notation we have denoted the subsequence by the same symbol.

We now apply the optimal partition notation (4) to the matrix Q in (21) and obtain the following formulas for $\tilde{Z}_B(\mu)$ and $\tilde{X}_N(\mu)$:

$$\tilde{Z}_B(\mu) = Q_B(\mu)\frac{\Omega_B(\mu)}{\mu}Q_B^T(\mu) + Q_{BN}(\mu)\frac{\Omega_N(\mu)}{\mu}Q_{BN}^T(\mu) + Q_{BT}(\mu)\frac{\Omega_T(\mu)}{\mu}Q_{BT}^T(\mu), \tag{23}$$

$$\tilde{X}_N(\mu) = Q_{NB}(\mu)\frac{\Lambda_B(\mu)}{\mu}Q_{NB}^T(\mu) + Q_N(\mu)\frac{\Lambda_N(\mu)}{\mu_k}Q_N^T(\mu) + Q_{NT}(\mu)\frac{\Lambda_T(\mu)}{\mu_k}Q_{NT}^T(\mu). \tag{24}$$

Moreover, setting (21) to the last of equations (3), and multiplying the resulting equation by $Q^T(\mu)$ from the left and by $Q(\mu)$ from the right we obtain $\Lambda(\mu)\Omega(\mu) = \mu I$ which yields

$$\frac{\Omega_B(\mu)}{\mu} = \Lambda_B^{-1}(\mu) \quad \text{and} \quad \frac{\Lambda_N(\mu)}{\mu} = \Omega_N^{-1}(\mu). \tag{25}$$

Consider (23), (24) and (25) along the subsequence $\{\mu_t\}$. Since by (22), $\Lambda_B(\mu_t) \to X_B^* \succ 0$ and $\Omega_N(\mu_t) \to Z_N^* \succ 0$ it follows from (25) that

$$\frac{\Omega_B(\mu_t)}{\mu_t} = \Lambda_B^{-1}(\mu_t) \to (X_B^*)^{-1} \succ 0, \quad \text{and} \quad \frac{\Lambda_N(\mu_t)}{\mu_t} = \Omega_N^{-1}(\mu_t) \to (Z_N^*)^{-1} \succ 0.$$

Since $Q(\mu_t) \to I_n$, we have that $Q_B(\mu_t) \to I_{|B|}$ and $Q_N(\mu_t) \to I_{|N|}$. Thus we have that

$$\lim_{t\to\infty} Q_B(\mu_t)\frac{\Omega_B(\mu_t)}{\mu}Q_B(\mu_t) = (X_B^*)^{-1} \succ 0 \quad \text{and} \quad \lim_{t\to\infty} Q_N(\mu_t)\frac{\Lambda_N(\mu_t)}{\mu_t}Q_N(\mu_t) = (Z_N^*)^{-1} \succ 0. \tag{26}$$

Since all the right hand side terms in (23) and (24) are positive semidefinite and $\tilde{Z}_B(\mu_t)$ and $\tilde{X}_N(\mu_t)$ are bounded, each right hand side term in (23) and (24) is bounded as $\{\mu_t\} \downarrow 0$. Thus, taking a subsequence of $\{\mu_t\}$ if necessary, we have a convergence of all terms in both (23) and (24). Moreover, (26) implies that $\tilde{Z}_B(\mu_t)$ and $\tilde{X}_N(\mu_t)$ converge to positive definite matrices as $t \to \infty$.

Substituting (23) and (24) into (20) yields

$$X_B^* \bullet \left(Q_B(\mu)\frac{\Omega_B(\mu)}{\mu}Q_B^T(\mu) + Q_{BN}(\mu)\frac{\Omega_N(\mu)}{\mu}Q_{BN}^T(\mu) + Q_{BT}(\mu)\frac{\Omega_T(\mu)}{\mu}Q_{BT}^T(\mu)\right)$$

$$+Z_N^* \bullet \left(Q_{NB}(\mu) \frac{\Lambda_B(\mu)}{\mu} Q_{NB}^T(\mu) + Q_N(\mu) \frac{\Lambda_N(\mu)}{\mu} Q_N^T(\mu) + Q_{NT}(\mu) \frac{\Lambda_T(\mu)}{\mu} Q_{NT}^T(\mu) \right) = n. \quad (27)$$

Taking the limit in (27) along the subsequence $\{\mu_t\}$ we obtain

$$X_B^* \bullet (X_B^*)^{-1} + Z_N^* \bullet (Z_N^*)^{-1} + X_B^* \bullet \lim_{t\to\infty} \left(Q_{BN}(\mu_t) \frac{\Omega_N(\mu_t)}{\mu_t} Q_{BN}^T(\mu_t) + Q_{BT}(\mu_t) \frac{\Omega_T(\mu_t)}{\mu_t} Q_{BT}^T(\mu_t) \right)$$

$$+Z_N^* \bullet \lim_{t\to\infty} \left(Q_{NB}(\mu_t) \frac{\Lambda_B(\mu_t)}{\mu_t} Q_{NB}^T(\mu_t) + Q_{NT}(\mu_t) \frac{\Lambda_T(\mu_t)}{\mu_t} Q_{NT}^T(\mu_t) \right) = n$$

It is easy to see that

$$X_B^* \bullet (X_B^*)^{-1} = |B|, \qquad Z_N^* \bullet (Z_N^*)^{-1} = |N|,$$

and

$$X_B^* \bullet \lim_{t\to\infty} \left(Q_{BN}(\mu_t) \frac{\Omega_N(\mu_t)}{\mu_t} Q_{BN}^T(\mu_t) + Q_{BT}(\mu_t) \frac{\Omega_T(\mu_t)}{\mu_t} Q_{BT}^T(\mu_t) \right) \geq 0 \quad (28)$$

$$Z_N^* \bullet \lim_{t\to\infty} \left(Q_{NB}(\mu_t) \frac{\Lambda_B(\mu_t)}{\mu_t} Q_{NB}^T(\mu_t) + Q_{NT}(\mu_t) \frac{\Lambda_T(\mu_t)}{\mu_t} Q_{NT}^T(\mu_t) \right) \geq 0. \quad (29)$$

Now, if $|B| + |N| = n$, then the inequalities in (28) and (29) must hold with equality and the statement (a) of the lemma is proved. If $|B| + |N| < n$, then at least one of the inequalities (28) and (29) holds as a strict inequality and thus the statement (b) of the lemma holds as well. $\square$

We now show that — even though (14) only holds in the case of strict complementarity — the quantities $\tilde{Z}_B(\mu)$ and $\tilde{X}_N(\mu)$ always converge to positive definite matrices as $\mu \downarrow 0$.

Theorem 3.1 *Both* $\tilde{Z}_B(\mu)$ *and* $\tilde{X}_N(\mu)$ *converge as* $\mu \downarrow 0$, *say* $\tilde{Z}_B^* := \lim_{\mu\downarrow 0} \tilde{Z}_B(\mu) \in S_{++}^{|B|}$ *and* $\tilde{X}_N^* := \lim_{\mu\downarrow 0} \tilde{X}_N(\mu) \in S_{++}^{|N|}$. *Moreover,*

(a) if $|B| + |N| = n$, *then* $\tilde{Z}_B^* = (X_B^*)^{-1}$ *and* $\tilde{X}_N^* = (Z_N^*)^{-1}$;

(b) if $|B| + |N| < n$, *then* $\tilde{Z}_B^* \neq (X_B^*)^{-1}$ *and/or* $\tilde{X}_N^* \neq (Z_N^*)^{-1}$.

Proof: In Lemma 3.1 we have proved that $\tilde{Z}_B(\mu)$ and $\tilde{X}_N(\mu)$ are bounded as $\mu \downarrow 0$. The proof of the convergence now follows the same pattern as the proof of convergence of the central path in [4]. The proof exploited the fact that the centering system of conditions defines a semialgebraic set. However, if we substitute $Z_B = \mu \tilde{Z}_B$ and $X_N = \mu \tilde{X}_N$ into the centering conditions, the new system of conditions defines a semialgebraic set as well. Hence, the same procedure as in the proof of Theorem A.3 in [4] yields the convergence $\tilde{Z}_B(\mu)$ and $\tilde{X}_N(\mu)$. The other claims of the theorem follow from Lemma 3.1. $\square$

As a consequence of this theorem and (17) we obtain that if there exists a strictly complementary solution, then the central path converges to the analytic center of the optimal set. The convergence to the analytic center in the case of strict complementarity has already been proved in [15] and [7] by different approaches that implicitly use (14) as well. The contribution of Theorem 3.1 was to give an alternative characterization of strict complementarity, that allows us to show why the usual convergence proofs for linear optimization cannot be extended to SDO.

4 The central path as a set of analytic centers

In this section we analyze another approach which uses the fact that for any $\mu > 0$ the central path $X(\mu), S(\mu), y(\mu)$ is the analytic center of the level set of the duality gap (see *e.g.* [6], Lemma 3.3). This means that $(X(\mu), Z(\mu), y(\mu))$ is a solution to the problem

$$\max_{\substack{X \in S^n_{++} \\ Z \in S^n_{++} \\ y \in \mathbb{R}^m}} \left\{ \ln\det X + \ln\det Z : \; A^i \bullet X = b^i \; (i = 1, \ldots, m), \; \sum_{i=1}^m A^i y_i + Z = C, \; C \bullet X - b^T y = \mu n \right\}. \tag{30}$$

We first observe that this problem can be separated into two problems that yield the primal and dual μ-centers respectively:

$$\max_{X \in S^n_{++}} \left\{ \ln\det X \; : \; A^i \bullet X = b^i \; (i = 1, \ldots, m), \; C \bullet X - b^T y(\mu) = \mu n \; \right\}, \tag{31}$$

and

$$\max_{\substack{Z \in S^n_{++} \\ y \in \mathbb{R}^m}} \left\{ \ln\det Z : \; \sum_{i=1}^m A^i y_i + Z = C, \; C \bullet X(\mu) - b^T y = \mu n \; \right\}. \tag{32}$$

Obviously, $X(\mu)$ and $Z(\mu)$ are the unique optimal solutions to (31) and (32) respectively.

We aim to rewrite these problems in terms of non-vanishing block components of the central path. So, we fix the particular block variables of X and Z at their optimal values, except for X_B and Z_N which are free under the conditions that

$$X_{B,\mu} := \begin{pmatrix} X_B & X_{BN}(\mu) & X_{BT}(\mu) \\ X_{NB}(\mu) & X_N(\mu) & X_{NT}(\mu) \\ X_{TB}(\mu) & X_{TN}(\mu) & X_T(\mu) \end{pmatrix}, \quad \text{and} \quad Z_{N,\mu} := \begin{pmatrix} Z_B(\mu) & Z_{BN}(\mu) & Z_{BT}(\mu) \\ Z_{NB}(\mu) & Z_N & Z_{NT}(\mu) \\ Z_{TB}(\mu) & Z_{TN}(\mu) & Z_T(\mu) \end{pmatrix}$$

are positive definite. The constraints in (31) then become

$$A^i_B \bullet X_B + \sum_{j \in \mathcal{I} - B} A^i_j \bullet X_j(\mu) = b^i \; (i = 1, \ldots, m), \tag{33}$$

$$C_B \bullet X_B + \sum_{j \in \mathcal{I} - B} C_j \bullet X_j(\mu) - b^T y(\mu) = \mu n, \tag{34}$$

and the constraints in (32) become

$$\sum_{i=1}^m A^i_j y_i + Z_j(\mu) = C_j \; (i \in \mathcal{I} - N), \tag{35}$$

$$\sum_{i=1}^m A^i_N y_i + Z_N = C_N, \tag{36}$$

$$C \bullet X(\mu) - b^T y = \mu n. \tag{37}$$

Thus we obtain the following two problems

$$\max_{X_{B,\mu} \in S^n_{++}} \left\{ \ln\det X_{B,\mu} \; : \; (33), \; (34) \text{ hold} \right\}, \tag{38}$$

$$\max_{Z_{B,\mu} \in S^n_{++}, y \in R^m} \left\{ \ln\det Z_{N,\mu} \; : \; (35), \; (36), \; (37) \text{ hold} \right\}. \tag{39}$$

Obviously $X(\mu)$ and $Z(\mu)$ are the optimal solutions to (38) and (39) respectively.

We first rewrite (38) in an equivalent form described in terms of X_B. It is easy to see that $X_{B,\mu}$ can be rewritten as

$$X_{B,\mu} := \begin{pmatrix} X_B & X_U(\mu) \\ X_U^T(\mu) & X_V(\mu) \end{pmatrix}$$

where

$$X_V(\mu) := \begin{pmatrix} X_N(\mu) & X_{NT}(\mu) \\ X_{TN}(\mu) & X_T(\mu) \end{pmatrix} \quad \text{and} \quad X_U(\mu) := \begin{pmatrix} X_{BN}(\mu) & X_{BT}(\mu) \end{pmatrix}.$$

Since $X_V(\mu) \in S_{++}^{n-|B|}$ we have that

$$X_{B,\mu} \in S_{++}^n \quad \Leftrightarrow \quad X_B - F_B(\mu) \in S_{++}^{|B|}$$

where

$$F_B(\mu) := X_U(\mu) X_V^{-1}(\mu) X_U^T(\mu) \in S_{++}^{|B|}. \tag{40}$$

Note that $X_B - F_B(\mu)$ is the Schur complement of $X_V(\mu)$ in $X_{B,\mu}$. By using the formula for the determinant of a block matrix we obtain

$$\det X_{B,\mu} = \det X_V(\mu) \det \left(X_B - F_B(\mu) \right).$$

Hence (38) can be rewritten as

$$\max_{X_B - F_B(\mu) \in S_{++}^{|B|}} \left\{ \ln \det (X_B - F_B(\mu)) \ : \ (33),\ (34) \text{ hold} \right\}, \tag{41}$$

and consequently $X_B(\mu)$ can be interpreted as the analytic center of the following set

$$P_{F(\mu)} := \left\{ X_B \ : \ (33), (34) \text{ hold, and } X_B - F_B(\mu) \in S_+^{|B|} \right\}. \tag{42}$$

Analogously, we obtain that (39) can be rewritten as

$$\max_{Z_N - F_N(\mu) \in S_{++}^{|N|}, y \in R^m} \left\{ \ln \det (Z_N - F_N(\mu)) \ : (35), (36), (37) \text{ hold} \right\}. \tag{43}$$

Here

$$F_N(\mu) := Z_U(\mu) Z_V^{-1}(\mu) Z_U^T(\mu) \in S_{++}^{|B|}, \tag{44}$$

where

$$Z_V(\mu) := \begin{pmatrix} Z_B(\mu) & Z_{BT}(\mu) \\ Z_{TB}(\mu) & Z_T(\mu) \end{pmatrix} \quad \text{and} \quad Z_U(\mu) := \begin{pmatrix} Z_{NB}(\mu) & Z_{NT}(\mu) \end{pmatrix}.$$

Obviously, $Z_N(\mu)$ is the analytic center of the set

$$D_{F(\mu)} := \left\{ (Z_N, y) \ : \ (35), (36), (37) \text{ hold, and } Z_N - F_N(\mu) \in S_+^{|N|} \right\}. \tag{45}$$

Lemma 4.1 *(a) For any $\mu > 0$, $X_B(\mu)$ is the analytic center of the set $P_{F(\mu)}$ defined by (42), where $F_B(\mu)$ is given by (40). Moreover $(X_B(\mu) - F(\mu))^{-1} \in \mathcal{R}(\mathcal{A}_B^T)$.*

(b) For any $\mu > 0$, $(Z_N(\mu), y(\mu))$ is the analytic center of the set $D_{F(\mu)}$ defined by (45), where $F_N(\mu)$ is given by (44). Moreover $A_N(Z_N(\mu) - F_N(\mu))^{-1} \in \mathcal{R}(\mathcal{A}_k(k \in \mathcal{I} - N))$.

Proof: Since $X(\mu)$ is an optimal solution to (31) and (38), from the derivation of (41) above it follows that $X_B(\mu)$ is an optimal solution of (41). By the optimality conditions of (41), there exist $u_\mu^i (i = 1, \ldots, m)$ and v_μ such that

$$(X_B(\mu) - F_B(\mu))^{-1} - \sum_{i=1}^{m} u_\mu^i A_B^i - v_\mu C_B = 0.$$

From this and from the range property (8) we have $(X_B(\mu) - F_B(\mu))^{-1} \in \mathcal{R}(\mathcal{A}_B^T)$ for any $\mu > 0$. The proof of (b) follows the same pattern. □

Lemma 4.2 *$F_B(\mu)$ and $F_N(\mu)$ given by (40) and (44) respectively converge as $\mu \downarrow 0$ and*

$$F_B^* := \lim_{\mu \downarrow 0} F_B(\mu) = X_B^* - (\tilde{Z}_B^*)^{-1} \quad \textit{and} \quad F_N^* := \lim_{\mu \downarrow 0} F_N(\mu) = Z_N^* - (\tilde{X}_N^*)^{-1}. \tag{46}$$

Moreover, $X_B^* - F_B^* \succ 0$, $Z_N^* - F_N^* \succ 0$, and

$$(X_B^* - F_B^*)^{-1} \in \mathcal{R}(\mathcal{A}_B^T), \qquad A_N(Z_N^* - F_N^*)^{-1} \in \mathcal{R}(\mathcal{A}_k(k \in \mathcal{I} - N)). \tag{47}$$

Proof: We first derive another description for $F_B(\mu)$ and $F_N(\mu)$. Applying the formula for the inverse of a block matrix (see *e.g.* [5]) to $X(\mu)$, and using the fact that $X_B(\mu) - F_B(\mu)$ is the Schur complement of $X_V(\mu)$ in $X(\mu)$, we obtain

$$\left(X^{-1}(\mu)\right)_B = (X_B(\mu) - F_B(\mu))^{-1}.$$

Combining this with (12), and taking the inversion yields $F_B(\mu) = X_B(\mu) - \tilde{Z}_B^{-1}(\mu)$. Analogously we obtain $F_N(\mu) = Z_N(\mu) - \tilde{X}_N^{-1}(\mu)$. The lemma now follows from Theorem 3.1 where the convergence of $\tilde{Z}_B(\mu)$ and $\tilde{X}_N(\mu)$ to positive definite matrices is ensured. The last claim follows from Lemma 4.1 by the closedness of the corresponding spaces. □

From (46) and Theorem 3.1 we obtain the following.

Corollary 4.1 $F_B^* \in S_+^{|B|}$ *and* $F_N^* \in S_+^{|N|}$. *If there exists a strictly complementary solution, then* $F_B^* = 0$ *and* $F_N^* = 0$. *In the absence of strict complementarity one has* $F_B^* \neq 0$ *and/or* $F_N^* \neq 0$.

Consider the following sets

$$\mathcal{P}_{F^*} = \left\{X \ : \ A_B^i \bullet X_B = b^i \ (i = 1, \ldots, m), \ X_B - F_B^* \in S_+^{|B|}, \ X_k = 0 \ (k \in \mathcal{I} - B)\right\},$$

$$\mathcal{D}_{F^*} = \left\{(Z, y) \ : \ \sum_{i=1}^{m} A_N^i y_i + Z_N = C_N, \quad Z_N - F_N^* \in S_+^{|N|}, \ \sum_{i=1}^{m} A_k^i y_i = C_k, \ Z_k = 0 \ (k \in \mathcal{I} - N)\right\}.$$

It is easy to see that $\mathcal{P}_{F^*} \subseteq \mathcal{P}^*$ and $\mathcal{D}_{F^*} \subseteq \mathcal{D}^*$. Moreover, both $\mathcal{P}_{F^*}$ and $\mathcal{D}_{F^*}$ are nonempty since $X^* \in \mathcal{P}_{F^*}$ and $(Z^*, y^*) \in \mathcal{D}_{F^*}$. It is easy to see that the analytic centers of these set are well defined as the unique solutions to the following problems

$$\max_{X_B - F_B^* \in S_{++}^{|B|}} \left\{\ln\det(X_B - F_B^*) \ : \ A_B^i \bullet X_B = b_i, \ i = 1, \ldots, m\right\}, \tag{48}$$

$$\max_{y \in \mathbb{R}^m, Z_N - F_N^* \in S_{++}^{|N|}} \left\{\ln\det(Z_N - F_N^*) \ : \sum_{i=1}^{m} A_N^i y_i + Z_N = C_N, \ \sum_{i=1}^{m} A_k^i y_i = C_k, \ k \in \ \mathcal{I} - N\right\}. \tag{49}$$

In fact, the uniqueness follows from the strict concavity of $\ln\det$. The existence follows from the Weierstrass theorem by the following construction: Since $X^* \in \mathcal{P}_{F^*}$ and $(Z^*, y^*) \in \mathcal{D}_{F^*}$, the feasible sets in both problems can be restricted by the following additional conditions:

$$\ln\det(X_B - F_B^*) \geq \ln\det(X_B^* - F_B^*) \quad \text{and} \quad \ln\det(Z_N - F_N^*) \geq \ln\det(Z_N^* - F_N^*)$$

respectively, without changing the optimal solutions. Now, the new feasible sets are not only bounded (as a consequence of the boundedness of $\mathcal{P}^*$ and $\mathcal{D}^*$), but also closed.

Theorem 4.1 *The limit point (X^*, Z^*, y^*) of the central path and the corresponding matrices F_B^* and F_N^* have the following properties:*

(i) X_B^* is the analytic center of $\mathcal{P}_{F^*}$ and

(ii) (Z^*, y^*) is the analytic enter of $\mathcal{D}_{F^*}$.

If there exists a strictly complementary solution, then $\mathcal{P}_{F^*} = \mathcal{P}^*$ and $\mathcal{D}_{F^*} = \mathcal{D}^*$.

Proof: The last part of the theorem follows from Corollary 4.1. We now prove (i). By using the optimality conditions of problem (48) we obtain that X_B^* should satisfy

$$(X_B^* - F_B^*)^{-1} - \sum_{i=1}^{m} u^i A_B^i = 0$$

for some $u^i (i = 1, \dots, m)$. This indeed holds, since $(X_B^* - F_B^*)^{-1} \in \mathcal{R}(A_B^T)$, by Lemma 4.2. The proof of (ii) is similar. $\square$

5 Examples

We provide two simple examples for which the central paths can be expressed in closed form. Both examples do not have strictly complementary solutions, and have multiple dual optimal solutions. For both examples we construct the corresponding sets $\mathcal{D}_{F^*}$. For the first example $\mathcal{D}_{F^*} \neq \mathcal{D}^*$, and the analytic center of $\mathcal{D}_{F^*}$ does not coincide with the analytic center of $\mathcal{D}^*$. Hence the dual central path does not converge to the analytic center of the dual optimal set. For the second example $\mathcal{D}_{F^*} = \mathcal{D}^*$, and thus the central path converges to the analytic center of the optimal set.

Example 1: Let $n = 4$, $m = 3$, $b = [0\; 0\; -1]^T$ and

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \; A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \; A_2 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \; A_3 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The dual problem maximizes $-y_3$ such that

$$Z = \begin{bmatrix} y_3 & y_2 & 0 & 0 \\ y_2 & y_1 & 0 & 0 \\ 0 & 0 & 1 - y_1 & 0 \\ 0 & 0 & 0 & y_2 \end{bmatrix} \succeq 0.$$

The optimal solutions are

$$Z^* = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & y_1 & 0 & 0 \\ 0 & 0 & 1 - y_1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \succeq 0.$$

The analytic center is where $y_1 = \frac{1}{2}$. The primal problem minimizes X_{33} such that

$$X = \begin{bmatrix} 1 & -\frac{1}{2}X_{44} & X_{13} & X_{14} \\ -\frac{1}{2}X_{44} & X_{33} & X_{23} & X_{24} \\ X_{13} & X_{23} & X_{33} & X_{34} \\ X_{14} & X_{24} & X_{34} & X_{44} \end{bmatrix} \succeq 0.$$

The unique optimal solution is

$$X^* = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

It can be easily computed that the central path is where

$$X_{13}(\mu) = X_{14}(\mu) = X_{23}(\mu) = X_{24}(\mu) = X_{34}(\mu) = 0,$$

$$y_1(\mu) = \frac{3}{5}, \quad y_2(\mu) = \sqrt{\frac{3}{10}\mu}, \quad y_3 = \frac{3}{2}\mu, \quad \text{and} \quad X_{33}(\mu) = \frac{5}{2}\mu, \quad X_{44}(\mu) = \sqrt{\frac{10}{3}\mu}.$$

Hence, the dual central path does not converge to the analytic center of the dual optimal set, since $y_1^* = \frac{3}{5}$. However, for this example we have

$$Z_V = \begin{bmatrix} y_3 & 0 \\ 0 & y_2 \end{bmatrix}, \quad Z_U = \begin{bmatrix} y_2 & 0 \\ 0 & 0 \end{bmatrix}, \quad F_N = \begin{bmatrix} \frac{y_2^2}{y_3} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & 0 \\ 0 & 0 \end{bmatrix} = F_N^*,$$

and thus $y_1^* = \frac{3}{5}$ is the analytic center of $\mathcal{D}_{F^*}$ where

$$\mathcal{D}_{F^*} = \left\{ (y, Z) \in \mathcal{D}^* \ : \ \begin{bmatrix} y_1 - \frac{1}{5} & 0 \\ 0 & 1 - y_1 \end{bmatrix} \succeq 0 \right\}.$$

□

Example 2: Let $n = 5$, $m = 3$, $b = [0\ 0\ -1]^T$ and

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \quad A_3 = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The dual problem maximizes $-y_3$ such that

$$Z = \begin{bmatrix} y_3 & y_2 & 0 & 0 & 0 \\ y_2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 - y_1 & 0 & 0 \\ 0 & 0 & 0 & y_1 & 0 \\ 0 & 0 & 0 & 0 & y_2 \end{bmatrix} \succeq 0.$$

The optimal solutions are

$$Z^* = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 - y_1 & 0 & 0 \\ 0 & 0 & 0 & y_1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \succeq 0.$$

The analytic center is where $y_1 = 1/2$. The primal problem minimizes $X_{22} + X_{33}$ such that

$$X = \begin{bmatrix} 1 & -\frac{1}{2}X_{55} & X_{13} & X_{14} & X_{15} \\ -\frac{1}{2}X_{55} & X_{22} & X_{23} & X_{24} & X_{25} \\ X_{13} & X_{23} & X_{33} & X_{34} & X_{35} \\ X_{14} & X_{24} & X_{34} & X_{33} & X_{45} \\ X_{15} & X_{25} & X_{35} & X_{45} & X_{55} \end{bmatrix} \succeq 0.$$

The unique optimal solution is

$$X^* = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

It can be easily computed that the primal–dual central path is given by

$$X_{13}(\mu) = X_{14}(\mu) = X_{15}(\mu) = X_{23}(\mu) = X_{24}(\mu) = X_{25}(\mu) = X_{34}(\mu) = X_{35}(\mu) = X_{45}(\mu) = 0,$$

$$y_1(\mu) = \frac{1}{2}, \quad y_2(\mu) = \sqrt{\frac{\mu}{2}}, \quad y_3 = \frac{3}{2}\mu, \quad \text{and} \quad X_{22}(\mu) = \frac{3}{2}\mu, \quad X_{33}(\mu) = 2\mu, \quad X_{55}(\mu) = 2\sqrt{\frac{\mu}{2}}.$$

Hence, the dual central path does converge to the analytic center of the dual optimal set. For this example we have

$$Z_V = \begin{bmatrix} y_3 & 0 \\ 0 & y_2 \end{bmatrix}, \quad Z_U = \begin{bmatrix} y_2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad F_N = \begin{bmatrix} \frac{y_2^2}{y_3} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = F_N^*.$$

We observe that

$$Z_N = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1-y_1 & 0 \\ 0 & 0 & y_1 \end{bmatrix} \quad \text{and} \quad Z_N - F_N^* = \begin{bmatrix} \frac{2}{3} & 0 & 0 \\ 0 & y_1 & 0 \\ 0 & 0 & 1-y_1 \end{bmatrix}.$$

That means that although $F_N^* \neq 0$, the dual central path converges to the analytic center. This is caused by the fact that the only nonzero element of F_N^* is at the position where Z_N has a fixed element (*i.e.* equal to 1) and hence the optimal set (being constrained by $Z_N \succ 0$) is the same as the set $\mathcal{D}_{\mathcal{F}^*}$ (constrained by $Z_N - F_N^* \succ 0$). This observation will be generalized in the next section. □

6 SDO problems with a special block diagonal structure

In this section we describe a class of SDO problems where the central path converges to the analytic center of the optimal set. We assume the data matrices to be in in the block diagonal form:

$$A^i = \begin{bmatrix} A_I^i & 0 \\ 0 & A_{II}^i \end{bmatrix}, \ (i = 1, \ldots, m), \quad C = \begin{bmatrix} C_I & 0 \\ 0 & C_{II} \end{bmatrix}, \quad A_I^i, C_I \in S^s, \quad \text{for some } s \leq n. \tag{50}$$

Denote $Z_I(y) = C_I - \sum_{i=1}^m A_I^i y_i$ and $Z_{II}(y) = C_{II} - \sum_{i=1}^m A_{II}^i y_i$.

It is easy to see that each dual feasible solution (Z, y) is of the block diagonal form with positive semidefinite matrices $Z_I(y)$ and $Z_{II}(y)$ on the diagonal. In what follows we will make the following assumption.

Assumption 6.1 *There exists $Z_I^* \succeq 0$ such that each dual optimal solution (Z, y) satisfies $Z_I(y) = Z_I^*$. Moreover, there exists an optimal solution (Z, y) for which $Z_{II}(y) \succ 0$.*

This assumption ensures that the corresponding optimal set can be characterized as

$$\mathcal{D}^* = \{(Z, y) \ : \ Z_I(y) = Z_I^*, \ Z_{II}(y) \succeq 0\}.$$

Theorem 6.1 *Let SDO be of the form (50) and satisfy Assumptions 1.1, 1.2 and 6.1. Then $\mathcal{D}^* = \mathcal{D}_{\mathcal{F}^*}$, and hence the dual central path $(y(\mu), Z(\mu))$ converges to the analytic center of $\mathcal{D}^*$.*

Proof: Without loss of generality we may assume that Z_I is partitioned as

$$Z_I = \begin{bmatrix} Z_{I11} & Z_{I12} \\ Z_{I12}^T & Z_{I22} \end{bmatrix},$$

where at any optimal solution (Z, y), it is $Z_{I11}(y) = 0$, $Z_{I12}(y) = 0$ and $Z_{I22}(y) = Z_{I22}^* \succ 0$. This and Assumption 6.1 imply that, for this example, the components Z_V, Z_U and Z_N of the optimal partition of Z are

$$Z_N = \begin{bmatrix} Z_{I22} & 0 \\ 0 & Z_{II} \end{bmatrix}, \quad Z_V = Z_{I11}, \quad \text{and} \quad Z_U = \begin{bmatrix} Z_{I12} \\ 0 \end{bmatrix}. \tag{51}$$

This implies that the optimal set is

$$\begin{aligned} \mathcal{D}^* &= \{(Z, y) \ : \ Z_V(y) = 0, \ Z_U(y) = 0, \quad Z_N(y) \succeq 0\} \\ &= \{(Z, y) \ : \ Z_{I11}(y) = 0, \ Z_{I12}(y) = 0, \ Z_{I22}(y) = Z_{I22}^*, \ Z_{II}(y) \succeq 0\}. \end{aligned}$$

Setting (51) into (44) we obtain the formula for $F_N(\mu)$, where by Lemma 4.2, $F_N(\mu)$ converges. That means we have

$$F_N(\mu) = \begin{bmatrix} Z_{I12}(\mu) Z_{I11}^{-1}(\mu) Z_{I12}^T(\mu) & 0 \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} F_{I22}^* & 0 \\ 0 & 0 \end{bmatrix} = F_N^*.$$

Since $Z_{I22}(y) = Z_{I22}^*$ at any optimal solution y, and $Z_{I22}^* - F_{I22}^* \succeq 0$ by Lemma 4.2, we obtain that the condition $Z_N(y) - F_N^* \succeq 0$ is equivalent with $Z_{II}(y) \succeq 0$. That means

$$\begin{aligned} \mathcal{D}_{F^*} &= \{(Z, y) \ : \ Z_V(y) = 0, \ Z_U(y) = 0, \quad Z_N(y) - F_N^* \succeq 0\} \\ &= \{(Z, y) \ : \ Z_{I11}(y) = 0, \ Z_{I12}(y) = 0, \ Z_{I22}(y) = Z_{I22}^*, \ Z_{II}(y) \succeq 0\} \quad = \mathcal{D}^* \end{aligned}$$

and the theorem is proved.

$\square$

7 Convex quadratically constraint quadratic optimization

In this section we show that a convex quadratically constrained quadratic program can be viewed as an SDO problem of the form (50) with Assumption 6.1, and hence the central path converges to the analytic center of the optimal set.

Consider the general convex program

$$(C) \qquad \min_{y\in\mathcal{C}}\{f_0(y) \; : \; f_i(y) \le 0, \quad i = 1, \dots, L\}$$

where $\mathcal{C}$ is an open convex subset of $\mathbb{R}^n$, and $f_i : R^n \to R$ $(i = 0, 1, \dots, L)$ are convex and smooth on $\mathcal{C}$. Let $\mathcal{C}^*$ be the set of optimal solutions which we assume to be nonempty and bounded.

In [11], the authors show the following (see also [2, 12, 13]).

Theorem 7.1 (McCormick and Witzgall [11]) *If the convex program (C) meets the Slater condition and the functions f_i are weakly analytic*[‡]*, then the central path of (C) converges to the (unique) analytic center of the optimal set.*

It therefore follows from Theorem 7.1 that the central path converges to the analytic center of the optimal set when the f_i's are convex quadratic functions. Our goal here is therefore only to show that the assumptions in Theorem 6.1 are indeed met for some interesting sub-classes of SDO problems.

We first prove a general result about the set of optimal solutions $\mathcal{C}^*$ of (C). By $I(y)$ we denote the index set of inequalities, which are active at an optimal point y.

Lemma 7.1 *Let $y^* \in \mathrm{ri}\ (\mathcal{C}^*)$. Then for each $i \in I(y^*) \cup \{0\}$ the following two properties hold*

$$\nabla f_i(y^*)^T \bar{y} = \nabla f_i(y^*)^T y^*, \quad \forall\ \bar{y} \in \mathcal{C}^* \tag{52}$$

$$\nabla f_i(y^*) = \nabla f_i(\bar{y}), \quad \forall\ \bar{y} \in \mathcal{C}^*. \tag{53}$$

Proof: Let $\bar{y}$ be an arbitrarily chosen point from $\mathcal{C}^*$ such that $\bar{y} \neq y^*$. Since $f_0(y)$ is the objective function, it is constant on $\mathcal{C}^*$. Moreover, since $\mathcal{C}^*$ is convex,

$$y_\lambda := y^* + \lambda(\bar{y} - y^*) \in \mathcal{C}^*, \quad \forall\ \lambda \in [/, \infty],$$

and hence the function

$$\phi(\lambda) := f_0(y^* + \lambda(\bar{y} - y^*))$$

is constant for any $\lambda \in [0, 1]$. Thus

$$\frac{d\phi(0)}{d\lambda^+} = \nabla f_0(y^*)^T(\bar{y} - y^*) = 0,$$

which implies (52) for $i = 0$. We now prove (52) for $i \in I(y^*)$. Since $y^* \in \mathrm{ri}\ (\mathcal{C}^*)$, we have that

$$y_\lambda := y^* + \lambda(\bar{y} - y^*) \in \mathcal{C}^*, \quad \forall\ \lambda \in (-\epsilon, \infty]$$

for some $\epsilon > 0$. We define

$$\phi(\lambda) := f_i(y^* + \lambda(\bar{y} - y^*)), \quad \forall\ \lambda \in (-\epsilon, 1].$$

We have

$$\phi(0) = 0, \quad \text{and} \quad \phi(\lambda) \le 0, \quad \text{for each} \quad \lambda \in (-\epsilon, 1].$$

However, since $\phi(\lambda)$ is differentiable, we obtain that

$$\frac{d\phi(0)}{d\lambda} = \nabla f_i(y^*)^T(\bar{y} - y^*) = 0 \tag{54}$$

which implies (52) for $i \in I(y^*)$. Moreover, since $f_i(y)$ is convex, we have

$$\nabla f_i(y^*)^T(\bar{y} - y^*) \le f_i(\bar{y}) - f_i(y^*).$$

[‡]A function is called weakly analytic if it has the following property: if it is constant on some line segment, then it is defined and constant on the entire line containing this line segment.

Setting (54) and $f_i(y^*) = 0$ into the last inequality, we obtain that $0 \leq f_i(\bar{y})$, which (together with the feasibility) implies that $f_i(\bar{y}) = 0$. Since $\bar{y}$ is an arbitrarily chosen point from $\mathcal{C}^*$, we have that $f_i(y)$ is constant on $\mathcal{C}^*$.

We now prove (53). For each $i \in I(y^*) \cup \{0\}$ we define

$$h(y) := f_i(y) - \nabla f_i(y^*)^T (y - y^*), \quad \forall\, y \in R^n.$$

Obviously, $h(y)$ is convex, $h(y^*) = f_i(y^*)$, and by (52), $h(y) = f_i(y)$ for each $y \in \mathcal{C}^*$. Since $f_i(y)$ is constant on $\mathcal{C}^*$, the function $h(y)$ is constant on $\mathcal{C}^*$ as well. Moreover, for any y

$$\nabla h(y) = \nabla f_i(y) - \nabla f_i(y^*) \tag{55}$$

and therefore $\nabla h(y^*) = 0$. Since $h(y)$ is convex, and $\nabla h(y^*) = 0$, $h(y)$ attains its minimum at y^*. However, since $h(y)$ is constant on $\mathcal{C}^*$, we have that $\nabla h(\bar{y}) = 0$ for any $\bar{y} \in \mathcal{C}^*$. Setting this into (55), we obtain (53). $\square$

Corollary 7.1 *Let $y^* \in$ ri $(\mathcal{C}^*)$, and for some $i \in I(y^*) \cup \{0\}$, $f_i(y)$ be convex and quadratic,* i.e.

$$f_i(y) = y^T P y - q^T y - r, \quad P \succeq 0.$$

Then both $P^{\frac{1}{2}} y$ and $q^T y$ are constant on $\mathcal{C}^*$.

Proof: We prove that for each $y \in \mathcal{C}^*$

$$P^{\frac{1}{2}} y = P^{\frac{1}{2}} y^*, \quad \text{and} \quad q^T y = q^T y^*. \tag{56}$$

Setting

$$\nabla f_i(y) = 2Py - q \tag{57}$$

into (53) we obtain $2Py - q = 2Py^* - q$, which implies

$$Py = Py^* \tag{58}$$

and hence the first part of (56) holds. Substituting (57) into (52) yields

$$y^{*T} P y - q^T y = y^{*T} P y^* - q^T x^*. \tag{59}$$

Using (58) and (59) we obtain the second part of (56). $\square$

Consider the convex program (C) where all functions $f_i (i = 0, \ldots, L)$ are convex quadratic, *i.e.*

$$f_i(y) = y^T P_i y - q_i^T y - r_i, \quad P_i \succeq 0. \tag{60}$$

Such a program is called convex quadratically constraint quadratic program and it is well-known that it can be equivalently reformulated to the convex quadratically constraint program:

$$(Q) \ : \ \max_{t,y} \{-t \ : \ f_0(y) \leq t, \ f_i(y) \leq 0, \ i = 1, \ldots, L\},$$

where $f_i, (i = 0, \ldots, L)$ are given by (60). As shown in *e.g.* [16] this problem can be rewritten as the SDO problem

$$(SDQ) \ : \ \max_{t,y} \{-t \ : \ Z_0(y,t) \succeq 0, \ Z_i(y) \succeq 0, \ i = 1, \ldots, L\},$$

where

$$Z_0(y,t) := \begin{bmatrix} I & P_0^{\frac{1}{2}} y \\ y^T P_0^{\frac{1}{2}} & q_0^T y + r_0 + t \end{bmatrix}, \quad Z_i(y) := \begin{bmatrix} I & P_i^{\frac{1}{2}} y \\ y^T P_i^{\frac{1}{2}} & q_i^T y + r_i \end{bmatrix}, \quad i = 1, \ldots, L.$$

This is an SDO program in the standard dual form (D), if we define Z as the block diagonal matrix with Z_i $(i = 0, \ldots, L)$ as diagonal blocks. The condition $Z_0(y,t) \succeq 0$ corresponds to $f_0(y) \leq t$, and the conditions $Z_i(y) \succeq 0$, $i = 1, \ldots, L$ correspond to the constraints $f_i(y) \leq 0$.

Theorem 7.2 *Let (SDQ) satisfy Assumptions 1.1 and 1.2. Then it also satisfies the other assumptions of Theorem 6.1. Hence $\mathcal{D}^* = \mathcal{D}_{\mathcal{F}^*}$, and the dual central path $(y(\mu), Z(\mu))$ converges to the analytic center of $\mathcal{D}^*$.*

Proof: Applying Corollary 7.1 to Z_0 we can see that both $P_0^{\frac{1}{2}} y$ and $r_0^T y$ are constant on $\mathcal{D}^*$. Moreover, also t is constant, and hence the entire block Z_0 can be considered as a part of the block Z_I from Assumption 6.1. We now consider a block Z_i for some $i \in \{1, \ldots, L\}$. If there exists $y^* \in \mathcal{D}^*$ such that $Z_i(y^*) \succ 0$, then this block is considered to be a part of Z_{II}. In the other case, $Z_i(y)$ is singular on $\mathcal{D}^*$, which means that the corresponding inequality $f_i(y) \leq 0$ is active at each $y^* \in \text{ri } (\mathcal{D}^*)$, and hence $i \in I(y^*)$, and we can apply Lemma 7.1. We obtain that all components of Z_i are constant on $\mathcal{D}^*$ and thus the entire block Z_i, for which $i \in I(y^*)$, can be considered to be a part of Z_I.

Since each block Z_i, $i = 0, \ldots, L$ was added either to Z_I, or to Z_{II}, the problem is of the form (50) and satisfies Assumption 6.1. Thus the dual central path converges to the analytic center of $\mathcal{D}^*$. $\square$

8 Concluding remarks

We have shown in this paper that the standard convergence proofs for the central path in LO fail for SDO, because the inversion property — that is essential in these proofs — holds if and only if strict complementarity holds. We have also seen that, in the absence of strict complementarity, the central path converges to the analytic center of a certain subset of the optimal set. Unfortunately, the description of this subset does not only depend on the problem data and the optimal (block) partition, and therefore does not give a nice geometrical characterization of the limit point. It would be interesting to understand whether such a characterization is possible.

We have also described a class of SDO problems with block diagonal structure where the central path does converge to the analytic center of the optimal set.

We conclude with some detailed remarks about the examples we have considered.

- Regarding the SDO reformulation (SDQ) of convex quadratic optimization problems: we have proved the convergence of the central path to the analytic center of the optimal set for the program in the standard dual form, but this result does not say anything about the situation for the corresponding dual counterpart in the standard primal form. In fact, the dual need not correspond to a quadratic program, and its central path need not converge to the analytic center of its optimal set. To see this, consider Example 1 where in the primal formulation we neglect the variables that vanish along the central path. Then the feasibility constraints can be described by one convex quadratic constraint $1/4x_{44}^2 - x_{33} \leq 0$, and two linear constraints $-x_{33} \leq 0$ and $-x_{44} \leq 0$. Hence the primal problem corresponds to a convex quadratic program. The corresponding dual problem, however, does not correspond to a convex quadratic program, and, as was shown in Section 5, the central path does not converge to the analytic center.

- The standard dual form of Example 1 is closely related to the nonlinear problem:

$$\max_{y \in \mathcal{C}} \left\{ -y_3 \ : \ \frac{y_2^2}{y_1} - y_3 \leq 0, \ y_1 \geq 0, \ 1 - y_1 \geq 0, \ y_2 \geq 0 \right\}, \tag{61}$$

  where $\mathcal{C}$ is the open half-space where $y_1 > 0$. The optimal set and central path of the standard dual form of Example 1 coincides with that of problem (61). In particular, the respective central paths do not converge to the analytic centers of the respective optimal sets. It is easy to show that all the functions in (61) are convex on $\mathcal{C}$. However, the function $\frac{y_2^2}{y_1} - y_3$ is not a weakly analytic function, and this is the reason why Theorem 7.1 does not apply here. In other words, it shows that the 'weakly analytic' requirement in Theorem 7.1 cannot simply be dropped.

- The simplicity of Example 1 allows us geometric insight: For positive μ the hyperbolic inequality 'pushes' the analytic center of the level set away from $y_1 = 0$, *i.e.* it has an influence on the description of the analytic center of the level set. However, at $\mu = 0$ this inequality disappears. Hence, it has no influence on the description of the analytic center of the optimal set. This is the reason why the central path does not converge to the analytic center of the optimal set for this example.

  One might deduce from this example that any appearance of a hyperbolic constraint courses a shift of the limit point away from the analytic center. However, this is not true as the following example shows.

  Example 3:

$$\max\left\{ -y_3 \;:\; \begin{bmatrix} y_1 & y_2 \\ y_2 & y_3 \end{bmatrix} \succeq 0,\; \begin{bmatrix} y_3 & y_2 \\ y_2 & (1-y_1) \end{bmatrix} \succeq 0,\; y_2 \geq 0 \right\}.$$

  It can easily be shown that the central path for this problem converges to the analytic center ($y_1^* = \frac{1}{2}$), and that $\mathcal{D}_{F^*} \neq \mathcal{D}^*$.

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