Limiting behavior of the central path in semidefinite optimization.

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Outline

1. The central path in SemiDefinite Optimization (SDO)
2. An overview of some recent results
3. Two illustrative examples of different behavior of the central path

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Notation:

S^n – the space of real symmetric $n \times n$ matrices

$$M \bullet N = \mathrm{trace}(M^T N) = \sum_{i,j} M_{ij} N_{ij}$$

$X \succeq 0$ (X symm. positive semidef.) $X \in S^n_+ \subset S^n$

$X \succ 0$ (X symm. positive definite) $X \in S^n_{++} \subset S^n$

Primal-dual pair of semidefinite programs:

$$(P) \qquad \begin{aligned} \min_{X \in S^n} \quad & C \bullet X \\ \text{s.t.} \quad & A^i \bullet X = b^i, \quad i = 1, \ldots, m, \\ & X \succeq 0, \end{aligned}$$

$$(D) \qquad \begin{aligned} \max_{y \in \mathbb{R}^m, Z \in S^n} \quad & b^T y \\ \text{s.t.} \quad & \sum_{i=1}^m A^i y^i + Z = C, \\ & Z \succeq 0. \end{aligned}$$

where $C,\ A^i \in S^n, \quad b \in R^m$.

Assumption 1:

A^i, $i = 1, ..., m$, are linearly independent.

Assumption 2:

Both (P) and (D) have strictly feasible solutions.

Facts:

(a) (P) and (D) have optimal solutions $\hat{X}$ and $(\hat{y}, \hat{Z})$;

(b) $C \bullet \hat{X} - b^T\hat{y} = \hat{X} \bullet \hat{Z} = 0$;

(c) $\hat{X}\hat{Z} = \hat{Z}\hat{X} = 0_{n\times n}$.

Definition:

An optimal solution pair $(\hat{X}, \hat{Z})$ is **strictly complementary** if rank $(\hat{X})$+rank $(\hat{Z}) = n$.

Fact:

The existence of a strictly complementary solution pair is not guaranteed in SDO.

The pair of μ - parametrized ($\mu > 0$) barrier problems:

$$(\mathrm{P}_\mu) \qquad \min_{X \in S^n_{++}} \quad C \bullet X - \mu \ln\det X$$
$$\text{s.t.} \quad A^i \bullet X = b^i, \quad i = 1, ..., m$$

$$(\mathrm{D}_\mu) \qquad \max_{y \in \mathbb{R}^m, Z \in S^n_{++}} \quad b^T y + \mu \ln\det Z$$
$$\text{s.t.} \quad \sum_{i=1}^m A^i y^i + Z = C$$

Fact: For any $\mu > 0$ there exists a unique optimal solution $(X(\mu), y(\mu), Z(\mu))$ to the pair (P_μ)-(D_μ).

Central path: $\{ \ (X(\mu), y(\mu), Z(\mu)) \quad : \quad \mu > 0\}$

Facts:

- The central path restricted to $0 < \mu \leq \bar{\mu}$ $(\bar{\mu} > 0)$ is bounded.
- Each of the accumulation points of the central path as $\mu \downarrow 0$ is an optimal solution for the pair (P)-(D).
- If the optimal set is a singleton, the central path converges to the (unique) optimal solution.

If the optimal set is not a singleton, the convergence of the central path is the question to be analyzed.

The situation in LO and LCP:

(a) Any accumulation point of the central path as $\mu \downarrow 0$ is the so-called analytic center of the optimal set.

(b) The analytic center of the optimal set is defined uniquely.

(a), (b) $\Rightarrow$ the convergence of the central path

Back to SDO:

Fact: The central path converges as $\mu \downarrow 0$ to the analytic center of the optimal set, provided a strictly complementary solution exists.

[De Klerk, Roos, Terlaky: Topics in SD&IPM, 18 (1998) 215-236]
[Luo, Sturm, Zhang: SIAM J. Opt. 8, (1998), 59-81]

Claim: The central path converges to the analytic center of the optimal set. (A serious gap in the proof!)

[Goldfarb, Scheinberg: SIAM J. Opt. 8 (1998), 871-886]

The claim disproved by Halická, de Klerk, Roos in *On the convergence of the central path in semidefinite optimization*, SIAM J. Opt. 12 (2002) 875-892

- The central path need not converge to the analytic center, when no strictly complementary solution exists. (Proof by a counter-example.)
- The central path converges as $\mu \downarrow 0$. (Proof by means of the theory of (semi)algebraic sets.)

Halická, De Klerk, Roos: *Limiting behavior of the central path in semidefinite optimization*, paper available at www.iam.fmph.uniba.sk/institute/halicka/shift4b.ps

- A posteriori characterization of the limit point
- Sufficient conditions for the convergence to the a. c.

In what follows:

- Two examples when no strictly compl. solution exists, and the central path does/does not converge to the a. c.

Example 1: $m = 3$, $n = 4$.

The dual problem **maximizes** $-y_3$ such that

$$\begin{bmatrix} y_3 & y_2 & 0 & 0 \\ y_2 & y_1 & 0 & 0 \\ 0 & 0 & 1-y_1 & 0 \\ 0 & 0 & 0 & y_2 \end{bmatrix} \succeq 0$$

The optimal solutions are

$$\hat{Z} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \hat{y}_1 & 0 & 0 \\ 0 & 0 & 1-\hat{y}_1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \succeq 0$$

The primal problem **minimizes** X_{33} such that

$$X = \begin{bmatrix} 1 & -\frac{1}{2}X_{44} & X_{13} & X_{14} \\ -\frac{1}{2}X_{44} & X_{33} & X_{23} & X_{24} \\ X_{13} & X_{23} & X_{33} & X_{34} \\ X_{14} & X_{24} & X_{34} & X_{44} \end{bmatrix} \succeq 0$$

The unique optimal solution is

$$\hat{X} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Asumptions 1 and 2 are satisfied,

no strictly complementary solution exists.

The dual optimal set:

$$\mathcal{D}^* = \{(y_1, y_2, y_3) \ : \ y_2 = y_3 = 0, \ y_1 \geq 0, \ 1 - y_1 \geq 0\}$$

The analytic center of $\mathcal{D}^*$ is:

$$\arg \max_{y \in \mathcal{D}^*} [\ln y_1 + \ln(1 - y_1)] = (\frac{1}{2}, 0, 0)^T$$

The dual barrier problem:

$$\max_{\substack{y_1, y_2, y_3 \in R \\ Z \succ 0}} \left\{ -y_3 + \mu \ln \det Z \ : \ Z = \begin{bmatrix} y_3 & y_2 & 0 & 0 \\ y_2 & y_1 & 0 & 0 \\ 0 & 0 & 1-y_1 & 0 \\ 0 & 0 & 0 & y_2 \end{bmatrix} \right\}$$

$$\Longleftrightarrow$$

$$\max_{\substack{y_1, y_2 \in R \\ y_3 > 0}} \left\{ -y_3 + \mu \left[\ln(y_3 y_1 - y_2^2) + \ln(1 - y_1) + \ln y_2 \right] \right\}$$

The dual central path:

$$y_1(\mu) = \frac{3}{5} \nrightarrow \frac{1}{2} \quad y_2(\mu) = \sqrt{\frac{3}{10} \mu} \to 0, \quad y_3(\mu) = \frac{3}{2} \mu \to 0$$

How to explain this?

The barrier problem $\Longleftrightarrow$

$$\max_{\substack{y_1, y_2\\ y_3}} \left\{ -y_3 + \mu \left[\ln y_3 + \ln\left(y_1 - \frac{y_2^2}{y_3}\right) + \ln(1 - y_1) + \ln y_2 \right] \right\}$$

$\Longrightarrow$

$$y_1(\mu) = \arg\max_{y_1} \left\{ \ln\left(y_1 - \frac{y_2^2(\mu)}{y_3(\mu)}\right) + \ln(1 - y_1) \right\}$$

where

$$\frac{y_2^2(\mu)}{y_3(\mu)} = \frac{\frac{3}{10}\mu}{\frac{3}{2}\mu} = \frac{1}{5}$$

The limit point is the analytic center of $\mathcal{S}$:

$$\mathcal{S} = \{(y_1, y_2, y_3) \ : \ y_2 = y_3 = 0,\ y_1 \geq \frac{1}{5},\ 1 - y_1 \geq 0\} \subset \mathcal{D}^*$$

This feature of the central path can be generalized:

Thm: The central path converges to the analytic center of a certain subset $\mathcal{S}$ of the optimal set (where the description of $\mathcal{S}$ depends on limiting behavior of "small components" of the central path).

Example 2: Maximize $b^T y$ such that

$$Z_i(y) := \begin{bmatrix} I & P_i^{\frac{1}{2}} y \\ y^T P_i^{\frac{1}{2}} & q_i^T y + r_i \end{bmatrix} \succeq 0, \quad i = 1, ..., L.$$

where $P_i \in S_+^m$, $q_i \in R^m$, $r_i \in R$ and Ass.1,2

Convex quadratic program:

$$\max_{y \in R^m} \left\{ b^T y \ : \ r_i + q_i^T y - y^T P_i y \geq 0 \ (i = 1, ..., L) \right\}$$

Thm: The central path $y(\mu)$ for Example 2 converges to the analytic center of the optimal set.

Proof:

- The central paths for both problems coincide:

$$y(\mu) = \arg\max_y \left\{ b^T y + \sum_{i=1}^{L} \ln\det Z_i(y) \right\}$$

$$= \arg\max_y \left\{ b^T y + \mu \sum_{i=1}^{L} \ln[r_i + q_i^T y - y^T P_i y] \right\}.$$

• Sufficient conditions for the convergence of the central path to the analytic center of the optimal set for general convex programs:

[Monteiro and Zhou: Comp. Optim. and Applic. 10: 51–77, 1998]
[McCormick and Witzgall: Math. Progr. (2001),]

• The restrictive functions $r_i + q_i^T y - y^T P_i y$ are concave and analytic on R^m

• Application of the results for general convex programs $\square$

Conclusion:

• Example for which the central path does not converge to the analytic center of the optimal set

• A special class of problems for which the (dual) central path always converges to the analytic center.

Slides available at:

http://www.iam.fmph.uniba.sk/institute/halicka/