Univerzita Komenského
Matematicko-fyzikálna fakulta
Ústav aplikovanej matematiky

Aplikácie asymptotických a regularizačných metód malého parametra

Habilitačná práca

Daniel Ševčovič

Bratislava August, 2000

Mojim Andrejkám

Obsah

1 Úvod

Natura non facit saltum.

Príroda nerobí skoky, tvrdí jeden z najväčších duchov všetkých čias nemecký filozof a učenec, autor Monadológie, Gottfried Wilhelm Leibniz. Azda presnejšie chcel povedať, že príroda *nerada* robí skoky, vyjadrujúc tým svoju vieru v robustnosť a stabilitu fyzikálnych systémov vystavovaných rozličným poruchám a výchylkám zo stabilného stavu. Bezpochyby je možné viesť polemiku o Leibnizovom výroku. Veď príroda predsa len robí skoky a prudké zmeny v správaní sa fyzikálnych systémov sú vďačným objektom výskumu v najrozmanitejších oblastiach bádania. Takéto zmeny sa často vysvetľujú na základe matematických modelov odrážajúcich realitu prostredníctvom obyčajných alebo parciálnych diferenciálnych rovníc s malým parametrom. Náhle zmeny charakteru riešení singulárne perturbovaných diferenciálnych rovníc vystihujú vznik pohraničnej a vnútornej vrstvy. Často sú potom východiskom pre interpretáciu rôznych očakávaných a želaných výsledkov - počnúc Zeeman-Hackenovou synergetikou až po nemenej odvážne pokusy L. Kvasza o modelovanie a vysvetľovanie epistomologických ruptúr vedeckých paradigiem ([Kvasz99]).

Teórie singulárnych perturbácii a asymptotických metód malého parametra skúmajú vzťahy perturbovaných a neperturbovaných systémov. Vo väčšine prípadov je ich hlavným cieľom ukázať, že v blízkosti riešenia neperturbovaného systému rovníc sa nachádza riešenie málo perturbovaného systému. Pritom riešenia neperturbovaného systému môžu pozostávať z niekoľkých častí, ktoré predstavujú pomalé resp. rýchle úseky, závisiac od toho aké sa použilo škálovanie. Cieľom tejto práce je poukázať na využitie techník malého parametra v singulárne perturbovaných a degenerovaných diferenciálnych rovniciach ako i v bifurkačnej analýze nelineárnych systémov. V práci sa autor pokúša priniesť niekoľko rôznorodých pohľadov na využitie asymptotických a regularizačných techník v troch oblastiach aplikácii, ktorými sa autor v ostatnom čase zaoberal. Hoci sa jedná o špeciálne problémy, autor sa nazdáva, že poskytnú čitateľovi pohľad na využitie týchto metód v rôznych aplikáciach, ktorých spoločným menovateľom je práve malý parameter a jeho využitie pri analýze skúmaných modelov. Na tomto mieste je nevyhnutné podotknúť, že nasledovné kapitoly obsahujú zámerne zjednodušené verzie tvrdení a ich predpokladov tak, aby sa nenarušila plynulá čitateľnosť predkladanej práce. Čitateľ sa môže s presnými formuláciami a výsledkami oboznámiť v Prílohe, ktorá je zostavená z troch dvojíc autorových článkov.

Malý parameter a teória singulárnych perturbácii. Tak znie názov prvej časti práce, ktorá poukazuje na možnosť využitia teórie singulárnych perturbácii pre systémy evolučných parciálnych diferenciálnych rovníc. Ako fyzikálny model je uvažovaná tzv. neinerciálna limita pre Johnson-Sagelman-Oldroydov model Poisseuleho prúdenia väzkopružnej kvapaliny. V tomto prípade malý parameter reprezentuje reálnu fyzikálnu veličinu - podiel Reynoldsovho a Deborahovho čísla. Ukazuje sa, že kvalitatívne vlastnosti riešení neinerciálnej limity, t.j. systému, v ktorom sa stráca malý parameter, sú blízke vlastnostiam riešení plného, t.j. perturbovaného systému rovníc s hoci nenulovým ale dostatočne malým "malým parametrom".

Malý parameter a regularizácia degenerovaných diferenciálnych rovníc. Druhá časť práce tvorí intermezzo medzi dvoma fyzikálnymi aplikáciami a prináša pohľad na možnosti regularizačných techník pri štúdiu tzv. degenerovaných úloh. Malý parameter teraz predstavuje mieru regularizácie pôvodného degenerovaného problému. Ako modelový príklad regularizačných techník sa v tejto práci uvažuje problém pohybu rovinných kriviek resp. fázových rozhraní, kde rýchlosť pohybu je mocninovou funkciou od krivosti pohybujúcej sa krivky.

Malý parameter a slabo nelineárna analýza fyzikálnych modelov. Záverečná tretia časť prináša pohľad na dnes už klasickú teóriu slabo nelineárnej analýzy fyzikálnych modelov. V tomto prípade malý parameter vystupuje latentne a je skrytý za tzv. rozvíjajúci parameter, podľa ktorého sa rozvíjajú poruchy, t.j. výchylky zo stabilného stavu, všetkých študovaných veličín systému. Slabo nelineárna analýza spočíva v rozvinutí riešenia do Taylorovho radu podľa malého parametra a následného porovnania koeficientov rozvoja. Ako modelový problém pre aplikáciu slabo nelineárnej analýzy je študovaný model magnetokonvekcie v rotujúcej vrstve.

Tres facet collegium.

August 2000
Daniel Ševčovič

2 Malý parameter a teória singulárnych perturbácii

Cieľom tejto kapitoly je poukázať na možnosti využitia metód malého parametra pri štúdiu limitného správania sa systému inerciálnych variet pre polotoky, ktoré sú generované systémom singulárne perturbovaných diferenciálnych rovníc v Banachovych priestoroch. Inerciálne variety sú konštruované ako grafy nad Banachovym priestorom a majú vlastnosť priťahovania každého riešenia daného systému rovníc. Hlavnou úlohou je ukázať všeobecný výsledok o tom, že pre malé hodnoty singulárneho parametra je inerciálna varieta a súčasne aj systém jej dotykových priestorov v blízkosti inerciálnej variety, ktorá zodpovedá redukovanému systému rovníc. V závere prvej časti tejto kapitoly aplikujeme získané výsledky na Johnson-Segalman-Oldroydov parabolicko-hyperbolický systém parciálnych diferenciálnych rovníc, ktorý modeluje planárne prúdenie ne-Newtonovskej kvapaliny. V druhej časti tejto kapitoly aplikujeme získané abstraktné výsledky za účelom konštrukcie disipatívnej spätnej väzby, ktorá stabilizuje zadaný výstupný funkcionál. Tento výsledok je potom aplikovaný opäť na singulárnej perturbovaný Johnson-Sagelman-Oldroyd model šmykového pohybu piestovo riadeného toku ne-Newtonovskej kvapaliny.

Výsledky častí 2.1 a 2.2 sú náplňou autorovho článku [Sevco97a] (pozri Prílohu 6.1.1) pojednávajúcom o hladkosti singulárnej limity invariantných variet singulárne perturbovaných systémov evolučných rovníc. Výsledky častí 2.3 a 2.4 sú obsiahnuté v ďalšom autorovom článku [Sevco97b] (pozri Prílohu 6.1.2) o existencii disipatívnej spätnej väzby pre singulárne perturbované systémy.

2.1 Abstraktný výsledok o blízkosti inerciálnych variet

V prácach [Sevco94, Sevco95, Sevco97a, Sevco97b] sa autor zaoberal otázkou blízkosti inerciálnych variet pre systémy evolučných diferenciálnych rovníc, ktoré majú štruktúru

$$\begin{aligned} U_t &= G_\varepsilon(U,S) \\ \varepsilon S_t + AS &= F_\varepsilon(U,S) \end{aligned} \tag{2.1.1}$$

kde $\varepsilon \geq 0$ je malý parameter, X, Y sú Banachove priestory, A je sektoriálny operátor Y, $F_\varepsilon : X \times Y^\alpha \to Y$; $G_\varepsilon : X \times Y^\alpha \to X$; sú dostatočne hladké nelinearity definované na zlomkových priestoroch, $\alpha \in [0,1)$, pričom $F_\varepsilon \to F_0, G_\varepsilon \to G_0$ keď $\varepsilon \to 0^+$.

Pripomeňme známy výsledok o tom, že systém (2.1.1) generuje C^1 polotok vo fázovom priestore $X \times Y^\alpha$ pre každé pevne zvolené $\varepsilon > 0$ (pozri Henry [Henry81]). Podľa práce Marion [Marion89] tento polotok má invariantnú exponenciálne rýchlo priťahujúcu varietu M_ε (inerciálnu varietu) za predpokladu, že Lipschitzova konštanta zobrazenia F_ε je dostatočne malá. Pripúšťame aj situácie, v ktorých M_ε nekonečnorozmerná Banachova varieta ([Marion89]). Naviac táto varieta môže byť konštruovaná ako graf nad Banachovym priestorom X, t.j. $M_\varepsilon = \{(U, \Phi_\varepsilon(U)), U \in X\}$ (pozri [Marion89]). Naviac Chow & Lu [Chow88] ukázali, že M_ε je vskutku C^k varieta za predpokladu, že F and G sú C^k hladké ohraničené funkcie.

V rámci geometrickej teórie singulárnych perturbácii bolo vynaložené značné úsilie na dôkaz spojitosti singulárnej limity $\varepsilon \to 0^+$ pre abstraktné systémy typu (2.1.1). Dobrú referenciu poskytuje kniha P.Bates [Bates98] venovaná problematike stability invariantných variet singulárne perturbovaných úloh, resp. kniha Mischenko et al. [Mish94] ako i práca Sviridyuk & Sukacheva [Svir90] a mnohé ďalšie.

Hlavným cieľom predkladanej práce je ukázať, že pre malé hodnoty singulárneho parametra $\varepsilon > 0$ je inerciálna varieta M_ε je C^1 blízko k variete $M_0 = \{(U,S), AS = F_0(U,S)\}$ zodpovedajúcej riešeniam kvázi-dynamickej úlohy $U_t = G_0(U,S)$ s väzbou $AS = F_0(U,S)$. Poznamenajme, že C^1 stabilita inerciálnych variet, resp. centro-nestabilných variet je užitočná vlastnosť s využitím v teórii Morse-Smaleovych vektorových polí. (pozri prácu Mora & Solà-Morales [Mora89]).

Tvrdenie 2.1. [Sevco97a, Theorem 3.9] *Predpokladajme, že úloha (2.1.1) vyhovuje predpokladom sformulovaných v hypotéze (H) z článku* [Sevco97a], Prílohu 6.2.1. *Potom existujú konštanty $\delta > 0$ a $\varepsilon_0 > 0$ také, že ak $\|D_S F_\varepsilon\|_{L(Y^\alpha,Y)} \leq \delta$, tak* *potom pre každé $\varepsilon \in [0, \varepsilon_0]$ existuje invariantná varieta M_ε pre polotok generovaný systémom (2.1.1),*

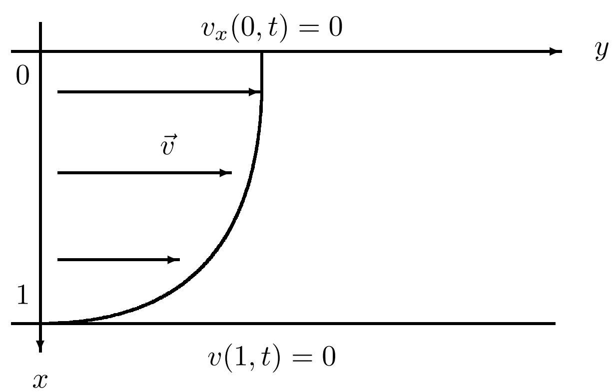
$$M_\varepsilon = \{(U, \Phi_\varepsilon(U)), U \in X\} \qquad kde \quad \Phi_\varepsilon \in C^1_{bdd}(X, Y^\alpha)\ ,$$

$$\Phi_\varepsilon \to \Phi_0 \quad ked' \quad \varepsilon \to 0^+ \quad v\ priestore \quad C^1_{bdd}(B, Y^\alpha)$$

pre každú ohraničenú otvorenú podmnožinu $B \subset X$. Ak $dim(X) = \infty$, tak M_ε je nekonečno rozmerná Banachova podvarieta vo fázovom priestore $X \times Y^\alpha$. Ak $dim(Y) = \infty$, tak $codim(M_\varepsilon) = \infty$. Ak naviac platí, že rezolventný operátor $A^{-1} : Y \to Y$ je kompaktný, tak varieta $M_\varepsilon, \varepsilon \in (0, \varepsilon_0]$, exponenciálne rýchlo priťahuje každú ohraničenú množinu počiatočných podmienok z $X \times Y^\alpha$.

2.2 Aplikácia na JSO model prúdenia tlakovo riadeného toku ne-Newtonovskej kvapaliny

V tejto časti budeme aplikovať získaný abstraktný výsledok na problém neinerciálnej limity pre Johnson-Segalman-Oldroyd model prúdenia ne-Newtonovskej kvapaliny. Pre jednoduchosť budeme uvažovať iba planárne Poisseuleho prúdenie ne-Newtonovskej kvapaliny, ktorou môže napríklad byť vysoko elastický a veľmi viskózny polymér. Kanál je rozšírený pozdĺž y osi, naprieč medzi $x \in [-1, 1]$. Tok kvapaliny uvažujeme symetrický vzhľadom $x = 0$ a kvapalina vyhovuje predpokladu o jednoduchom strihovom režime. Premenné sú nezávislé od y a teda $\vec{v} = (0, v(t, x))$ (pozri Obr. 2.1).



Obr. 2.1

Tenzor extra napätia je určený na základe Johnson-Segalman-Oldroyd konštitučného zákona (pozri [Malkus91]). Na základe práce Malkus, Nohel and Plohr [Malkus91] potom pohyb takejto kvapaliny môže opísaný bezrozmerným systémom parabolicko-hyperbolických rovníc

$$\begin{aligned} \varepsilon v_t - v_{xx} =& \sigma_x + f \\ \sigma_t =& -\sigma + (1+z)v_x \quad (x,t) \in (0,1) \times (0,\infty) \\ z_t =& -z - \sigma v_x \end{aligned} \tag{2.2.1}$$

$$v_x(0,t) = v(0,t) = \sigma(0,t) = 0 \quad \text{pre každé } t \geq 0$$

kde $v = v(x,t)$, je skalár rýchlosti, σ je tzv. extra strihové napätie, z je rozdiel normálových napätí. Okrajové podmienky zodpovedajú predpokladu o nekľzavosti kvapaliny na stenách $x = \pm 1$ a symetričnosti vzhľadom na os $x = 0$ (pozri Obr. 2.1). V prípade tlakovo riadeného pohybu je tlakový gradient f vopred zadaný.

Malý parameter $\varepsilon > 0$ je v tomto modeli proporcionálny podielu Reynoldsovho a Deborahovho (Weissenbergovho) čísla. Dôležité je upozorniť, že na základe reologických experimentov Vinogradova *et. al.* [Vinog72] je toto číslo veľmi malé, rádovo $O(10^{-12})$. Tento poznatok dáva možnosť úvahe o tzv. *neinerciálnej aproximácii* $\varepsilon = 0$. Na základe tejto úvahy boli Malkus, Nohel a Plohr [Malkus91] schopní vysvetliť veľa dôležitých a v experimentoch pozorovaných fenoménov akými sú napr. prietrž toku, hysterézia, tvarová pamäť a či latencia.

Hoci výsledky dosiahnuté pre redukovaný systém rovníc (2.2.1) s $\varepsilon = 0$ poskytli uspokojivé odpovede na otázky reológov, napriek tomu stále ostávala otvorená otázka o oprávnenosti neinerciálnej aproximácie systému (2.2.1). V článku [Nohel93] sa Nohel a Pego pokúsili dokázať správnosť neinerciálnej aproximácie prostredníctvom Morse-Conleyho teórie. Dokázali, že pre $\varepsilon \to 0$ riešenia konvergujú bodovo k riešeniu neinerciálnej aproximácie systému (2.2.1). Cieľom tejto kapitoly je poskytnúť alternatívny pohľad na túto problematiku. V našom prístupe sa budeme opierať o abstraktný výsledok dosiahnutý v časti 2.1. Aplikáciou Tvrdenie 2.1 dostávame

Tvrdenie 2.2. [Sevco97a, Theorem 3.11] *Pre každé dostatočne malé $\varepsilon > 0$ nelineárny systém rovníc (2.1.1), modelujúci strihový pohyb Poisseuleho planárneho prúdenia Johnson-Segalman-Oldroydovej kvapaliny,*

i) má nekonečno rozmernú nekonečno ko-rozmernú lokálnu invariantnú varietu M_ε, ktorá exponenciálne rýchlo priťahuje všetky riešenia systému (2.1.1).

ii) Existuje $R_0 > 1$ také, že každé riešenie systému (2.1.1) po istom čase vstúpi do gule o polomere R_0 vo fázovom priestore $(L_\infty(0,1))^2 \times W_B^{1/2}(0,1)$;

iii) $M_\varepsilon = \{(\sigma, z, \Phi_\varepsilon(\sigma,z)), (\sigma,z) \in B_{R_0}\}$, $\Phi_\varepsilon \in C^1_{bdd}(B_{R_0}, W_B^{1,2}(0,1))$ kde $B_{R_0} = \{(\sigma,z) \in (L_\infty(0,1))^2,\ \|\sigma\|^2_\infty + \|z\|^2_\infty < R_0\}$;

iv) $\Phi_\varepsilon \to 0$ keď $\varepsilon \to 0^+$ v topológii priestoru $C^1_{bdd}(B_{R_0}, W_B^{1,2}(0,1))$.

Dôkaz tvrdenia sa nachádza v Prílohe 6.1.2. Z hľadiska odôvodnenia správnosti neinerciálnej aproximácie za najdôležitejšiu časť predošlého tvrdenia môžeme považovať bod iv), ktorého zmysel tkvie v tom, že asymptotické správanie sa riešení systému (2.1.1) s $\varepsilon > 0$ veľmi malým je blízke asymptotickému správaniu sa riešení redukovaného systému rovníc, v ktorom $\varepsilon = 0$.

2.3 Existencia disipatívnej spätnej väzby

V tejto časti práce budeme študovať špecifický problém stabilizácie riešení singulárne perturbovaného systému rovníc v Banachovych priestoroch. Cieľom je dokázať existenciu a skúmať kvalitatívne vlastnosti disipatívnej spätnej väzby, ktorá stabilizuje zadaný výstupný funkcionál. Abstraktný výsledok bude potom aplikovaný v nasledujúcej časti na Johnson-Sagelman-Oldroyd model piestovo riadeného pohybu ne-Newtonovskej kvapaliny.

Študovať budeme singulárne perturbovaný systém evolučných rovníc

$$\begin{aligned} x_t &= G_\varepsilon(x, y, z) \\ \varepsilon y_t + By &= F_\varepsilon(x, y, z) \end{aligned} \tag{2.3.1}$$

kde $0 \le \varepsilon \ll 1$ je malý parameter, $x \in X, y \in Y$, X a Y sú Banachove priestory, B je sektoriálny operátor v Y. Skúmať budeme špecifický spätno väzobný mechanizmus, ktorý môže byť vyjadrený prostredníctvom funkcie spätnej väzby

$$z = \Xi(x)$$

kde Ξ je hladká funkcia z X do iného Banachoveho priestoru Z. Inými slovami povedané, požadujeme, aby spätná väzba $z = \Xi(x)$ závisela iba od tzv. pomalej premennej x.

Význam spätnej väzby tkvie v nasledovnom pozorovaní. V mnohých aplikovaných prípadoch totižto štruktúra redukovaného systému rovníc (2.3.1) s $\varepsilon = 0$ umožňuje explicitne nájsť syntézu $z = \Xi_0(x)$ s vlastnosťou, že zadaný výstupný funkcionál Q_0 sa asymptoticky stabilizuje na nule, t.j. $Q_0(x(t), y(t))) \to 0$ keď $t \to \infty$. Takýto príklad konštrukcie môže čitateľ nájsť v nasledovnej kapitole.

Problém teraz spočíva v zodpovedaní otázky, či sa táto vlastnosť dá preniesť aj na plný systém singulárne perturbovaných rovníc, v ktorom malý parameter je vskutku malý ale nenulový. Riešiť teda má zmysel otázku existencie spätnej väzby $\Xi = \Xi_\varepsilon$ stabilizujúcej výstupný funkcionál Q_ε pozdĺž trajektórii plného systému singulárne perturbovaných rovníc (2.3.1). Zároveň je zmysluplné sa pýtať, či spätná väzba

$\Xi = \Xi_\varepsilon$ je blízko spätnej väzby $\Xi = \Xi_0$ redukovaného systému. Hlavný výsledok autorovho článku [Sevco97b] (pozri Prílohu 6.1.2) rieši tento problém nasledovným spôsobom.

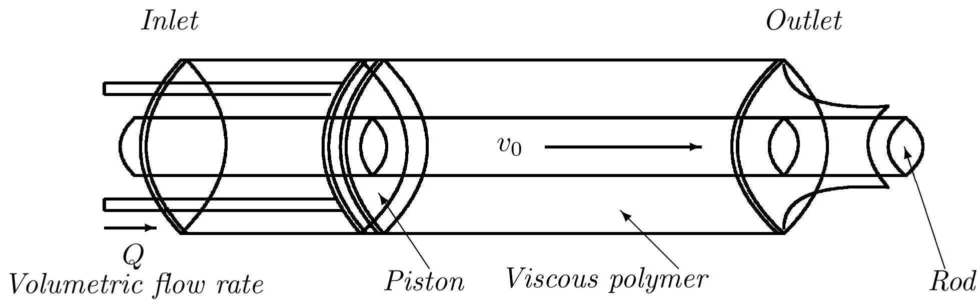
TVRDENIE 2.3. [Sevco97b, Theorem 1.1] *Predpokladajme, že sú splnené predpoklady (H1)-(H4), (5.1) z článku* [Sevco97b], Príloha 6.1.2. *Potom pre každé dostatočne malé* $\varepsilon > 0$

a) systém (2.3.1) pripúšťa disipatívnu spätnú väzbu $\Xi_\varepsilon \in C^1_{bdd}(B, Z) \cap C^{0,1}(X, Z)$ *pričom*
b) $\lim_{\varepsilon\to 0^+} \Xi_\varepsilon = \Xi_0$ *v* $C^1_{bdd}(B, Z)$ *pre každú otvorenú ohraničenú podmnožinu* $B \subset X$. *Naviac*
c) spätná väzba $z = \Xi_\varepsilon(x)$ *stabilizuje zadaný výstupný funkcionál* Q_ε*, t.j.* $\lim_{t\to\infty} Q_\varepsilon(x(t), y(t)) = 0$ *pre každé riešenie* $(x(.), y(.))$ *systému (2.3.1).*
d) Polotok generovaný riešeniami systému (2.3.1) je asymptoticky Q_ε *- asymptoticky obmedzený na* C^1 *hladkú inerciálnu varietu* M_ε*. Táto varieta* M_ε *je* C^1 *blízko k variete* M_0 *redukovaného systému pre dostatočne malé* $\varepsilon > 0$.

2.4 Aplikácia na JSO model prúdenia piestovo riadeného toku ne-Newtonovskej kvapaliny

V tejto časti sa budeme zaoberať aplikáciou Tvrdenia 2.3 v teórii prúdenia ne-Newtonovských kvapalín. Podobne ako v časti 2.2 budeme ako fyzikálny model opäť uvažovať Johnson-Segalman-Oldroydov model prúdenia kvapaliny. ktorý bol s úspechom použitý na opis prietokových nestabilít objavujúcich sa pri analýze prúdenia ne-Newtonovskej kvapaliny (Malkus, Nohel, Plohr and Tzavaras Malkus91], [Nohel90], Grob [Grob94]), Aarts, Van de Ven [Aarts95]). Táto časť práce bola motivovaná najmä reologickými experimentami Lima & Schowaltera [Lim89]. Ich experimentálne dáta poskytli evidenciu o vzniku oscilatorického režimu pre tlakový gradient v prípade, že objemový prietok presiahne istú kritickú hranicu. V práci [Malkus93] Malkus, Nohel & Plohr vyvinuli matematickú teóriu schopnú vysvetliť tento jav práve na základe JSO modelu, v ktorom tlakový gradient je funkciou strihových napätí. Istý nedostatok ich prístupu spočíval v tom, že (podobne ako už bolo spomenuté v kapitole 2.2) sa uvažovala iba tzv. neinerciálna aproximácia problému, kde $\varepsilon = 0$. Cieľom nasledovných riadkov je poskytnúť tvrdenie o existencii a stabilite disipatívnej spätnej väzby aj v prípade, že $\varepsilon > 0$ je dostatočne malý parameter.

Obrázok 2.2 poskytuje technologický pohľad na piestovo riadený tok ne-Newtonovskej kvapaliny.



Obr. 2.2

V práci [Sevco97b] sa autor zaoberal JSO modelom tlakovo riadeného prúdenia ne-Newtonovskej kvapaliny

$$\begin{aligned} \varepsilon v_t - v_{xx} &= \sigma_x + f \\ \sigma_t + \sigma &= (1+z)v_x \qquad (x,t) \in (0,1)\times(0,\infty) \\ z_t + z &= -\sigma v_x \\ v_x(0,t) = v(0,t) &= \sigma(0,t) = 0 \quad \text{pre } t \geq 0 \end{aligned} \tag{2.4.1}$$

kde význam jednotlivých veličín je ten istý ako v časti 2.2. Úlohu malého parametra opäť zohráva podiel Reynoldsovho a Deborahovho čísla $\varepsilon > 0$, ktorý je rádovo $O(10^{-12})$. Na rozdiel od modelu tlakovo riadeného pohybu budeme teraz uvažovať piestovo riadený model, v ktorom tlakový gradient nie je vopred zadaný môže byť vo všeobecnosti funkciou času, resp. veličín σ, z. Tok kvapaliny má však v tomto prípade ďalšie obmedzenie v predpísanom prietoku, t.j. $Q(t) = \int_0^1 v(t,x)\,dx$ má byť rovné predpísanej hodnote Q_{fix}. Pre redukovaný systém rovníc Malkus et al. odvodili, že tlakový gradient je nelokálnou funkciou prietokových premenných

$$f = \Xi_0(\sigma, z) = 3Q_{fix} - 3\int_0^1 x\sigma(x)\,dx$$

(pozri [Malkus93, (FB)]). Rozsiahle numerické simulácie z článku [Malkus93] ukázali, že tento tvar väzby je schopný vysvetliť zaujímavý jav oscilatorického režimu priebehu tlakového gradientu $f = f(t)$, ktorý sa skutočne podarilo experimentálne zachytiť v reologických experimentoch Lima & Showaltera [Lim89].

Hlavným výsledkom tejto časti je tvrdenie o existencii a stabilite disipatívnej spätnej väzby v prípade, že $\varepsilon > 0$ je dostatočne malý parameter. Voľne povedané, toto tvrdenie hovorí, že pre každé dostatočne malé $0 < \varepsilon \ll 1$ existuje spätná väzba systému (2.4.1), vyjadrená skrze tlakový gradient $f = f_\varepsilon(\sigma, z)$, ktorá stabilizuje objemový prietok na zadanej hodnote Q_{fix}. Naviac zobrazenie f_ε je C^1 blízko k funkcionálu $f_0 = \Xi_0(\sigma, z)$ skonštruovaného Malkusom et al. v [Malkus93].

Presná formulácia tvrdenia a predpokladov sa nachádzajú vo výsledku [Sevco97b, Theorem 6.3] (pozri Prílohu 6.1.2). Dôkaz sa opiera o abstraktný výsledok z kapitoly 2.3.

3 Malý parameter a regularizácia degenerovaných diferenciálnych rovníc

V tejto kapitole sa sústredíme na ďalší aspekt využitia metód malého parametra. Cieľom je poukázať na jeho využitie v regularizácii degenerovaných parabolických rovníc. Metóda regularizácie je užitočným nástrojom nielen z uhla pohľadu analytických dôkazov existencie riešenia, ale poskytuje aj návod na konštrukciu stabilných numerických schém riešenia degenerovaných úloh. Regularizačné techniky zavedenia malého parametra použijeme na problém dôkazu existencie riešenia opisujúceho pohyb rovinných kriviek. Analýza pohybu rovinných kriviek podľa krivosti je veľmi užitočná pre lepšie pochopenie dynamiky pohybu fázových rozhraní. Obsah tejto kapitoly je detailne rozpracovaný v autorových článkoch [Mik00, Mik99] (Prílohy 6.2.1 a 6.2.2), ktoré sú spoločným dielom s K.Mikulom.

3.1 Analýza pohybu rovinných kriviek podľa nelineárnej funkcie krivosti

Študovať budeme pohyb rovinných kriviek spĺňajúcich geometrickú rovnicu

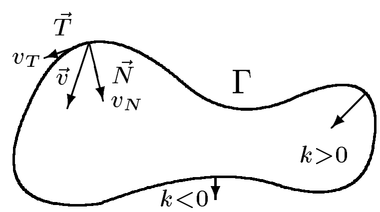
$$v = \beta(k, \nu) \tag{3.1}$$

kde v je normálová rýchlosť pohybu, k je krivosť krivky a ν je dotykový uhol (pozri Obr. 3.1).

V posledných rokoch geometrické rovnice tvaru (3.1) pritiahli mnoho pozornosti ako z teoretického, tak aj z praktického pohľadu ich využitia. Aplikácie rovnice (3.1) vznikajú v rozličných fyzikálnych modeloch, najčastejšie opisujúcich problém fázových rozhraní. V tomto prípade rovnica (3.1) zodpovedá Gibbs-Thomsonovmu zákonu kryštalického rastu podchladeného kryštálu (pozri [Gurtin93, Schm96, Benes98]). V sérii článkov [Angen89, Angen94, Angen96] Angenent a Gurtin študovali pohyb fázových rozhraní podľa zákona

$$\mu(\nu, v)v = h(\nu)k - g$$

kde μ je kinematický koeficient a veličiny h, g majú pôvod v konštitučných zákonoch opisu fázovej hranice. Závislosť normálovej rýchlosti v na krivosti k je daná skrze povrchové napätie. Na druhej strane závislosť na uhle ν (orientácia rozhrania) vnáša anizotropické efekty do modelu. Vo všeobecnosti kinematický koeficient μ môže samotný závisieť od rýchlosti. Tento fakt dáva oprávnenie na štúdium všeobecnej závislosti rýchlosti $v = \beta(k, \nu)$ na krivosti k a uhle ν. Lineárny prípad keď $\beta(k, \nu)$ závisí lineárne na krivosti k bol detailne analyzovaný v prácach [Gage86, Abresch86, Angen91a, Gray87]. Numerické aspekty riešenia zase boli skúmané v prácach [Dziuk94, Deck97, Mik96, Osher88, Mik99, Schm96, Mik96, Mik97, Girao95, Girao94, Ush00, Kim94, Kim97, Moiss98, Seth90, Seth96, Handl98, Cagin90, Elliot96, Benes98].



Obr. 3.1.

S rovnicami typu (3.1) sa môžeme stretnúť aj teórii spracovania obrazu, kde sa s úspechom používa tzv. afínna škálovacia funkcia, ktorá zodpovedá výberu $v = \beta(k) = k^{1/3}$. Tieto modely boli študované najmä v prácach Sapiro and Tannenbaum ([Sap94]) a Alvarez, Guichard, Lions and Morel ([Alvarez93, Alvarez94]) a iných autorov (pozri napr. [Alvarez93, Sap94, Angen98]). Anizotropické modely boli analyzované v prácach ([Kass87, Casse97]). Dobrú referenciu na využitie geometrickej rovnice (3.1) v rozmanitých oblastiach použitia poskytuje prehľadná kniha Sethiana [Seth96].

V tejto časti sa budeme zaoberať predovšetkým prípadom, keď

$$\beta(k, \nu) = \gamma(\nu)|k|^{m-1}k \tag{3.2}$$

kde $\gamma(\nu) > 0$ je daná funkcia anizotropie prostredia $m > 0$. Z analytického uhla pohľadu je cieľom ukázať lokálnu existenciu regulárnej triedy rovinných kriviek vyhovujúcich rovnici (3.1) pre oba prípady $0 < m < 1$ ako i $1 < m \leq 2$. Prípad rýchlej difúzie je rozšírením výsledku Angenent, Sapiro and Tannenbaum [Angen98], ktorý analyzuje iba prípad $k = 1/3$. V prípade pomalej degenerovanej difúzie $1 < m \leq 2$ budeme potrebovať dodatočný geometrický predpoklad na počiatočnú krivku. Poznamenajme, že hodnota exponentu $m = 2$ sa javí byť kritickou nielen kvôli našim výsledkom, ale aj z dôvodu výsledkov Andrewsa [Andr98], ktorý ukázal, že hodnota $m = 2$ je kritická v zmysle, že pre vyššie mocniny už nemusí existovať klasické riešenie.

Pripomeňme, že predpoklady na základe ktorých Angenent v sérii článkov [Angen90a, Angen90b] dokázal lokálnu existenciu klasického riešenia sa nedajú bezprostredne aplikovať na degenerovaný prípad (3.2), $m \neq 1$. Zhruba povedané, kľúčový predpoklad jeho teórie je

$$0 < \lambda_- \leq \beta'_k(k, \nu) \leq \lambda_+ < \infty, \tag{3.3}$$

kde $\lambda_\pm > 0$ sú konštanty. V ďalšom článku Angenent, Sapiro a Tannenbaum [Angen98] ukázali lokálnu existenciu klasického riešenia aj pre singulárny prípad $\beta(k) = k^{1/3}$. Ich argument je založený na regularizácii funkcie $\beta(k, \nu) = \gamma(\nu)|k|^{m-1}k$

$$\beta \leftrightarrow \beta^\varepsilon$$

tak, aby regularizovaná funkcia β^ε vyhovovala predpokladu (3.3) a naviac aby $\beta^\varepsilon \to \beta$ keď $\varepsilon \to 0^+$. **Malý parameter** teda hrá úlohu regularizačného parametra. V práci uvažujeme špecifickú regularizáciu mocninovej funkcie (3.2) v tvare

$$\beta^\varepsilon(k, \nu) = m\gamma(\nu) \int_0^k (\varepsilon^2 + \xi^2)^{\frac{m-1}{2}}\, d\xi \qquad \text{ak } \ 0 < m \leq 1;$$

$$\beta^\varepsilon(k, \nu) = \beta(k, \nu) + \varepsilon k \qquad \text{ak } \ m > 1$$

Myšlienka metódy ďalej spočíva v tom, že prostredníctvom a-priórnych odhadov nezávisiacich na malom regularizačnom parametri ε sa dá ukázať existencia regulárnej limity riešenia pre $\varepsilon \to 0^+$. Tieto odhady sú založené na Nash-Moserovej iteračnej metóde na získanie L^∞ odhadu pre gradient normálovej rýchlosti. Ďalej v tejto kapitole ukazujeme možnosť rozšírenia výsledku Angenenta et al. [Angen98] pre $k = 1/3$ aj na prípad $0 < m < 1$ ako i $1 < m < 2$. Hlavný výsledok sa dá zhrnúť do nasledovného tvrdenia 3.1. Presná formulácia sa dá nájsť v Prílohe 6.2.1. Je určitou zaujímavosťou, že ako dôležitý geometrický predpoklad v prípade degenerovanej pomalej difúzie $1 < m < 2$ sa ukázala požiadavka, aby každý inflexný bod počiatočnej krivky Γ^0 (ak nejaké inflexné body vôbec má) mal nanajvýš $2 + \frac{1}{m-1}$ rád kontaktu so svojou dotyčnicou. Príkladom takéhoto inflexného bodu je počiatok Bernoulliho lemniskáty $(x^2 + y^2)^2 = 4xy$. V tomto príklade je uvedená geometrická podmienka splnená vtedy a len vtedy ak $0 < m < 2$.

TVRDENIE 3.1. [Mik00, Theorem 6.3] *Predpokladajme, že* $\beta(k,\nu) = \gamma(\nu)|k|^{m-1}k$ *kde* $0 < m \leq 2$, γ *a* $\Gamma^0 = Image(x^0)$ *vyhovujú predpokladom* [Mik00, Theorem 6.3]. *Potom existuje* $T > 0$ *a systém rovinných kriviek* $\Gamma^t = Image(x(.,t)), t \in [0,T]$, *taký, že*

a) $x, \partial_u x \in (C(\overline{Q_T}))^2$, $\partial_u^2 x$, $\partial_t x$, $\partial_u \partial_t x \in (L^\infty(Q_T))^2$ *kde* $Q_T = (0,1) \times (0,T)$*;*

b) tok $\Gamma^t = Image(x(.,t)), t \in [0,T]$ *regulárnych rovinných kriviek spĺňa geometrickú rovnicu*

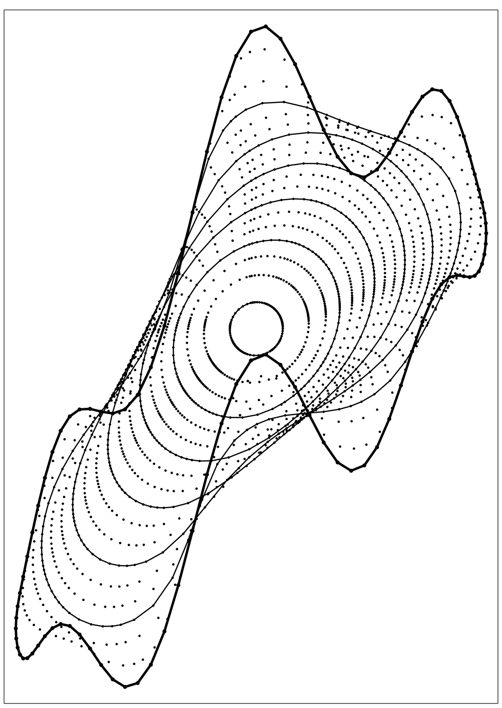
$$\partial_t x = \beta \vec{N} + \alpha \vec{T} \ ,$$

kde $\vec{N}, \vec{T}$ *sú normálový, resp. tangenciálny vektor ku krivke* Γ^t, $\beta = \beta(k,\nu)$ *a* α *je tangenciálna rýchlosť zachovávajúca relatívnu dĺžku*

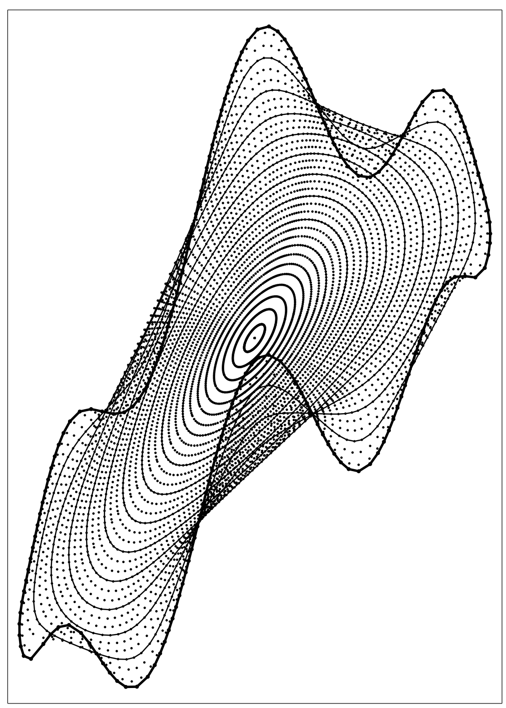
$$\frac{|\partial_u x(u,t)|}{L^t} = \frac{|\partial_u x^0(u)|}{L^0}$$

pre každé $(u,t) \in Q_T$ *kde* L^t *je celková dĺžka krivky* Γ^t.

Výsledky analytickej časti sú potom aplikované za účelom konštrukcie efektívnej numerickej schémy na riešenie geometrickej rovnice (3.1). Na záver tejto kapitoly prinášame ukážky numerických simulácii. Ďalšie príklady si môže čitateľ pozrieť v článkoch [Mik99, Mik00], ktoré tvoria Prílohy 6.2.1 a 6.2.2.



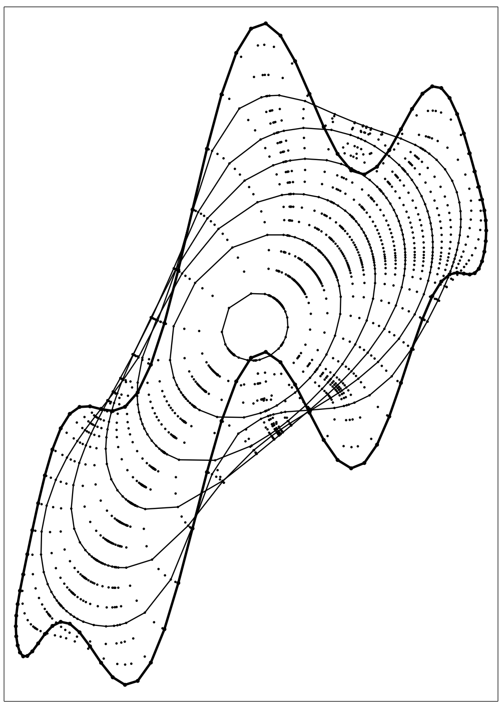
Obr. 3.2. $\beta(k) = k$



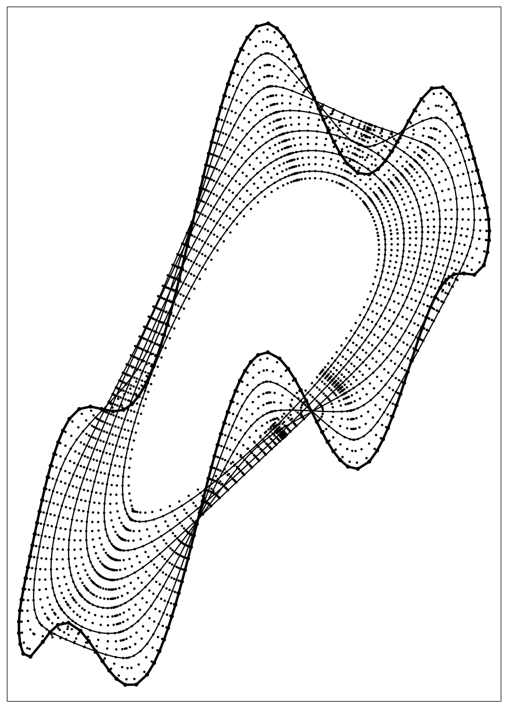
Obr. 3.3. $\beta(k) = k^{1/3}$

Na obrázku 3.3 je zreteľné, že limitným profilom krivky je elipsa. Toto pozorovanie je plne v súlade s teóriou o afínnom škálovaní pochádzajúcej od Sapira a Tannenbauma [Sap94]. Podľa týchto výsledkov je pre prípad $k = 1/3$ a generickej počiatočnej krivky limitným profilom vždy elipsa. Poznamenajme, že tvrdenie sa dá rozšíriť aj na iné hodnoty parametra m vystupujúceho v funkcii $\beta(k) = |k|^{m-1}k$ (pozri Ushijima a Yazaki [Ush00]), pričom kritickými sú hodnoty v tvare $m = 1/(n^2 - 1)$.

Poznamenajme, že predošlé obrázky boli počítané pomocou numerickej schémy, ktorá využívala netriviálnu tangenciálnu zložku rýchlosti α. Hoci tangenciálna rýchlosť z analytického hľadiska nemení geometrický tvar kriviek, jej význam spočíva v numerickej implementácii. Mnoho autorov uskutočnilo výpočty pohybu kriviek bez uvažovania tangenciálnej rýchlosti. Výsledky však viedli k rôznym numerickým nestabilitám spôsobeným nerovnomerným zahusťovaním numerických bodov. Na Obr. 3.4 a 3.5 môžeme vidieť ako numerická schéma používajúca nulovú tangenciálnu rýchlosť vedie k nestabilitám, hoci počiatočná krivka a normálová rýchlosť β sú tie isté ako v experimentoch na Obr. 3.2 resp. Obr. 3.3.



Obr. 3.4. $\beta(k) = k$, nulová tangenciálna rýchlosť



Obr. 3.5. $\beta(k) = k^{1/3}$, nulová tangenciálna rýchlosť

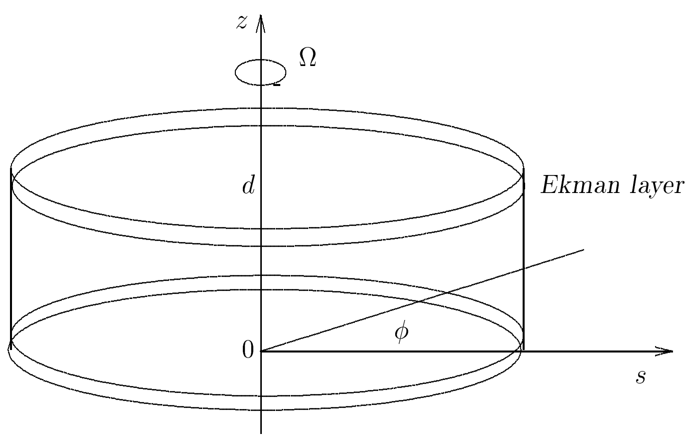
4 Malý parameter a slabo nelineárna analýza fyzikálnych modelov

Cieľom záverečnej kapitoly je poukázať na ďalšie dôležité využitie techník malého parametra v aplikovaných problémoch. Náplňou kapitoly je slabo nelineárna metóda analýza nelineárnych fyzikálnych problémov. Ťažisko tejto metódy spočíva v rozvinutí všetkých veličín modelu do Taylorovho radu podľa mocnín malého parametra, dosadením týchto rozvojov do fyzikálneho modelu a následného porovnania koeficientov pri rovnakých mocninách malého parametra. Rozvíjané veličiny sa chápu ako poruchy od stabilného stavu fyzikálneho systému. Výsledkom porovnania koeficientov rozvoja sú potom napr. amplitúdové rovnice, normálne formy pre štúdium rôznych bifurkácii, resp. Ginzburg-Landauove modulačné rovnice. Z matematického hľadiska slabo nelineárna analýza sa dá interpretovať ako lokálna bifurkačná analýza a malý parameter zodpovedá rozvíjajúcemu alebo bifurkačnému parametru. Konkrétne sa budeme zaoberať aplikáciou metódy slabo nelineárnej analýzy na model magnetokonvekcie, ktorá podrobnejšie diskutovaná v Prílohách 6.3.1 a 6.3.2. Obsah týchto kapitol tvoria spoločné články s M.Revallom. J.Brestenským a S.Ševčíkom [Reval97, Reval99].

4.1 Slabo nelineárna analýza modelu rotujúcej magnetokonvekcie

Význam štúdia modelov rotujúcej magnetokonvekcie tkvie najmä v naplnení dlhodobej snahy vedcov o vysvetlenie vzniku a dlhodobej existencie zemského magnetizmu a efektu zemského dynama. Magnetokonvekcia v rotujúcej vrstve, tak ako ju popísal Taylor [Taylor63] a ďalej skúmali Soward [Sow79, Sow86], Skinner [Skin88, Skin91], Fearn a Proctor [Fearn94a, Fearn94b], sa stala síce veľmi zjednodušeným no predsa len východiskovým modelom pre štúdium vzniku oscilačných nestabilít. Práve oscilačné nestability sú považované za jedny z možných východísk pre vysvetlenie dlhodobo udržiavaného magnetického poľa našej planéty - Zemského dynama. Podľa Cowlingovej teórie by sa totižto jednoduché magnetické polia Zeme (čisto poloidálne, resp. čisto toroidálne) exponenciálne rýchlo utlmili a jediná šanca na vznik dynama je netriviálna kombinácia oboch týchto zložiek polí. Nie je však cieľom tejto kapitoly pojednávať o týchto širších súvislostiach a preto sa v ďalšom obmedzíme iba na štúdium Taylorovho modelu rotujúcej magnetokonvekcie v horizontálnej vrstve.

Metódu slabo nelineárnej analýzy využijeme pri štúdiu modelu magnetokonvekcie v rotujúcej horizontálnej cylindrickej vrstve naplnenej vodivou kvapalinou. O vrstve predpokladáme, že je na spodnej časti zahrievaná a nerovnomerne stratifikovaná, čo znamená, že tepelný gradient nie je konštantný pozdĺž šírky vrstvy. Ďalej sa o vrstve predpokladá, že je preniknutá azimutálnym magnetickým poľom. Pri výbere zodpovedajúceho fyzikálneho systému rovníc sme vychádzali z modelu opisujúceho tzv. Ekmanovu vrstvu, ktorá sa rozprestiera pozdĺž horizontálnych mechanických hraníc. V tomto modeli jedinú nelinearitu predstavuje tzv. modifikovaná Taylorova podmienka, ktorá bola odvodená J.B. Taylorom v práci [Taylor63].



Obr. 4.1 Rotujujúca vrstva v azimutálnom magnetickom poli

Základné polia sú reprezentované nulovým rýchlostným rýchlostným poľom, azimutálnym magnetickým poľom a nehomogénne stratifikovaným teplotným poľom s parabolickým profilom stratifikácie, t.j.

$$\boldsymbol{U}_0 = \boldsymbol{0}, \qquad \boldsymbol{B}_0 = B_M \frac{s}{d}\,\hat{\phi}, \qquad T_0 = T_l - \Delta T\,\frac{z}{d}\left(1 - \frac{z-d}{2\,z_M^* - d}\right).$$

Rešpektujúc cylindrickú geometriu modelu budeme všetky veličiny vyjadrovať v cylindrických súradniciach - výška $z \in (0, d)$, polomer $s \in (0, s_n)$ a azimutálny uhol $\phi \in (0, 2\pi)$. Jednotkové vektory budú označené ako $\hat{z}, \hat{s}, \hat{\phi}$. Výchylky polí od základného stavu označíme ako $\boldsymbol{u}, \boldsymbol{b}, \tilde{\theta}$.

Pri odvodzovaní modelu sme ďalej uvažovali Bussinesqueovu neinerciálnu aproximáciu a Ekmanovu neviskóznu limitu. Prvý predpoklad vedie na statickú rovnicu zachovania lineárneho momentu a druhý zase k zanedbaniu viskózneho člena v hlavnom objeme kvapaliny. Pri druhom predpoklade je nutné poznamenať, že viskozita hrá úlohu práve v pohraničnej tzv. Ekmanovej vrstve. Efekt viskozity (Ekmanovo nasávanie) sa však premietne do rovníc práve vďaka modifikovanej Taylorovej podmienke (pozri nižšie). Systém riadiacich rovníc pre malé výchylky $\boldsymbol{u}, \boldsymbol{b}, \tilde{\theta}$ od základného stavu sa potom skladá so statickej rovnice rovnováhy Lorentzovej, Coriolisovej a gravitačnej sily, indukčnej rovnice a tepelnej rovnice

$$
\begin{aligned}
\hat{\boldsymbol{z}} \times \boldsymbol{u} &= -\nabla p + \Lambda \left[(\nabla \times s\,\hat{\phi}) \times \boldsymbol{b} + (\nabla \times \boldsymbol{b}) \times s\,\hat{\phi} \right] + R\,\tilde{\theta}\,\hat{\boldsymbol{z}} \\
\partial_t \boldsymbol{b} - \nabla \times (s\,\Omega \hat{\phi} \times \boldsymbol{b}) &= \nabla \times (\boldsymbol{u} \times s\,\hat{\phi}) + \nabla^2 \boldsymbol{b} \\
q \left(\partial_t \tilde{\theta} + (s\,\Omega\,\hat{\phi} \cdot \nabla)\,\tilde{\theta} \right) &= -\boldsymbol{u} \cdot \nabla T_0 + \nabla^2 \tilde{\theta}
\end{aligned}
\tag{4.1}
$$

a podmienok solenoidálnosti polí

$$
\nabla \cdot \boldsymbol{b} = 0, \qquad \nabla \cdot \boldsymbol{u} = 0\ .
$$

Hraničné podmienky zodpovedajú pevným mechanickým, perfektne elektrickým a tepelne vodivým horizontálnym vrstvám

$$
\boldsymbol{u}_z = \tilde{\theta} = \boldsymbol{b}_z = 0, \hat{z} \times \frac{\partial b}{\partial z} = 0 \qquad \text{pre } z = 0, d\ .
$$

Uvedený systém je už v bezrozmernom tvare, pričom R reprezentuje modifikované Rayleghovo číslo, Λ Elsasserovo číslo, E Ekmanovo číslo, q Robertsovo číslo.

Ako už bolo spomenuté, jedinou nelinearitou vstupujúcou do systému riadiacich rovníc je tzv. modifikovaná Taylorova podmienka udávajúca vzťah medzi výchylkou magnetického poľa $\boldsymbol{b}$ a uhlovou rýchlosťou Ω geostrofického toku. Táto závislosť môže byť vyjadrená skrze

$$
\Omega = \frac{\Lambda}{2\pi (2E)^{1/2} s} \int_0^d \int_0^{2\pi} [(\nabla \times \boldsymbol{b}) \times \boldsymbol{b}]_\phi \, d\phi \, dz
$$

t.j. Ω je úmerná vertikálnemu a azimutálnemu priemeru azimutálnej zložky Lorentzovej sily.

Ďalší postup pri analýze systému (3.1) spočíva v dvoch krokoch:

1) analýza linearizovaného problému
2) slabo nelineárnej analýze

1) *Linearizovaný problém* zodpovedá systému rovníc (4.1), v ktorom je uhlová rýchlosť Ω nulová. Riešenie linearizovaného systému rovníc možno hľadať metódou separácie premenných v tvare

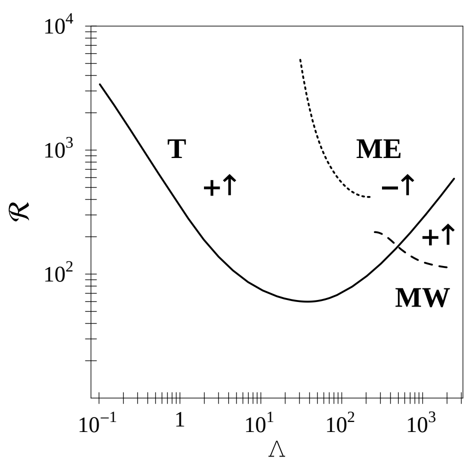
$$\tilde{f}(z,s,\phi,t) = \text{ Re } \{A(\varepsilon^2 t)\, f(z) J_m(ks) \exp\, (im\phi + \lambda t)\},$$

kde symbol f reprezentuje každú z hľadaných funkcií $\boldsymbol{u}, \boldsymbol{b}, \tilde{\theta}$. Ďalej m predstavuje azimutálne vlnové číslo, k radiálne vlnové čislo, $\lambda = i\sigma$ je komplexná frekvencia, A je komplexná amplitúda, J_m je Besselova funkcia prvého druhu m-tého rádu, ε je malý rozvíjajúci parameter, ktorého význam bude zrejmý z nasledovnej slabo nelineárnej analýzy. Linearizovaný problém sa potom dá stručne zapísať v operátorovom tvare

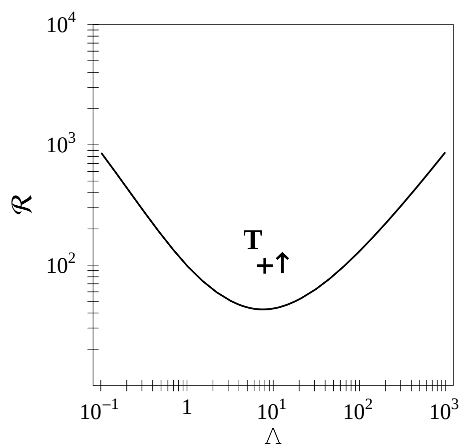
$$L_{R,k,\lambda}\Psi = 0 \tag{4.2}$$

kde vektor $\Psi = (\boldsymbol{u}, \boldsymbol{b}, \tilde{\theta})$ a $L_{R,k,\lambda}$ je nesamoadjungovaný diferenciálny operátor druhého rádu v premennej z.

Štúdiom riešení linearizovaného problému sa zaoberal Ševčík v práci [Sevcik89]. Výsledkom tejto analýzy bolo stanovenie vlastnej funkcie Ψ_1 ako i kritických hodnôt bifurkačných parametrov R_c, k_c, λ_c pre rôzne hodnoty fyzikálnych parametrov Λ, q, m. Nasledovné obrázky závislosti kritického Rayleghovho čísla na Elssaserovom čísle sú prebraté s autorovho článku [Reval99] a ich podrobnejší opis sa dá nájsť v Prílohe 6.3.1.



Obr. 4.2. Kritické hodnoty R_c tepelného, mag. východného a západného módu pre $m = 1, q = 0.005$



Obr. 4.3. Kritické hodnoty R_c tepelného a mag. východného módu pre $m = 1, q = 0.5$

2) *Slabo nelineárna analýza* je zameraná na štúdium riešení v blízkosti stacionárneho riešenia pre hodnoty parametrov R, k, λ blízkym kritickým hodnotám R_c, k_c, λ_c.

Podobne ako v prípade linearizovaného problému (4.2), v ktorom $\Omega = 0$, tak aj plne nelineárny systém rovníc (4.1) sa dá zapísať v abstraktnom tvare

$$L_{R,k,\lambda}\Psi = N(A(\tau), \dot{A}(\tau), \Psi) \; , \tag{4.4}$$

kde $\tau = \varepsilon^2 t$, $\dot{A} = \frac{d}{d\tau}A$ a výraz N obsahuje všetky nelinearity systému (4.1). Ako sme už skôr naznačili, postup slabo nelineárnej analýzy spočíva v hľadaní riešenia systému (4.4) v tvare radu

$$\Psi = \Psi_1 + \varepsilon\Psi_2 + \varepsilon^2\Psi_3 + \ldots \; , \qquad A(\tau) = \varepsilon A_1(\tau) + \varepsilon^2 A_2(\tau) + \varepsilon^3 A_3(\tau) \; .$$

Podobne sa aj parametre R, k, λ sa rozvinú do Taylorovho radu

$$R = R_c + \varepsilon R_1 + \varepsilon^2 R_2 + \ldots \; , \qquad \lambda = \lambda_c + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + \ldots \; .$$

Dosadením týchto rozvojov do systému (4.4) a následným porovnaním koeficientov pri rovnakých mocninách ε dostávame vzťahy medzi jednotlivými koeficientami rozvoja. V prvom ráde ε^1 sa vlastne jedná o linearizovanú rovnicu (4.2). V druhom ráde získame podmienky, ktoré implikujú, že $R_1 = \lambda_1 = 0, \Psi_2 = 0$. Tento fakt sa dá nahliadnuť aj vďaka Z_2 symetrii problému (4.1). Rozhodujúcou je preto podmienka v treťom ráde, kde dostávame obyčajnú diferenciálnu rovnicu pre amplitúdu

$$\frac{d}{dt}A(\varepsilon^2 t) = \alpha(R - R_c)A(\varepsilon^2 t) - \beta|A(\varepsilon^2 t)|^2 A(\varepsilon^2 t) \; . \tag{4.5}$$

Kvadrát **malého parametra** ε sa dá interpretovať ako rozdiel Rayleghovho čísla a kritického Rayleghovho čísla R_c, t.j. $\varepsilon^2 \approx (R - R_c)$. Koeficienty α, β sa dajú analyticky určiť s podmienok riešiteľnosti testovaním s vlastnou funkciou Ψ^+ adjungovaného problému k (4.2). Detaily tohto postupu sa dajú nájsť v prácach [Brest97, Reval97, Reval99].

Význam amplitúdovej rovnice (4.5) spočíva v možnosti analyzovať vznik oscilačných nestabilít pre hodnoty $R > R_c$. Táto rovnica predstavuje normálnu formu pre Hopfovu bifurkáciu periodickej orbity. Na Obr. 4.2 a 4.3 sú znázornené výsledky tejto analýzy. Symbol + resp. − označuje zónu super resp. sub kritickej Hopfovej bifurkácie. Horné a dolné šípky symbolizujú nárast resp. pokles vo frekvencii nelineárnych oscilácii. Viac o fyzikálnych dôsledkoch tejto analýza sa môže čitateľ dozvedieť v prácach [Brest97, Reval97, Reval99].

5 Literatúra

[Aarts95] A. Aarts, A. Van de Ven, *Transient behaviour and instability points of the Poiseuille flow of a KBKZ fluid*, J. of Engineering Mathematics **29** (1995), 371–392.

[Abresch86] U.Abresch, J.Langer, *The normalized curve shortening flow and homothetic solutions*, J. Diff. Geom. **23** (1986), 175–196.

[Alvarez93] L.Alvarez, F.Guichard, P.L.Lions, J.M.Morel, *Axioms and Fundamental Equations of Image Processing*, Archive for Rat.Mech.Anal. **123** (1993), 200–257.

[Alvarez94] L.Alvarez, J.M.Morel, *Formalization and computational aspects of image analysis*, Acta Numerica (1994), 1–59.

[Andr98] B. Andrews, *Evolving convex curves*, Calc. Var. Partial Differ. Equ. **7** (1998), 315–371.

[Angen89] S.B.Angenent, M.E.Gurtin, *Multiphase thermomechanics with an interfacial structure 2. Evolution of an isothermal interface*, Arch. Rat. Mech. Anal. **108** (1989), 323–391.

[Angen90a] S.B.Angenent, *Parabolic equations for curves on surfaces I: Curves with p-integrable curvature.*, Annals of Mathematics **132, No.3** (1990), 451–483.

[Angen90b] S.B. Angenent, *Nonlinear analytic semiflows*, Proc. R. Soc. Edinb., Sect. A **115** (1990), 91–107.

[Angen91a] S.B.Angenent, *On the formation of singularities in the curve shortening flow*, J. Diff. Geom. **33** (1991), 601–633.

[Angen94] S.B.Angenent, M.E.Gurtin, *Anisotropic motion of a phase interface. Well-Posedness of the initial value problem and qualitative properties of the interface*, J. Reine Angew. Math. **446** (1994), 1–47.

[Angen96] S.B.Angenent, M.E.Gurtin, *General contact angle conditions with and without kinetics*, Quarterly of Appl. Math. **54** (1996), 557–569.

[Angen98] S. B. Angenent, G. Sapiro, A. Tannenbaum, *On affine heat equation for non-convex curves*, Journal of the Amer. Math. Soc. **11** (1998), 601–634.

[Bates98] P. Batesm Kening Lu, Chongchum Zeng, *Existence and persistence of invariant manifolds for semiflows in Banach spaces*, Memoirs of AMS, No. 645, 1998.

[Benes98] M. Beneš, K. Mikula, *Simulations of anisotropic motion by mean curvature – comparison of phase field and sharp interface approaches*, Acta Math. Univ. Comenianae **67** (1998), 17–42.

[Boda88] Boďa J., *Thermal and magnetically driven instabilities in a non-constantly stratified fluid layer*, Geophys. Astrophys. Fluid Dynamics **44** (1988), 77–90.

[Brest94] Brestenský J., Ševčík S., *Mean electromotive force due to magneto- convection in rotating horizontal layer with rigid boundaries*, Geophys. Astrophys. Fluid Dynamics **77** (1994), 191–208.

[Brest97] Brestenský J., Revallo M., Ševčovič D., *Finite amplitude magnetoconvection determined by modified Taylor's constraint*, Acta Astron. et Geophys. Univ. Comenianae **18** (1997), 1–18.

[Brun94] P. Brunovský, D. Ševčovič, *Explanation of spurt for a non-Newtonian fluid by a diffusion term*, Quarterly of Appl. Math. **52** (1994), 401–426.

[Cagin90] G. Caginalp, *The dynamics of a conserved phase field system: Stefan-like, Hele-Shaw, and Cahn-Hilliard models as asymptotic limits*, IMA J. Appl. Math. **44** (1990), 77–94.

[Casse97] V. Caselles, R. Kimmel, G. Sapiro, C. Sbert, *Minimal surfaces based object segmentation*, IEEE Trans. Pattern Analysis and Machine Intelligence **19** (1997), 394–398.

[Chow88] S.-N.Chow, K.Lu, *Invariant manifolds for flows in Banach spaces*, J. of Differential Equations **74** (1988), 285–317.

[Deck97] K. Deckelnick, *Weak solutions of the curve shortening flow*, Calc. Var. Partial Differ. Equ. **5** (1997), 489–510.

[Dziuk94] G.Dziuk, *Convergence of a semi discrete scheme for the curve shortening flow*, M^3AS **4, No. 4** (1994), 589–606.

[Dziuk99] G. Dziuk, *Discrete anisotropic curve shortening flow*, SIAM J. Numer. Anal. **36** (1999), 1808–1830.

[Elliot96] C. M. Elliott, M. Paolini, R. Schätzle, *Interface estimates for the fully anisotropic Allen–Cahn equation and anisotropic mean curvature flow*, Mathematicals Models and Methods in Applied Sciences **6** (1996), 1103–1118.

[Fearn94a] Fearn D.R. (1994), *Nonlinear Planetary Dynamos, M.R.E. Proctor and A.D. Gilbert eds. Lectures on Solar and Planetary Dynamos*, Cambridge University Press.

[Fearn94b] Fearn D.R., Proctor M.R.E., Sellar C.C., *Nonlinear magnetoconvection in a rapidly rotating sphere and Taylor's constraint*, Geophys. Astrophys. Fluid Dynamics **77** (1994), 111–132.

[Foias88] C.Foias, G.R. Sell, R. Temam, *Inertial manifolds for nonlinear evolutionary equations*, J. of Differential Equations **73** (1988), 309–353.

[Gage86] M.Gage, R.S.Hamilton, *The heat equation shrinking convex plane curves*, J. Diff. Geom. **23** (1986), 285–314.

[Girao94] P. M. Girão, R. V. Kohn, *Convergence of a crystalline algorithm for the heat equation in one dimension and for the motion of a graph by weighted curvature*, Numer. Math. **67** (1994), 41–70.

[Girao95] P. M. Girão, *Convergence of a crystalline algorithm for the motion of a simple closed convex curve by weighted curvature*, SIAM J. Numer. Anal. **32** (1995), 886–899.

[Gray87] M.Grayson, *The heat equation shrinks embedded plane curves to round points*, J. Diff. Geom **26** (1987), 285–314.

[Grob94] M. Grob, *Flow instabilities in a polymer extrusion process*, PhD theses, Eidhoven Institute of Technology, 1994.

[Gurtin93] M. Gurtin, *Thermomechanics of Evolving Phase Boundaries in the Plane*, Clarendon Press, Oxford, 1993.

[Handl98] A. Handlovičová, K. Mikula, A. Sarti, *Numerical solution of parabolic equations related to level set formulation of mean curvature flow*, Computing and Visualization in Science **1** (1998), 179–182.

[Henry81] D.Henry, *Geometric theory of semilinear parabolic equations. Lecture Notes in Math. 840*, Springer Verlag, Berlin, New York, Heidelberg, 1981.

[Kass87] M.Kass, A.Witkin, D.Terzopoulos, *Snakes: active contours models*, Proc. First International Computer Vision Conference (1987), 259–269.

[Kim94] M. Kimura, *Accurate numerical scheme for the flow by curvature*, Appl. Math. Letters **7** (1994), 69–73.

[Kim97] M. Kimura, *Numerical analysis for moving boundary problems using the boundary tracking method*, Japan J. Indust. Appl. Math. **14** (1997), 373–398.

[Kvasz99] L. Kvasz, *Gramatika zmeny*, Chronos, Bratislava, 1999.

[Lim89] F.J. Lim, W.R. Schowalter, *Wall slip of narrow molecular weight distribution polybutadienes*, J. of Rheology **33** (1989), 1359–1382.

[Malkus91] D.S. Malkus, J.A. Nohel, B.J. Plohr, *Analysis of new phenomena in shear flow of non-Newtonian fluids*, SIAM J. Appl. Math. **51** (1991), 899–929.

[Malkus93] D.S. Malkus, J.A. Nohel, B.J. Plohr, *Approximation of piston-driven flows of a non-Newtonian fluid*, Control Theory, Dynamical Systems, and Geometry of Dynamics. Elworthy, Everitt, and Lee, Eds., Marcel Dekker (1993), 173–192.

[Marion89] M.Marion, *Inertial manifolds associated to partly dissipative reaction - diffusion systems*, J. of Math. Anal. Appl. **143** (1989), 295–326.

[Mik96] K.Mikula, J.Kačur, *Evolution of convex plane curves describing anisotropic motions of phase interfaces*, SIAM J. Sci. Comput. **17, No. 6** (1996).

[Mik97] K.Mikula, *Solution of nonlinear curvature driven evolution of plane convex curves*, Applied Numerical Mathematics **21** (1997), 1–14.

[Mik99] K. Mikula, D. Ševčovič, *Solution of nonlinearly curvature driven evolution of plane curves*, Applied Numerical Mathematics **31** (1999), 191–207.

[Mik00] K. Mikula, D. Ševčovič, *Evolution of plane curves driven by a nonlinear function of curvature and anisotrophy*, SIAM J. Appl. Math., to appear.

[Mish94] Mishchenko E.F., Kolesov Yu.,S., Kolesov A.,Yu., Rozov N.K., *Asymptotic methods in singularly perturbed systems*, Consultants Bureau, New York & London, 1994.

[Moiss98] L. Moissan, *Affine plane curve evolution: A fully consistent scheme*, IEEE Transactions on Image Processing **7** (1998), 411–420.

[Mora89] X.Mora, J.Solà-Morales,, *The singular limit dynamics of semilinear damped wave equation*, J. of Differential Equations **78** (1989), 262–307.

[Nohel90] J.A. Nohel, R.L. Pego, A.E. Tzavaras, *Stability of discontinuous steady states in shearing motion of a non-newtonian fluid*, Proc. of the Royal Society of Edinburgh **115A** (1990), 39–59.

[Nohel93] J.A. Nohel, R.L. Pego,, *Nonlinear stability and asymptotic behavior of shearing motions of a non-Newtonian fluid*, SIAM J. of Math. Anal. **24** (1993), 911–942.

[Osher88] S.Osher, J.Sethian, *Fronts propagating with curvature dependent speed: algorithm based on Hamilton-Jacobi formulation*, J. Comput. Phys. **79** (1988), 12–49.

[Reval97] Revallo M., Ševčovič D., Brestenský J., *Analysis of the model of magnetoconvection with nonlinearity due to modified Taylor's constraint*, Acta Astron. et Geophys. Univ. Comenianae **19** (1997), 317–335.

[Reval99] Revallo M., Ševčovič D., Brestenský J, S.Ševčík, *Viscously controlled nonlinear magnetoconvection in a non-uniformly stratified horizontal fluid layer*, Physics of the Earth and Planetary interiors **111** (1999), 83-92.

[Sap94] G.Sapiro, A.Tannenbaum, *On affine plane curve evolution*, J.Funct.Anal. **119, No. 1** (1994), 79–120.

[Schm96] A.Schmidt, *Computation of three dimensional dendrites with finite elements*, J. Comput. Phys. **125, No. 2** (1996), 293–312.

[Sevcik89] Ševčík S., *Thermal and magnetically driven instabilities in a non-constantly stratified rapidly rotating fluid layer with azimuthal magnetic field*, Geophys. Astrophys. Fluid Dynamics **49** (1989), 195–211.

[Sevco91] D. Ševčovič, *Limiting behavior of global attractors for singularly perturbed beam equations*, Comment. Math. Univ. Carolinae **32** (1991), 45–60.

[Sevco94] D. Ševčovič, *Limiting behaviour of invariant manifolds for a system of singularly perturbed evolution equations*, Math. Methods in the Appl. Sci. **17** (1994), 643–666.

[Sevco95] D. Ševčovič, *The C^1 stability of slow manifolds for a system of singularly perturbed evolution equations*, Comment. Math. Univ. Carolinae **36** (1995), 89–107.

[Sevco97a] D. Ševčovič, *Smoothness of the singular limit of inertial manifolds of singularly perturbed evolution equations*, Nonlinear Analysis: TMA. **26** (1997), 199–215.

[Sevco97b] D. Ševčovič, *Dissipative feedback synthesis for a singularly perturbed model of a piston driven flow of a non-Newtonian fluid*, Math. Methods in the Appl. Sci. **20** (1997), 79–94.

[Sevco99] D. Ševčovič, *Analysis of the free boundary for the Black-Scholes equation*, to appear in: Euro Journal of Applied Mathematics.

[Seth90] J. Sethian, *Numerical algorithm for propagating interfaces: Hamilton-Jacobi equations and conservation laws*, J. Differential. Geom. **31** (1990), 131–161.

[Seth96] J. Sethian, *Level Set Methods. Evolving Interfaces in Geometry, Fluid Mechanics, Computer Vision, and Material Science*, Cambridge University Press, 1996.

[Skin88] Skinner P.H., A.M. Soward A.M., *Convection in a rotating magnetic system and Taylor's constraint*, Geophys. Astrophys. Fluid Dynamics **44** (1988), 91–116.

[Skin91] Skinner P.H., Soward A.M., *Convection in a rotating magnetic system and Taylor's constraint II*, Geophys. Astrophys. Fluid Dynamics **60** (1991), 335–356.

[Sow79] Soward A.M., *Thermal and magnetically driven convection in a rapidly rotating fluid layer*, J. Fluid Mech. **90** (1979), 669–684.

[Sow86] Soward A.M., *Non-linear marginal convection in a rotating magnetic system*, Geophys. Astrophys. Fluid Dynamics **35** (1986), 329–371.

[Stam99] R. Stamicar, D. Ševčovič, J. Chadam, *The early exercise boundary for the American put near expiry: numerical approximation*, submitted (1999).

[Svir90] G.A.Sviridyuk, T.G.Sukacheva, *Cauchy problem for a class of semilinear equations of Sobolev type*, Sibirskii Matem. Zhurnal **31** (1990), 120–127.

[Taylor63] Taylor J.B., *The magnetohydrodynamics of a rotating fluid and the Earth's dynamo problem*, Proc. R. Soc. London **A274** (1963), 274–283.

[Ush00] T. Ushijima, S. Yazaki, *Convergence of a crystalline algorithm for the motion of a closed convex curve by a power of curvature* $V = K^\alpha$, SIAM J. Numer. Anal. **37** (2000), 500–522.

[Vander87] A.Vanderbauwhede, S.A.Van Gils, *Center manifolds and contraction on a scale of Banach spaces*, J. of Funct. Analysis **72** (1987), 209–224.

[Vinog72] G.Vinogradov, A.Malkin, Yu.Yanovskii, E.Borisenkova, B.Yarlykov and G.Berezhnaya, *Viscoelastic properties and flow of narrow distribution polybutadienes and polyisoprenes*, J. Polymer Sci., Part A-2 **10** (1972), 1061-1084.

6 Prílohy

Príloha 6.1.1

Reprint práce:

D. Ševčovič: *Smoothness of the singular limit of inertial manifolds of singularly perturbed evolution equations.* Nonlinear Analysis: TMA, 28 (1997), 199-215.

Pergamon

Nonlinear Analysis, Theory, Methods & Applications, Vol. 28, No. 1, pp. 199–215, 1997

0362-546X/96 \$15.00 + 0.00

0362-546X(95)00139-5

SMOOTHNESS OF THE SINGULAR LIMIT OF INERTIAL MANIFOLDS OF SINGULARLY PERTURBED EVOLUTION EQUATIONS

DANIEL ŠEVČOVIČ

Institute of Applied Mathematics, Comenius University, Mlynská dolina, 842 15, Bratislava, Slovak Republic

(*Received* 23 *May* 1994; *received for publication* 7 *July* 1995)

Key words and phrases: Inertial manifolds, smoothness of the singular limit, constitutive models of shearing motions.

1. INTRODUCTION

The aim of this paper is to investigate the singular limit behavior of inertial manifolds of the following singularly perturbed system of evolution equations in Banach spaces

$$\begin{aligned} U_t &= G_\varepsilon(U, S) \\ \varepsilon S_t + AS &= F_\varepsilon(U, S), \end{aligned} \tag{1.1}$$

where $\varepsilon \geq 0$ is a small parameter, X, Y are Banach spaces, A is a sectorial operator in a Banach space Y, Y^α is the fractional power space and $F_\varepsilon: X \times Y^\alpha \to Y$; $G_\varepsilon: X \times Y^\alpha \to X$; are smooth bounded functions, $\alpha \in [0, 1)$, $F_\varepsilon \to F_0$, $G_\varepsilon \to G_0$ as $\varepsilon \to 0^+$. It is well known that the above system of equations generates a C^1 semi-flow $\mathcal{S}_\varepsilon$ in the phase-space $X \times Y^\alpha$ for any $\varepsilon > 0$ (cf. Henry [1]). According to Marion [2] the semi-flow $\mathcal{S}_\varepsilon$ possesses an invariant attracting manifold $\mathfrak{M}_\varepsilon$ (inertial manifold) provided that the Lipschitz constant of F_ε is sufficiently small. This manifold can be constructed as a Lipschitz continuous graph over the Banach space X, i.e. $\mathfrak{M}_\varepsilon = \{(U, \Phi_\varepsilon(U)), U \in X\}$ (see [2]). From the results due to Chow and Lu [3] it follows that $\mathfrak{M}_\varepsilon$ is a C^k manifold whenever F and G are C^k bounded functions. Notice that, in contrast to the usual definition of an inertial manifold (see, e.g. [4]), we allow $\mathfrak{M}_\varepsilon$ to be an infinite dimensional Banach manifold.

In the geometric singular perturbation theory much effort is being spent in order to justify the continuity of the singular limit ε tends to 0^+ (see, e.g. Sviridyuk and Sukacheva [5]). The purpose of this paper is to examine the smoothness of the singular limit behavior of inertial manifolds $\mathfrak{M}_\varepsilon$ as $\varepsilon \to 0^+$. The main goal is to show that, for small values of $\varepsilon > 0$, the inertial manifold $\mathfrak{M}_\varepsilon$ is C^1 close to the manifold $\mathfrak{M}_0 = \{(U, S), AS = F_0(U, S)\}$ corresponding to the quasi-dynamic problem $U_t = G_0(U, S)$ with the constraint $AS = F_0(U, S)$. Notice that the C^1 stability of inertial or centre unstable manifolds is a useful tool in the theory of Morse–Smale vector fields (cf. Mora and Solà-Morales [6]). We hope that C^1 stability result can be also applied in the theory of linearization at a steady state like, e.g. extension of the Hartman–Grobman lemma from the reduced problem, $\varepsilon = 0$ to the perturbed system with $\varepsilon > 0$ small enough. Neverthless, such applications of the results obtained are not discussed here.

The idea of construction of an inertial manifold for (1.1) is based on the well-known Lyapunov–Perron method of integral equations. This method is combined with a nonlocal approach using the graph transform which is applied to solutions of the singularly perturbed equation in (1.1). We then seek an inertial manifold as the union of all solutions of (1.1) growing exponentially at $-\infty$. By contrast to the usual functional space setting (see, e.g. Chow and Lu [3], Foias *et al.* [4] or Miklavčič [7]) an essential rôle is played by better smoothing properties of integral kernels enabling us to operate with Hölderian spaces of curves instead of usual continuous ones. We first study the singularly perturbed equation $\varepsilon S_t + AS = F_\varepsilon(U, S)$ and prove that there is a solution operator $S = \phi_\varepsilon(U)$ in the space of globally defined solutions. It, however, turns out that the derivative of this mapping becomes continuous at $\varepsilon = 0$ only when the mapping ϕ_ε operates on the space of Hölder continuous curves growing exponentially at $-\infty$ (see lemma 3.2). To construct an attractive invariant manifold $\mathcal{M}_\varepsilon$ as a C^1 graph of $\Phi_\varepsilon : X \to Y^\alpha$ we then apply the method of integral equations to the equation $U_t = G_\varepsilon(U, \phi_\varepsilon(U))$. In order to prove that $\mathcal{M}_\varepsilon$ is C^1 close to $\mathcal{M}_0$ for $0 < \varepsilon \ll 1$ we make use of the two parameter contraction principle due to Mora and Solà-Morales [6, theorem 5.1] covering differentiability and continuity of a family of nonlinear mappings operating between a pair of Banach spaces.

[redacted] (1.1) is a semilinear equation $U_t + BU = G(U, S)$ and the nonlinearity F only depends on the U-variable. The last assumption makes the analysis of the singularly perturbed equation considerably easier. The results obtained in [8] are not capable to cover some applied problems like, e.g. a flow of viscous media governed by a constitutive equation of differential type. Such an application is discussed in Section 4 of this paper.

The paper is organized as follows. In Section 2 we recall some useful results regarding properties of functional spaces of Hölder continuous curves growing exponentially at $-\infty$. In Section 3 we prove that $\Phi_\varepsilon \to \Phi_0$ in the C^1 topology as $\varepsilon \to 0^+$. The main result of this paper is contained in theorem 3.9. Section 4 is devoted to an application of the results obtained to a singular perturbation problem arising in the study of the so-called Johnson–Segalman–Oldroyd model of shearing motions of a non-Newtonian fluid. Following the paper by Malkus *et al.* [9] the motions of the channel Poisseule flow of a highly elastic and very viscous fluid (like, e.g. a polymer) can be described, in a satisfactory manner, by a system of parabolic-hyperbolic equations of the form

$$\begin{aligned} \varepsilon v_t - v_{xx} &= \sigma_x + f \\ \sigma_t &= -\sigma + (1 + z)v_x \\ z_t &= -z - \sigma v_x, \end{aligned} \tag{1.2}$$

where $v = v(t, x)$, $x \in [0, 1]$, is the velocity of the channel flow between two parallel plates, σ is the extra shear stress, z is the difference of normal stresses, f is the pressure gradient driving the flow. The number $\varepsilon > 0$ is proportional to the ratio of the Reynolds number and Deborah number and according to rheological experiments due to Vinogradov *et al.* [10] this number is very small, of the order of magnitude $O(10^{-12})$. It gives rise to the inertialess approximation $\varepsilon = 0$. Based on such an approximation, Malkus *et al.* [9] were able to explain several striking phenomena like spurt, hysteresis, shape memory and latency observed in rheological experiments (see also [11, 12]). Using the new variable $S := v_x + \sigma + fx$, S is the total stress tensor, system (1.2) can be rewritten in the general form (1.1) with $F_\varepsilon = O(\varepsilon)$ as $\varepsilon \to 0^+$. In [13], Nohel and Pego have justified the inertialess approximation by a clever application of the Morse–Conley theory. They proved that any solution of (1.2) converges pointwise for $x \in [0, 1]$

to a solution of the inertialess approximation as $\varepsilon \to 0^+$. The purpose of Section 4 is to give another justification of the inertialess approximation by means of C^1 closeness of infinite dimensional inertial manifolds. It is hoped that the C^1 stability result of inertial manifolds can be also applied to the problem of a piston driven flow studied recently by Malkus *et al.* [11]. Based on careful numerical simulations, their results indicate the Hopf bifurcation phenomenon in a piston driven Johnson–Segalman–Oldroyd fluid. Any information about C^1 stability can be a useful tool in order to prove that the Hopf bifurcation extends to the full system of governing equations with $\varepsilon > 0$ sufficiently small.

2. PRELIMINARIES

As usual, for Banach spaces E_1, E_2 and $\eta \in (0, 1]$ we denote $C^k_{bdd}(E_1, E_2)$ the Banach space consisting of the mappings $F: E_1 \to E_2$ which are k-times Fréchet differentiable and such that $F, \ldots, D^kF$ are bounded and uniformly continuous, the norm being given by $\|F\|_k := \sum_{i=0}^k \sup|D^iF|$. $C^{k+\eta}_{bdd}(E_1, E_2)$ will denote the Banach space consisting of the mappings $F \in C^k_{bdd}(E_1, E_2)$ such that D^kF is η-Hölder continuous, the norm being given by

$$\|F\|_{k,\eta} := \|F\|_k + \sup_{\substack{x \neq y \\ x,y \in E_1}} \frac{\|D^kF(x) - D^kF(y)\|}{\|x - y\|^\eta}.$$

Let $\mathfrak{X}$ be a Banach space and $\mu \in R$. Following the notation of [3, 6, 7] we denote

$$C^-_\mu(\mathfrak{X}) := \left\{ u : C((-\infty, 0], \mathfrak{X}), \text{ and } \|u\|_{C^-_\mu(\mathfrak{X})} := \sup_{t \le 0} e^{\mu t}\|u(t)\|_{\mathfrak{X}} < \infty \right\}.$$

The linear space $C^-_\mu(\mathfrak{X})$ endowed with the norm $\|\cdot\|_{C^-_\mu(\mathfrak{X})}$ is a Banach space. If $\mu \le \nu$ then the embedding operator $J_{\mu,\nu}: C^-_\mu(\mathfrak{X}) \to C^-_\nu(\mathfrak{X})$ is continuous and $\|J_{\mu,\nu}\| \le 1$.

For any $\rho \in (0, 1]$, $a \in (0, 1]$ and $\mu \ge 0$, we furthermore denote

$$C^-_{\mu,\rho,a}(\mathfrak{X}) = \left\{ u \in C^-_\mu(\mathfrak{X}); [u]_{\mu,\rho,a} = \sup_{\substack{t \le 0 \\ h \in (0,a]}} \frac{\|e^{\mu t}u(t) - e^{\mu(t-h)}u(t-h)\|}{h^\rho} < \infty \right\}.$$

Let

$$\|u\|_{C^-_{\mu,\rho,a}}(\mathfrak{X}) := \|u\|_{C^-_\mu(\mathfrak{X})} + [u]_{\mu,\rho,a} \qquad \text{for any } u \in C^-_{\mu,\rho,a}(\mathfrak{X}).$$

The space $C^-_{\mu,\rho,a}(\mathfrak{X})$ endowed with the norm $\|\cdot\|_{C^-_{\mu,\rho,a}}$ is a Banach space continuously embedded into $C^-_\mu(\mathfrak{X})$ with an embedding constant equal to 1. Furthermore, the space $C^-_{\mu,\rho,a}(\mathfrak{X})$ is continuously embedded into $C^-_{\nu,\rho,a}(\mathfrak{X})$ for any $0 \le \mu \le \nu$ and $\rho \in (0, 1]$, its embedding constant being less or equal to $\max\{1, (\nu - \mu)a^{1-\rho}\}$ (see [8]).

Let E_1, E_2 be Banach spaces and $F: E_1 \to E_2$ be a bounded and Lipschitz continuous mapping, E_1, E_2 be Banach spaces. Denote

$$\tilde{F}: C^-_\mu(E_1) \to C^-_\mu(E_2)$$

a mapping defined as $\tilde{F}(u)(t) := F(u(t))$ for any $t \le 0$ and $u \in C^-_\mu(E_1)$. By [6, lemma 5.1], for every $\mu \ge 0$, the mapping $\tilde{F}$ is bounded and Lipschitzian with $\sup|\tilde{F}| \le \sup|F|$ and $\mathrm{Lip}(F) \le \mathrm{Lip}(F)$. If $F: E_1 \to E_2$ is Fréchet differentiable then $\tilde{F}: C^-_\mu(E_1) \to C^-_\mu(E_2)$ need not be necessarily differentiable. Nevertheless, the following result holds.

LEMMA 2.1 [8, lemma 2.8, 14, lemma 5]. If $F: E_1 \to E_2$ is Fréchet differentiable with $DF: E_1 \to L(E_1, E_2)$ bounded and uniformly continuous, then, for every $\nu > \mu$, $\nu > 0$, the mapping $\tilde{F}: C_\mu^-(E_1) \to C_\nu^-(E_2)$ ($\tilde{F}: C_{\mu,p,a}^-(E_1) \to C_\nu^-(E_2)$) is Fréchet differentiable, its derivative being given by $D\tilde{F}(u)h = DF(u(\cdot))h(\cdot)$ and $D\tilde{F}: C_\mu^-(E_1) \to L(C_\mu^-(E_1), C_\nu^-(E_2))$ ($D\tilde{F}: C_{\mu,p,a}^-(E_1) \to L(C_{\mu,p,a}^-(E_1), C_\nu^-(E_2))$) is bounded and uniformly continuous.

Throughout Sections 2 and 3 we adopt the following hypothesis

$$
\text{(H)} \qquad \begin{cases} X, Y \text{ are real Banach spaces:} \\ A \text{ is a sectorial operator in } Y, \operatorname{Re}\sigma(A) > \omega > 0; \\ \text{there exist } \alpha \in [0,1) \text{ and } \eta \in (0,1) \text{ such that} \\ G_\varepsilon \in C_{bdd}^1(X \times Y^\alpha; X),\ F_\varepsilon \in C_{bdd}^{1+\eta}(X \times Y^\alpha, Y) \text{ for any } \varepsilon \in [0, \varepsilon_0]; \\ F_\varepsilon \to F_0,\ G_\varepsilon \to G_0 \text{ as } \varepsilon \to 0^+ \text{ in the respective topologies.} \end{cases}
$$

We refer to [1, Chapter 1] for the definition of a sectorial operator, fractional power spaces Y^α, $\alpha \geq 0$, and their basic properties. We denote $\|\cdot\|_\alpha$ the norm in Y^α given by $\|u\|_\alpha = \|A^\alpha u\|$, $u \in Y^\alpha = D(A^\alpha)$.

By a globally defined solution of (1.1), $\varepsilon > 0$, with initial data $(U_0, S_0) \in X \times Y^\alpha$ we mean a function $(U(\cdot), S(\cdot)) \in C([0,T]; X \times Y^\alpha) \cap C^1((0,T); X \times Y^\alpha)$ for any $T > 0$ such that $(U(0), S(0)) = (U_0, S_0)$; $U(t), S(t)) \in X \times D(A)$ for $t > 0$ and $(U(\cdot), S(\cdot))$ satisfies (1.1) for any $t > 0$. The global existence and uniqueness of solutions of (1.1), for initial data belonging to the phase-space $X \times Y^\alpha$ follow from [1, theorems 3.3.3 and 3.3.4].

In case the function F_0 satisfies the condition $\|D_S F_0\| \|A^{\alpha-1}\| < 1$ the set $\mathfrak{M}_0 = \{(U,S), AS = F_0(U,S)\}$ is an embedded Banach manifold in $X \times Y^\alpha$. More precisely, there is a C_{bdd}^1-function $\Phi_0: X \to Y^\alpha$ such that

$$\mathfrak{M}_0 = \{(U, \Phi_0(U)) \in X \times Y^\alpha, U \in X\}. \tag{2.1}$$

By a solution of (1.1), $\varepsilon = 0$, we mean a function $U \in C([0,T]; X) \cap C^1((0,T); X)$ for any $T > 0$, $U(0) = U_0$ and $U(\cdot)$ satisfies the equation $U_t = G_0(U, \Phi_0(U))$. Since G_0 is assumed to be Lipschitz continuous the global existence and uniqueness of solutions to (1.1) with $\varepsilon = 0$ is again assured by the above references to Henry's lecture notes.

In summary, we have shown that the system $(1.1)_\varepsilon$, $\varepsilon > 0$ generates a semi-flow $\mathcal{S}_\varepsilon(t)$, $t \geq 0$; $\mathcal{S}_\varepsilon(t)(U_0, S_0) = (U(t), S(t))$, on the phase-space $X \times Y^\alpha$. The system $(1.1)_0$ defines a semi-flow $\mathcal{S}_0(t)$, $t \geq 0$, $\mathcal{S}_0(t)(U_0, \Phi_0(U_0)) = (U(t), \Phi_0(U(t)))$, on the embedded manifold $\mathfrak{M}_0 \subset X \times Y^\alpha$.

3. EXISTENCE AND SMOOTHNESS OF THE SINGULAR LIMIT OF INVARIANT MANIFOLDS

Before proving the existence and smoothness of the singular limit of inertial manifolds of (1.1) we need several auxiliary lemmas. First, let us examine solutions of the following linear equation

$$\varepsilon S_t + AS = f \tag{3.1$_\varepsilon$}$$

belonging to the space $C_{\nu,p,a}^-(Y^\alpha)$. We will also study the limiting case of $(3.1)_\varepsilon$ when $\varepsilon = 0$, i.e.

$$AS = f \tag{3.1$_0$}$$

and examine behavior of solutions when $\varepsilon \to 0^+$.

Denote by $\mathfrak{X}_\nu$ and $\mathfrak{X}_{\nu,\rho}$, $\nu > 0$, $0 < \rho \le 1$, $a \in (0, 1]$ the following Banach spaces of bounded linear operators

$$\mathfrak{X}_\nu = L(C_\nu^-(Y), C_\nu^-(Y^\alpha)), \qquad \mathfrak{X}_{\nu,\rho} = L(C_{\nu,\rho,a}^-(Y), C_\nu^-(Y^\alpha)). \tag{3.2}$$

LEMMA 3.1 [8, lemma 3.1]. Assume that the operator A fulfils the hypothesis (H). Then, for any $\varepsilon \in [0, \varepsilon_0]$, $0 < \nu < \omega\varepsilon_0^{-1}$, and $f \in C_\nu^-(Y)$ there is the unique solution $S \in C_\nu^-(Y^\alpha)$ of $(3.1)_\varepsilon$ given by $S = L_\varepsilon f$, where

$$L_\varepsilon f(t) = \frac{1}{\varepsilon}\int_{-\infty}^{t} \exp(-A(t-s)/\varepsilon) f(s)\,\mathrm{d}s, \quad (\varepsilon > 0); \qquad L_0 f(t) = A^{-1}f(t) \quad (\varepsilon = 0).$$

for $t \le 0$. The linear operator L_ε belongs to the space $\mathfrak{X}_\nu$ as well as to $\mathfrak{X}_{\nu,\rho}$, $0 < \rho \le 1$, and there is a $K_0 > 0$ such that $\|L_\varepsilon\|_{\mathfrak{X}_{\nu,\rho}} \le \|L_\varepsilon\|_{\mathfrak{X}_\nu} \le K_0(\omega - \nu\varepsilon_0)^{\alpha-1}$ for any $\varepsilon \in [0, \varepsilon_0]$, $0 < \nu\varepsilon_0 < \omega$. Moreover, $L_\varepsilon \to L_0$ as $\varepsilon \to 0^+$ in the space $\mathfrak{X}_{\nu,\rho}$, $0 < \rho \le 1$.

LEMMA 3.2. Let $0 < (1+\eta)\mu \le \kappa < \omega\varepsilon_0^{-1}$, $0 < \rho \le 1$ and $0 < a \le 1$. Assume that there is a $\theta < 1$ such that $\|L_\varepsilon\|_{\mathfrak{X}_\mu}\|D_S F_\varepsilon(U, S)\|_{L(Y^\alpha, Y)} \le \theta$ for any $U \in X$, $S \in Y^\alpha$ and $\varepsilon \in [0, \varepsilon_0]$. Then, for any $U \in C_\mu^-(X)$ there is the unique solution $S = \phi_\varepsilon(U) \in C_\mu^-(Y^\alpha)$ of the equation $S = L_\varepsilon F_\varepsilon(U, S)$. Moreover, there exists a $K_1 > 0$ such that, for any $\varepsilon \in [0, \varepsilon_0]$,

(i) $\|\phi_\varepsilon(U_1) - \phi_\varepsilon(U_2)\|_{C_{\mu,(Y^\alpha)}^-} \le \|L_\varepsilon\|_{\mathfrak{X}_\mu}\|F_\varepsilon\|_1(1-\theta)^{-1}\|U_1 - U_2\|_{C_\mu^-(X)}$;

(ii) $\lim_{\varepsilon\to 0^+}\phi_\varepsilon(U) = \phi_0(U)$ in $C_\mu^-(Y^\alpha)$ uniformly w.r. to $U \in \mathcal{B}$, where $\mathcal{B}$ is an arbitrary bounded subset of $C_{\mu,\rho,a}^-(X)$;

(iii) $\phi_\varepsilon \in C_{bdd}^1(C_\mu^-(X), C_\kappa^-(Y^\alpha))$, $\|\phi_\varepsilon\|_1 \le K_1$ and there is a $d\phi_\varepsilon \in L(C_\mu^-(X), C_\mu^-(Y^\alpha))$ with the property $D\phi_\varepsilon = J_{\mu,\kappa}d\phi_\varepsilon$, $\|d\phi_\varepsilon\| \le \|L_\varepsilon\|_{\mathfrak{X}_\mu}\|F_\varepsilon\|_1(1-\theta)^{-1}$;

(iv) $\lim_{\varepsilon\to 0^+}\phi_\varepsilon = \phi_0$ in $C_{bdd}^1(\mathcal{B}, C_\kappa^-(Y^\alpha))$ for any bounded and open subset $\mathcal{B}$ of $C_{\mu,\rho,a}^-(X)$.

Remark 3.3. It follows from the proof of [8, lemma 3.1] that $\|L_\varepsilon - L_0\|_{\mathfrak{X}_{\mu,\rho}} = O(\varepsilon^r)$ as $\varepsilon \to 0^+$ for any $0 < r < 1$. The author was able to prove neither C^1 differentiability nor Lipschitz continuity of L_ε with respect to ε at $\varepsilon = 0$.

Remark 3.4. We remind ourselves that in the case $\varepsilon = 0$ the mapping Φ_0 defined in (2.1) coincides with ϕ_0 in the sense that $\phi_0(U)(t) = \Phi_0(U(t))$ for any $U \in C_\mu^-(X)$ and $t \le 0$.

Proof of lemma 3.2. Under the assumption $\|L_\varepsilon\|_{\mathfrak{X}_\mu}\|D_S F_\varepsilon\|_{L(Y^\alpha, Y)} \le \theta < 1$ the existence of the solution operator $S = \phi_\varepsilon(U)$ as well as its Lipschitz continuity (i) follows from the parameterized contraction principle.

To prove (ii), we first find an estimate of the norm of $\|\phi_0(U)\|_{C_{\mu,\rho,a}^-(Y^\alpha)}$ in terms of $U \in C_{\mu,\rho,a}^-(X)$. To this end, we put $S(t) = \phi_0(U)(t)$. Then, for any $t \le 0$, $h \in (0, a]$, we have

$$\begin{aligned} e^{\mu t}S(t) - e^{\mu(t-h)}S(t-h) &= (e^{\mu t} - e^{\mu(t-h)})A^{-1}F_0(U(t), S(t)) \\ &\quad + e^{\mu(t-h)}A^{-1}(F_0(U(t), S(t)) - F_0(U(t-h), S(t-h))). \end{aligned}$$

Notice that, for any $t \le 0$, $h \in (0, a]$,

$$\begin{aligned} \|W(t) - W(t-h)\|_E &\le e^{-\mu t}\|e^{\mu t}W(t) - e^{\mu(t-h)}W(t-h)\|_E + (1 - e^{-\mu h})\|W(t-h)\|_E \\ &\le K_1 e^{-\mu t}\|W\|_{C_{\mu,\rho,a}^-(E)} h^\rho, \end{aligned} \tag{3.3}$$

where E stands either for X or Y^α and $K_1 = K_1(\mu) > 0$. Thus,

$$\|e^{\mu t}S(t) - e^{\mu(t-h)}S(t-h)\|_{Y^\alpha} \le K_1\|U\|_{C^-_{\mu,\rho,a}(X)}h^\rho + \|A^{\alpha-1}\|\|D_S F_0\|\|S(t) - S(t-h)\|_{Y^\alpha} e^{\mu(t-h)}.$$

Since $\|S\|_{C^-_\mu(Y^\alpha)} \le \|A^{\alpha-1}\|\|F_0\|_0$ and $\|L_0\|_{\mathfrak{X}_\mu}\|D_S F_0\| \le \theta < 1$ the above inequality yields the estimate

$$\|\phi_0(U)\|_{C^-_{\mu,\rho,a}(Y^\alpha)} \le K_1(1 + \|U\|_{C^-_{\mu,\rho,a}(X)}). \tag{3.4}$$

Arguing similarly as above one can show $\|F_0(U, S)\|_{C^-_{\mu,\rho,a}(Y)} \le K_1(1 + \|U\|_{C^-_{\mu,\rho,a}(X)} + \|S\|_{C^-_{\mu,\rho,a}(Y^\alpha)})$. Hence,

$$\|F_0(U, \phi_0(U))\|_{C^-_{\mu,\rho,a}(Y)} \le K_1(1 + \|U\|_{C^-_{\mu,\rho,a}(X)}). \tag{3.5}$$

As $\phi_\varepsilon(U) = L_\varepsilon F_\varepsilon(U, \phi_\varepsilon(U))$ we obtain

$$(1-\theta)\|\phi_\varepsilon(U) - \phi_0(U)\|_{C^-_\mu(Y^\alpha)} \le \|L_\varepsilon - L_0\|_{\mathfrak{X}_{\mu,\rho}}\|F_0(U, \phi_0(U))\|_{C^-_{\mu,\rho,a}(Y)} + \|L_\varepsilon\|_{\mathfrak{X}_\mu}\|F_\varepsilon(U, \phi_0(U)) - F_0(U, \phi_0(U))\|_{C^-_\mu(Y)}.$$

By lemma 3.1, (H) and (3.5) we obtain $\lim_{\varepsilon\to 0^+}\phi_\varepsilon(U) = \phi_0(U)$ in $C^-_\mu(Y^\alpha)$ uniformly w.r. to $U \in \mathcal{B}$ where $\mathcal{B}$ is arbitrary bounded subset of $C^-_{\mu,\rho,a}(X)$.

(iii) For any $U, W \in C^-_\mu(X)$ we denote

$$D\phi_\varepsilon(U)W := [I - L_\varepsilon D_S F_\varepsilon(U(\cdot), \phi_\varepsilon(U)(\cdot))]^{-1}L_\varepsilon D_U F_\varepsilon(U(\cdot), \phi_\varepsilon(U)(\cdot))W. \tag{3.6}$$

A straightforward calculation yields

$$\begin{aligned}\phi_\varepsilon(U+W) - \phi_\varepsilon(U) - D\phi_\varepsilon(U)W &= B_\varepsilon[F_\varepsilon(U+W, \phi_\varepsilon(U)) - F_\varepsilon(U, \phi_\varepsilon(U)) - D_U F_\varepsilon(U, \phi_\varepsilon(U))W] \\ &\quad + B_\varepsilon[F_\varepsilon(U+W, \phi_\varepsilon(U+W)) - F_\varepsilon(U+W, \phi_\varepsilon(U)) \\ &\quad - D_S F_\varepsilon(U, \phi_\varepsilon(U))(\phi_\varepsilon(U+W) - \phi_\varepsilon(U))] =: I_1 + I_2,\end{aligned}$$

where

$$B_\varepsilon := [I - L_\varepsilon D_S F_\varepsilon(U(\cdot), \phi_\varepsilon(U)(\cdot))]^{-1}L_\varepsilon.$$

Obviously, $\|B_\varepsilon\|_{\mathfrak{X}_\nu} \le (1-\theta)^{-1}\|L_\varepsilon\|_{\mathfrak{X}_\nu}$ for $\nu = \mu$ or $\nu = \kappa$, $\varepsilon \in [0, \varepsilon_0]$. Furthermore, by lemma 2.1, we have $\|I_1\|_{C^-_\kappa(Y^\alpha)} = o(\|W\|_{C^-_\mu(X)})$ as $\|W\| \to 0$. On the other hand, as $F_\varepsilon \in C^{1+\eta}_{bdd}$ and $0 < (1+\eta)\mu \le \kappa$ we have

$$\begin{aligned}\|I_2\|_{C^-_\kappa(Y^\alpha)} &= O(\|W\|^\eta_{C^-_\mu} + \|\phi_\varepsilon(U+W) - \phi_\varepsilon(U)\|^\eta_{C^-_\mu})\|\phi_\varepsilon(U+W) - \phi_\varepsilon(U)\|_{C^-_\mu} \\ &= o(\|W\|_{C^-_\mu}).\end{aligned}$$

Hence, $\phi_\varepsilon \in C^1_{bdd}(C^-_\mu(X), C^-_\kappa(Y^\alpha))$; $D\phi_\varepsilon(U)W = J_{\mu,\kappa}d\phi_\varepsilon(U)W$, where the mapping $W \mapsto d\phi_\varepsilon(U)W$ is defined by the right-hand side of (3.6) and so $\|d\phi_\varepsilon\| \le \|L_\varepsilon\|_{X_\mu}\|F_\varepsilon\|_1(1-\theta)^{-1}$.

Finally, we prove the assertion (iv). Let $\mathcal{B} \subset C^-_{\mu,\rho,a}(X)$ be an arbitrary bounded set. With regard to (ii) it is sufficient to show the uniform convergence $D\phi_\varepsilon(U) \to D\phi_0(U)$ as $\varepsilon \to 0^+$ for $U \in \mathcal{B}$. For any $U \in C^-_{\mu,\rho,a}(X)$ we have

$$D\phi_\varepsilon(U) - D\phi_0(U) = (B_\varepsilon - B_0)D_U F_0(U, \phi_0(U)) \\ + B_\varepsilon[D_U F_\varepsilon(U, \phi_\varepsilon(U) - D_U F_0(U, \phi_0(U))].$$

Now one can readily verify that

$$B_\varepsilon - B_0 = B_\varepsilon[D_S F_\varepsilon(U, \phi_\varepsilon(U)) - D_S F_0(U, \phi_0(U))]B_0 \\ + [I - L_\varepsilon D_S F_\varepsilon(U, \phi_\varepsilon(U))]^{-1}(L_\varepsilon - L_0)(I + D_S F_0(U, \phi_0(U))B_0).$$

Furthermore,

$$D_S F_\varepsilon(U, \phi_\varepsilon(U)) - D_S F_0(U, \phi_0(U)) = D_S[F_\varepsilon(U, \phi_\varepsilon(U)) - F_0(U, \phi_\varepsilon(U))] \\ + D_S[F_0(U, \phi_\varepsilon(U)) - F_0(U, \phi_0(U))].$$

Thus,

$$\|D_S F_\varepsilon(U(t), \phi_\varepsilon(U)(t)) - D_S F_0(U(t), \phi_0(U)(t))\|_{L(Y^\alpha, Y)} \\ \leq \|F_\varepsilon - F_0\|_1 + \|F_0\|_{1+\eta}\|\phi_\varepsilon(U)(t) - \phi_0(U)(t)\|^\eta_{Y^\alpha}.$$

Since $0 < (1 + \eta)\mu \leq \kappa$, we obtain

$$\|D_S F_\varepsilon(U, \phi_\varepsilon(U)) - D_S F_0(U, \phi_0(U))\|_{L(C^-_\mu(Y^\alpha), C^-_\kappa(Y))} \\ \leq \|F_\varepsilon - F_0\|_1 + \|F_0\|_{1+\eta}\|\phi_\varepsilon(U) - \phi_0(U)\|^\eta_{C^-_\mu(Y^\alpha)}.$$

However, the right-hand side of the above inequality tends to 0 as $\varepsilon \to 0^+$ uniformly w.r. to $u \in \mathcal{B}$. Similarly, one has

$$\|D_U F_\varepsilon(U, \phi_\varepsilon(U)) - D_U F_0(U, \phi_0(U))\|_{L(C^-_\mu(X), C^-_\kappa(Y))} \to 0 \qquad \text{as } \varepsilon \to 0^+$$

uniformly w.r. to $U \in \mathcal{B}$. Now we notice that $\|B_0 D_U F_0(U, \phi_0(U))\|_{L(C^-_\mu(X), C^-_\mu(Y^\alpha))} \leq K_1$ and

$$\|[I + D_S F_0(U, \phi_0(U))B_0]D_U F_0(U, \phi_0(U))\|_{L(C^-_{\mu,\rho,a}(X), C^-_{\mu,\rho,a}(Y))} \leq K_1(1 + \|U\|^\eta_{C^-_{\mu,\rho,a}(X)}).$$

Indeed, let us denote

$$A(t) := [I + D_S F_0(U(t), \phi_0(U)(t))B_0]D_U F_0(U(t), \phi_0(U)(t)), \qquad t \leq 0.$$

Then, by (3.3) and (3.4),

$$\|A(t) - A(t - h)\|_{L(X,Y)} \leq K_1(\|U(t) - U(t - h)\|^\eta_X + \|\phi_0(U)(t) - \phi_0(U)(t - h)\|^\eta_{Y^\alpha}) \\ \leq K_1 e^{-\mu\eta t}h^{\eta\rho}(1 + \|U\|^\eta_{C^-_{\mu,\rho,a}(X)}).$$

As $0 < (1 + \eta)\mu \leq \kappa$ we obtain $\|A(\cdot)W\|_{C^-_{\kappa,\nu\rho,a}(Y)} \leq K_1\|W\|_{C^-_{\mu,\rho,a}(X)}(1 + \|U\|^\eta_{C^-_{\mu,\rho,a}(X)})$ for any $W \in C^-_{\mu,\rho,a}(X)$. According to lemma 3.2 it is now obvious that $D\phi_\varepsilon(U) \to D\phi_0(U)$ as $\varepsilon \to 0^+$ uniformly w.r. to $U \in \mathcal{B}$. The proof of lemma 3.2 is complete. ■

We will construct an inertial manifold $\mathcal{M}_\varepsilon$ for the semi-flow $\mathcal{S}_\varepsilon$ as the union of all Hölder continuous curves growing exponentially at $-\infty$, i.e.

$$\mathcal{M}_\varepsilon = \{(Y(\tau), \tau \in R, Y \in C^-_{\mu,\rho,a}(X \times Y^\alpha), (U(\cdot), S(\cdot)) = Y(\cdot) \text{ solves } (1.1)\} \tag{3.7}$$

for some $\mu > 0$, $\rho \in (0, 1)$ and $a \in (0, 1]$. The invariance property of $\mathfrak{M}_\varepsilon$ under the semiflow $\mathcal{S}_\varepsilon(t)$, $t \geq 0$, generated by system (1.1) is obvious. According to lemmas 3.1 and 3.2 $(U(\cdot), S(\cdot)) \in C^-_{\mu,\rho,a}(X \times Y^\alpha)$ is a solution of (1.1) if and only if it satisfies the following integral equation

$$U(t) = x + \int_0^t G_\varepsilon(U(s), \phi_\varepsilon(U)(s))\, ds =: T_\varepsilon(x, U)(t) \qquad \text{for any } t \leq 0 \tag{3.8}$$

for some $x \in X$. Using the invariance property of $\mathfrak{M}_\varepsilon$ we can write the set $\mathfrak{M}_\varepsilon$ as

$$\mathfrak{M}_\varepsilon = \{(x, \phi_\varepsilon(U)(0)),\ x \in X,\ U = T_\varepsilon(x, U) \in C^-_{\mu,\rho,a}(X)\}. \tag{3.9}$$

In what follows we will investigate the existence and the limiting behavior of fixed points of the two parameter family of mappings

$$T_\varepsilon(x, \cdot)\colon C^-_{\mu,\rho,a}(X) \to C^-_{\mu,\rho,a}(X), \qquad \varepsilon \in [0, \varepsilon_0],\ x \in X, \tag{3.10}$$

defined by the right-hand side of (3.8). We are going to prove that $T_\varepsilon(x, \cdot)$ is a uniform contraction. If $\|L_\varepsilon\|_{\mathfrak{X}_\mu}\|D_S F_\varepsilon\| \leq \theta < 1$ then by lemma 3.2(i), we have

$$\begin{aligned} &\|G_\varepsilon(U_1, \phi_\varepsilon(U_1)) - G_\varepsilon(U_2, \phi_\varepsilon(U_2))\|_{C^-_\mu(X)} \\ &\quad \leq \|G_\varepsilon\|_1(1 + K_0\|F_\varepsilon\|_1(\omega - \mu\varepsilon_0)^{\alpha-1}(1 - \theta)^{-1})\|U_1 - U_2\|_{C^-_\mu(X)}, \end{aligned} \tag{3.11}$$

where $K_0 > 0$ is a constant independent of $0 < \mu < \omega\varepsilon_0^{-1}$. Assume that $0 < \rho < 1$ and $\nu > 0$. Then the linear operator

$$\mathfrak{J}\colon g \mapsto \int_0^t g(s)\, ds, \qquad \mathfrak{J}\colon C^-_\nu(X) \to C^-_{\nu,\rho,a}(X)$$

is bounded its norm being estimated by

$$\|\mathfrak{J}\|_{L(C^-_\nu(X),\, C^-_{\nu,\rho,a}(X))} \leq \frac{2}{\nu} \tag{3.12}$$

provided that $a = a(\nu) > 0$ sufficiently small (c.f. [8, lemma 3.2,c]).

By the next lemma 3.5 we will show that under an additional assumption on $D_S F_\varepsilon$ the following hypotheses are fulfilled

(T)
- (1) there is $\theta < 1$ with the property $\|T_\varepsilon(x, U_1) - T_\varepsilon(x, U_2)\|_{\mathfrak{U}} \leq \theta\|U_1 - U_2\|_{\mathfrak{U}}$ for any $x \in \mathfrak{X}$, $U_1, U_2 \in \mathfrak{U}$ and $\varepsilon \in [0, \varepsilon_0]$;
- (2) there is a $Q < \infty$ such that $\|T_\varepsilon(x_1, U) - T_\varepsilon(x_2, U)\|_{\mathfrak{U}} \leq Q\|x_1 - x_2\|_X$ for any $x_1, x_2 \in X$, $U \in \mathfrak{U}$ and $\varepsilon \in [0, \varepsilon_0]$;
- (3) for any bounded open subset $B \subset X$,
$$\sup_{x \in B}\|T_\varepsilon(x, U_0(x)) - T_0(x, U_0(x))\|_{\mathfrak{U}} \to 0 \ \varepsilon \to 0^+$$
where $U_\varepsilon(x)$, $x \in X$, $\varepsilon \in [0, \varepsilon_0]$, is the unique fixed point of $T_\varepsilon(x, U) = U$ in $\mathfrak{U}$

on the Banach spaces

$$\mathfrak{U} := C^-_{\mu,\rho,a}(X), \qquad \bar{\mathfrak{U}} := C^-_{\kappa,\rho,a}(X), \qquad 0 < (1 + \eta)\mu < \kappa < \omega\varepsilon_0^{-1}, \tag{3.13}$$

where $0 < \rho < 1$ is fixed and $a \in (0, 1]$ is such that the estimate (3.12) holds for both values $\nu = \mu$ as well as $\nu = \kappa$.

LEMMA 3.5. Assume that the hypotheses (H) are fulfilled. Then there is a positive number $\delta > 0$ such that if $\|D_S F_\varepsilon\|_{L(Y^\alpha, Y)} \leq \delta$ for any $\varepsilon \in [0, \varepsilon_0]$ then there exists an invariant manifold $\mathfrak{M}_\varepsilon$, $\varepsilon \in [0, \varepsilon_0]$, for the semi-flow $\mathcal{S}_\varepsilon$ generated by the system of equations (1.1). This manifold is a graph, $\mathfrak{M}_\varepsilon = \{(x, \Phi_\varepsilon(x)), x \in X\}$, where $\Phi_\varepsilon: X \to Y^\alpha$ is a bounded Lipschitz continuous function. Moreover, for any bounded subset $B \subset X$, $\lim_{\varepsilon \to 0^+} \Phi_\varepsilon(x) = \Phi_0(x)$ uniformly w.r. to $x \in B$.

If, in addition, the operator A has a compact resolvent $A^{-1}: Y \to Y$ then the manifold $\mathfrak{M}_\varepsilon$ is also exponentially attractive, i.e. there is a $\mu > 0$ such that $\mathrm{dist}((U(t), S(t)), \mathfrak{M}_\varepsilon) = O(e^{-\mu t})$ when $t \to \infty$ for any solution $(U(\cdot), S(\cdot))$ of (1.1), $\varepsilon \in (0, \varepsilon_0]$.

Proof. According to lemma 3.2, for any $\mu > 0$, we can choose an $\varepsilon_0 = \varepsilon(\mu) \ll 1$ such that $\|L_\varepsilon\|_{\mathfrak{X}_\mu} \leq K_0(\omega - \mu\varepsilon_0)^{\alpha-1} \leq K_0(\omega/2)^{\alpha-1}$ for any $\varepsilon \in [0, \varepsilon_0(\mu)]$. Let $0 < \delta \ll 1$ be such that $K_0(\omega/2)^{\alpha-1}\delta < 1$. Now, if we suppose $\|D_S F_\varepsilon\| \leq \delta$, $\varepsilon \in [0, \varepsilon_0]$, we obtain the estimate (3.11) for the Lipschitz constant of the mapping $C_\mu^-(X) \ni U \mapsto G_\varepsilon(U, \phi_\varepsilon(U)) \in C_\mu^-(X)$ with some $\theta = K_0(\omega/2)^{\alpha-1}\delta < 1$. With regard to (3.12) one can furthermore choose $\mu \gg 1$ large enough and such that the mapping $T_\varepsilon(x, \cdot): \mathcal{U} \to \mathcal{U}$ fulfils the hypothesis $(T)_1$. The Lipschitz constant Q of the mapping $x \mapsto T_\varepsilon(x, U)$ is equal to 1. Let $U_0 = U_0(x)$ be the unique fixed point of $U_0 = T_0(x, U_0)$. Then, for any bounded and open subset $B \subset X$, we have $\|U_0(x)\|_{\mathcal{U}} \leq \|x\|_X + \|\mathfrak{J}\|_{L(C_\mu^-(X), \mathcal{U})}\|G_0\|_0 \leq K_0(B)$ for every $x \in B$. Moreover,

$$\|T_\varepsilon(x, U_0(x)) - T_0(x, U_0(x))\|_{\mathcal{U}} \leq \|\mathfrak{J}\|\|G_\varepsilon(U_0, \phi_\varepsilon(U_0)) - G_0(U_0, \phi_0(U_0))\|_{C_\mu^-(X)}$$
$$\leq (2/\mu)\|G_\varepsilon\|_1\|\phi_\varepsilon(U_0) - \phi_0(U_0)\|_{C_\mu^-(X)} + O(\|G_\varepsilon - G_0\|).$$

Due to lemma 3.3(ii), we know that $\|\phi_\varepsilon(U_0(x)) - \phi_0(U_0(x))\|_{C_\mu^-(Y^\alpha)} \to 0$ as $\varepsilon \to 0^+$ uniformly w.r. to $x \in B$ and so the hypothesis $(T)_3$ is also satisfied.

Define $\Phi_\varepsilon(x) := \phi_\varepsilon(U_\varepsilon(x))(0)$. According to (3.9) we have that the set $\mathfrak{M}_\varepsilon$ is a graph over the Banach space X, i.e. $\mathfrak{M}_\varepsilon = \{(x, \Phi_\varepsilon(x)), x \in X\}$ and moreover, as the mapping $x \mapsto U_\varepsilon(x)$ and ϕ_ε are Lipschitz continuous, Φ_ε is Lipschitz continuous as well. Hence, $\mathfrak{M}_\varepsilon$ is an invariant Lipschitz manifold for the semi-flow S_ε generated by (1.1), $\varepsilon \in (0, \varepsilon_0]$. Since $\|L_0\|_{\mathfrak{X}_\mu} = \|A^{\alpha-1}\|$ we have $\|A^{\alpha-1}\|\|D_S F_0\| \leq \theta < 1$ and so, by definition of a solution of (1.1), $\varepsilon = 0$, the set $\mathfrak{M}_0$ defined by (2.1) is an invariant manifold for the semi-flow $\mathcal{S}_0$. With regard to remark 3.4, we again have $\Phi_0(x) = \phi_0(U_0(x))(0)$.

Let $B \subset X$ be a bounded subset. From $(T)_1$ and $(T)_3$ it follows that $U_\varepsilon(x) \to U_0(x)$ as $\varepsilon \to 0^+$ uniformly w.r. to $x \in B$. Then by lemma 3.2(i),(ii), we have $\Phi_\varepsilon(x) \to \Phi_0(x)$ in Y^α as $\varepsilon \to 0^+$ uniformly w.r. to $x \in B$.

The proof of exponential attractivity of $\mathfrak{M}_\varepsilon$, $\varepsilon \in (0, \varepsilon_0]$ is similar, in spirit, to that of the paper by Chow and Lu [3, theorem 5.1]. In fact, it follows the lines of known proofs of existence of stable invariant foliation to a centre-unstable manifold. Let $\varepsilon \in (0, \varepsilon_0]$ be fixed. Given a solution $(\bar{U}, \bar{S})$ of (1.1) we want to find a solution $(U^*, S^*) \in \mathfrak{M}_\varepsilon$ with the property $(U, S) \in C_\mu^+(X \times Y^\alpha)$ for some $\mu > 0$ where $U = U^* - \bar{U}$, $S = S^* - \bar{S}$ and C_μ^+ is the Banach space

$$C_\mu^+(X \times Y^\alpha) := \left\{ f \in C(R^+, X \times Y^\alpha),\ \|f\|_{C_\mu^+} = \sup_{t \geq 0} e^{\mu t}\|f(t)\|_{X \times Y^\alpha} < \infty \right\}.$$

Obviously, the existence of such a solution would imply that the $\mathfrak{M}_\varepsilon$ manifold has the exponential tracking property and as a consequence we would have $\mathrm{dist}((U(t), S(t)), \mathfrak{M}_\varepsilon) = O(e^{-\mu t})$ when $t \to \infty$, i.e. $\mathfrak{M}_\varepsilon$ is an exponentially attractive invariant manifold.

Now, one easily verifies that (U, S) belongs to C_μ^+, $\mu > \|G_\varepsilon\|_1$, if and only if the following integral equations are satisfied

$$
\begin{aligned}
U(t) &= \int_\infty^t G_\varepsilon(\bar{U}(s) + U(s), \bar{S}(s) + S(s)) - G_\varepsilon(\bar{U}(s), \bar{S}(s))\,\mathrm{d}s =: \mathfrak{F}^U(U, S)(t) \\
S(t) &= \exp(-At/\varepsilon)\xi + \frac{1}{\varepsilon}\int_0^t \exp(-A(t-s)/\varepsilon)[F_\varepsilon(\bar{U}(s) + U(s), \bar{S}(s) + S(s)) \\
&\quad - F_\varepsilon(\bar{U}(s), \bar{S}(s))]\,\mathrm{d}s \qquad \text{for any } t \geq 0
\end{aligned}
\tag{3.14}
$$

for some $\xi \in Y^\alpha$. The operator $\mathfrak{F}^U$ defined by the right-hand side of the first equation in (3.14) is well posed on the space $C_\mu^+(X \times Y^\alpha)$ with values in $C_\mu^+(X)$. Moreover, the mapping $U \mapsto \mathfrak{F}^U(U, S)$ is a uniform contraction in $C_\mu^+(X)$ provided that $\mu > \|G_\varepsilon\|_1$. More precisely, one has

$$\|\mathfrak{F}^U(U_1, S) - \mathfrak{F}^U(U_2, S)\|_{C_\mu^+(X)} \leq \|G_\varepsilon\|_1 \mu^{-1} \|U_1 - U_2\|_{C_\mu^+(X)}$$

and, similarly,

$$\|\mathfrak{F}^U(U, S_1) - \mathfrak{F}^U(U, S_2)\|_{C_\mu^+(X)} \leq \|G_\varepsilon\|_1 \mu^{-1} \|S_1 - S_2\|_{C_\mu^+(Y^\alpha)}.$$

By the parameterized contraction principle there is a mapping $h: C_\mu^+(Y^\alpha) \to C_\mu^+(X)$ such that, for any $S \in C_\mu^+(Y^\alpha)$, $U \in C_\mu^+(X)$ is a solution of $U = \mathfrak{F}^U(U, S)$ iff $U = h(S)$. The Lipshitz constant of the mapping h can be estimated as

$$\|h(S_1) - h(S_2)\|_{C_\mu^+(X)} \leq \frac{\|G_\varepsilon\|_1}{\mu - \|G_\varepsilon\|_1} \|S_1 - S_2\|_{C_\mu^+(Y^\alpha)}. \tag{3.15}$$

It means that $(U, S) \in C_\mu^+$ is a solution of (3.14) iff $U = h(S)$ and S solves the equation

$$S(t) = \exp(-At/\varepsilon)\xi + \frac{1}{\varepsilon}\int_0^t \exp(-A(t-s)/\varepsilon) f(S)(s)\,\mathrm{d}s = \mathfrak{F}^S(\xi, S)$$

for any $t \geq 0$, where $f(S)(s) := F_\varepsilon(\bar{U}(s) + h(S)(s), \bar{S}(s) + S(s)) - F_\varepsilon(\bar{U}(s), \bar{S}(s))$. Since

$$
\begin{aligned}
\|f(S_1) - f(S_2)\|_{C_\mu^+(Y)} &\leq \|F_\varepsilon\|_1 \|h(S_1) - h(S_2)\|_{C_\mu^+(X)} + \|D_S F_\varepsilon\| \|S_1 - S_2\|_{C_\mu^+(Y^\alpha)} \\
&\leq (\|F_\varepsilon\|_1 \|G_\varepsilon\|_1 (\mu - \|G_\varepsilon\|_1)^{-1} + \|D_S F_\varepsilon\|) \|S_1 - S_2\|_{C_\mu^+(Y^\alpha)}
\end{aligned}
$$

the mapping $S \mapsto \mathfrak{F}^S(\xi, S)$ is a uniform contraction on $C_\mu^+(Y^\alpha)$ with respect to $\xi \in Y^\alpha$, provided that $\mu \gg 1$ is large enough and $\|D_S F_\varepsilon\| \leq \delta \ll 1$ is sufficiently small for $\varepsilon \in (0, \varepsilon_0]$, $\varepsilon_0 \ll 1$.

For a given $\xi \in Y^\alpha$, we denote $S^\xi \in C_\mu^+(Y^\alpha)$ the unique fixed point of $S = \mathfrak{F}^S(\xi, S)$. Again, due to the parameterized contraction principle the mapping $\xi \mapsto S^\xi$ is Lipschitzian and, hence, the mapping $Y^\alpha \ni \xi \mapsto (U^\xi, S^\xi) \in C_\mu^+(X \times Y^\alpha)$, $U^\xi := h(S^\xi)$, is Lipschitz continuous as well. Finally, if we denote

$$g(\xi) := \bar{U}(0) + U^\xi(0), \qquad \xi \in Y^\alpha,$$

then the mapping $g\colon Y^\alpha \to X$ is also Lipschitz continuous. We recall that $(U^*(0), S^*(0)) \in \mathfrak{M}_\varepsilon$ iff $S^*(0) = \Phi_\varepsilon(U^*(0))$. However, the last condition is satisfied if and only if

$$\xi = S^*(0) - \bar{S}(0) \in Y^\alpha$$

is a solution of

$$\Phi_\varepsilon(g(\xi)) - \bar{S}(0) = \xi. \tag{3.16}$$

Now, if we suppose that $A^{-1}\colon Y \to Y$ is a compact linear operator than, by [1, chapter 1] the embedding $Y^\beta \hookrightarrow\hookrightarrow Y^\alpha$ is compact whenever $\alpha < \beta$. We then claim that the mapping $X \ni x \mapsto \Phi_\varepsilon(x) \in Y^\alpha$ has a compact range. Indeed, by lemmas 3.2 and 3.3 we know that

$$\Phi_\varepsilon(x) = \frac{1}{\varepsilon}\int_{-\infty}^{0} \exp(As/\varepsilon) F_\varepsilon(U_\varepsilon(x)(s), \phi_\varepsilon(U_\varepsilon(x))(s))\,\mathrm{d}s.$$

This yields the estimate

$$\|\Phi_\varepsilon(x)\|_{Y^\beta} \le K_0 \|F_\varepsilon\|_0 \varepsilon^{-1} \int_{-\infty}^{0} (-s/\varepsilon)^{-\beta}\, \mathrm{e}^{\omega s/\varepsilon}\,\mathrm{d}s =: K_0(\beta) < \infty \qquad \text{for any } x \in X$$

for any $\alpha \le \beta < 1$. Then the mapping $Y^\alpha \ni \xi \mapsto \Phi_\varepsilon(g(\xi)) - \bar{S}(0) \in Y^\alpha$ is compact and Lipschitz continuous. Moreover, it takes a ball $B(0, R) \subset Y^\alpha$ into itself, $R = K_0(\alpha) + \|\bar{S}(0)\|_{Y^\alpha}$. Due to the Schauder fixed point theorem there is a solution $\xi \in Y^\alpha$ of (3.16). In other words, there exists a $(U^*(0), S^*(0)) \in \mathfrak{M}_\varepsilon$, $U^*(0) = g(\xi)$, $S^*(0) = \bar{S}(0) + \xi$, such that $\|\bar{U}(t) - U^*(t)\|_X + \|\bar{S}(t) - S^*(t)\|_{Y^\alpha} = O(\mathrm{e}^{-\mu t})$ when $t \mapsto \infty$. It completes the proof of lemma 3.5. ■

Remark 3.6. In case the Lipschitz constants of Φ_ε and g are less than 1, equation (3.16) can be solved by means of the Banach fixed point theorem (see, [3, theorem 5.1]). Since we have provided no bounds on the Lipschitz constant of Φ_ε we cannot apply a contraction principle here. This is why we have to turn to Schauder's fixed point principle and therefore the compactness of A^{-1} is needed in our proof.

In the following we will show that this family of fixed points $U_\varepsilon(x)$ and their derivatives $D_x U_\varepsilon(x)$ depend continuously on $\varepsilon > 0$ when ε tends to 0^+ uniformly w.r. to $x \in B$, where $B \subset X$ is an arbitrary bounded subset.

The proof uses abstract results due to Mora and Solà-Morales [6] regarding the limiting behavior of fixed points of a two-parameter family of nonlinear mappings. The main difficulty is that the mapping $(U, S) \mapsto (G_\varepsilon(U, S), F_\varepsilon(U, S))$ from the space $C^-_{\mu,\rho,a}(X \times Y^\alpha)$ into $C^-_{\mu,\rho,a}(X \times Y)$ need not be generally C^1 differentiable and, therefore, $T_\varepsilon(x, \cdot)\colon C^-_\mu(X) \to C^-_\mu(X)$ need not be C^1 as well. According to lemma 3.1 one can, however, expect that it becomes differentiable after composition with an embedding operator $J_{\mu,\kappa}$ for some $0 < \mu < \kappa$. This is why we need a version of a contraction theorem covering the case in which differentiability involves a pair of Banach spaces.

Consider a two parameter family of mappings $T_\varepsilon(x, \cdot)\colon \mathcal{U} \to \mathcal{U}$, $\varepsilon \in [0, \varepsilon_0]$, $x \in X$, where X is a Banach space. We assume that the Banach space $\mathcal{U}$ is continuously embedded into a Banach space $\bar{\mathcal{U}}$ through a linear embedding operator J. We also denote $\bar{T}_\varepsilon := JT_\varepsilon$ and $\bar{U}_\varepsilon(x) := JU_\varepsilon(x)$.

Now a slightly modified version of [6, theorem 5.1] reads as follows.

THEOREM 3.7 [8, theorem 3.6]. Besides the hypothesis (T) we assume also that the mappings $\bar{T}_\varepsilon: X \times \mathcal{U} \to \bar{\mathcal{U}}$, $\varepsilon \in [0, \varepsilon_0]$ satisfy the following conditions:

(1) for any $\varepsilon \in [0, \varepsilon_0]$, $\bar{T}_\varepsilon$ is Fréchet differentiable with $D\bar{T}_\varepsilon: X \times \mathcal{U} \to L(X \times \mathcal{U}, \bar{\mathcal{U}})$ bounded and uniformly continuous and there exist mappings

$$d_U T_\varepsilon; X \times \mathcal{U} \to L(\mathcal{U}, \mathcal{U}); \quad \bar{d}_U T_\varepsilon: X \times \mathcal{U} \to L(\bar{\mathcal{U}}, \bar{\mathcal{U}}); \quad d_x T_\varepsilon: X \times \mathcal{U} \to L(X, \mathcal{U})$$

such that

$$D_U \bar{T}_\varepsilon(x, U) = J d_U T_\varepsilon(x, U) = \bar{d}_U T_\varepsilon(x, U) J, \qquad D_x \bar{T}_\varepsilon(x, U) = J d_x T_\varepsilon(x, U)$$

$$\|d_U T_\varepsilon(x, U)\|_{L(\mathcal{U}, \mathcal{U})} \le \theta, \qquad \|\bar{d}_U T_\varepsilon(x, U)\|_{L(\bar{\mathcal{U}}, \bar{\mathcal{U}})} \le \theta, \qquad \|d_x T_\varepsilon(x, U)\|_{L(X, \mathcal{U})} \le Q;$$

(2) for any B bounded and open subset of X, $D\bar{T}_\varepsilon(x, U) \to D\bar{T}_0(x, U)$ as $\varepsilon \to 0^+$ uniformly for $(x, U) \in \{(x, U_\varepsilon(x)), x \in B, \varepsilon \in [0, \varepsilon_0]\}$.

Then the mappings $\bar{U}_\varepsilon: X \to \bar{\mathcal{U}}$ have the following properties:

(a) for any $\varepsilon \in [0, \varepsilon_0]$; $\bar{U}_\varepsilon: X \to \bar{\mathcal{U}}$ is Fréchet differentiable, with $D\bar{U}_\varepsilon: X \to L(X, \bar{\mathcal{U}})$ bounded and uniformly continuous,

(b) for any B bounded and open subset of X, $D\bar{U}_\varepsilon(x) \to D\bar{U}_0(x)$ as $\varepsilon \to 0^+$ uniformly with respect to $x \in B$.

In order to apply theorem 3.7 we choose the Banach spaces defined in (3.13). The space $\mathcal{U}$ is continuously embedded into $\bar{\mathcal{U}}$ through the linear embedding operator

$$J = J_{\mu,\kappa}: C^-_{\mu,\rho,a}(X) = \mathcal{U} \to C^-_{\kappa,\rho,a}(X) = \bar{\mathcal{U}}.$$

If we suppose that the assumptions of lemma 3.2 are satisfied then the mapping ϕ_ε is well defined and, hence, we can introduce the mapping $\mathcal{G}_\varepsilon: \mathcal{U} \to C^+_\mu(X)$

$$\mathcal{G}_\varepsilon(U)(s) := G_\varepsilon(U(s), \phi_\varepsilon(U)(s)) \qquad \text{for any } U \in \mathcal{U} \text{ and } s \le 0. \tag{3.17}$$

Now assume that $B \subset X$ is a bounded subset and define the set

$$\mathcal{B}_B := \{U_\varepsilon(x), x \in B, \varepsilon \in [0, \varepsilon_0]\}.$$

Since $U_\varepsilon(x) = T_\varepsilon(x, U_\varepsilon(x)) = x + \mathfrak{J}\mathcal{G}_\varepsilon(U_\varepsilon(x))$ and both $\mathcal{G}_\varepsilon$ and $\mathfrak{J}$ are bounded, we obtain

$$\mathcal{B}_B \text{ is a bounded subset of } \mathcal{U}. \tag{3.18}$$

Lemmas 2.1 and 3.2 enables us to conclude that

$$\bar{\mathcal{G}}_\varepsilon := J_{\mu,\kappa}\mathcal{G}_\varepsilon \in C^1_{bdd}(\mathcal{U}, C^-_\kappa(X)), \qquad \varepsilon \in [0, \varepsilon_0] \tag{3.19}$$

and there exists a mapping $d\mathcal{G}_\varepsilon: \mathcal{U} \to L(\mathcal{U}, C^-_\mu(X))$ such that $D\bar{\mathcal{G}}_\varepsilon = J_{\mu,\kappa} d\mathcal{G}_\varepsilon$,

$$d\mathcal{G}_\varepsilon(U)W = D_U G_\varepsilon(U(\cdot), \phi_\varepsilon(U)(\cdot))W + D_S G_\varepsilon(U(\cdot), \phi_\varepsilon(U)(\cdot)) d\phi_\varepsilon(U)W$$

We also remind ourselves that $\mathcal{U} \hookrightarrow C^-_\mu(X)$ and so $\phi_\varepsilon \in C^1_{bdd}(\mathcal{U}, C^-_{(1+\eta)\mu}(Y^\alpha))$, $D\phi_\varepsilon = J_{\mu,(1+\eta)\mu} d\phi_\varepsilon$ (see lemma 3.2(iii)).

LEMMA 3.8. $D\bar{\mathcal{G}}_\varepsilon(U) \to D\bar{\mathcal{G}}_0(U)$ as $\varepsilon \to 0^+$ uniformly with respect to $U \in \mathcal{B}_B$.

Proof. First observe that

$$D\bar{\mathcal{G}}_\varepsilon(U)W = J_{\mu,\kappa}D_U G_\varepsilon(U(\cdot), \phi_\varepsilon(U)(\cdot))W + J_{(1+\eta)\mu,\kappa}D_S G_\varepsilon(U(\cdot), \phi_\varepsilon(U)(\cdot))D\phi_\varepsilon(U)W$$

for any $U \in \mathcal{B}_B$ and $W \in \mathcal{U}$. By lemma 3.2(ii), we know that $\lim_{\varepsilon\to 0^+}(U, \phi_\varepsilon(U)) = (U, \phi_0(U))$ in $C_\mu^-(X \times Y^\alpha)$ uniformly w.r. to $U \in \mathcal{B}_B$. According to lemma 2.1, $\tilde{G}_\varepsilon \in C^1_{bdd}(C_\nu(X \times Y^\alpha), C_\kappa(X))$ for $\nu = \mu$ or $\nu = (1+\eta)\mu$ and $D\tilde{G}_\varepsilon(U, \phi_\varepsilon(U)) \to D\tilde{G}_0(U, \phi_0(U))$ as $\varepsilon \to 0^+$ uniformly w.r. to $U \in \mathcal{B}_B$. Now the proof of lemma follows from the fact that $D\tilde{G}_\varepsilon(U, S) = J_{\nu,\kappa}DG_\varepsilon(U, S)$. ■

Now we are in a position to apply theorem 3.7 to the family of operators $\{T_\varepsilon\}$. To do so we define the following operators

$$d_U T_\varepsilon : X \times \mathcal{U} \to L(\mathcal{U}, \mathcal{U}), \qquad d_x T_\varepsilon : X \times \mathcal{U} \to L(X, \mathcal{U}), \qquad \bar{d}_U T_\varepsilon : X \times \mathcal{U} \to L(\bar{\mathcal{U}}, \bar{\mathcal{U}})$$

as follows

$$d_U T_\varepsilon(x, U) := \mathfrak{J} d\mathcal{G}_\varepsilon(U); \qquad d_x T_\varepsilon(x, u) := I_X; \qquad \bar{d}_U T_\varepsilon(x, U) := \bar{\mathfrak{J}}\bar{d}\mathcal{G}_\varepsilon(U),$$

where the linear operators $\mathfrak{J} \in L(C_\mu(X), \mathcal{U})$ and $\bar{\mathfrak{J}} \in L(C_\kappa(X), \bar{\mathcal{U}})$ were introduced in (3.12), $\nu = \mu$ or $\nu = \kappa$, respectively. Furthermore, if we denote

$$\bar{T}_\varepsilon := J_{\mu,\kappa}T_\varepsilon : X \times \mathcal{U} \to \bar{\mathcal{U}} \qquad \text{and} \qquad \bar{U}_\varepsilon(x) := J_{\mu,\kappa}U_\varepsilon(x)$$

then we obtain from (3.18), (3.19) and lemma 3.5,

$$\bar{T}_\varepsilon \in C^1_{bdd}(X \times \mathcal{U}, \bar{\mathcal{U}}) \qquad \text{and} \qquad \lim_{\varepsilon\to 0^+} D\bar{T}_\varepsilon(x, U) = D\bar{T}_0(x, U)$$

uniformly for $(x, U) \in \{(x, U_\varepsilon(x)), x \in B, \varepsilon \in [0, \varepsilon_0]\}$. Under the assumptions of lemma 3.5 we also know that the mappings $d_U T_\varepsilon$, $d_x T_\varepsilon$ and $\bar{d}_U T_\varepsilon$ satisfy all the hypotheses of theorem 3.7 with some $0 < \theta < 1$ and $Q = 1$, provided that $\mu \gg 1$ is large enough.

Finally, we recall that the mapping Φ_ε was defined as $\Phi_\varepsilon(x) = \phi_\varepsilon(U_\varepsilon(x))(0)$ (see lemma 3.5). With regard to theorem 3.7 and lemma 3.2(iii), (iv); the mapping $X \ni x \mapsto \phi_\varepsilon(U_\varepsilon(x)) \in C_{\bar{\mu}}^-(Y^\alpha)$ becomes C^1_{bdd} differentiable, for some $\bar{\mu} > \kappa$, and $\phi_\varepsilon(U_\varepsilon(x)) \to \phi_0(U_0(x))$, $x \in B$, as $\varepsilon \to 0^+$ in the respective topology. Hence, $\Phi_\varepsilon \to \Phi_0$ as $\varepsilon \to 0^+$ in $C^1_{bdd}(B, Y^\alpha)$ where $B \subset X$ is arbitrary bounded open subset.

Summarizing all the preceding results we can state the main result of this paper.

THEOREM 3.9. Assume that the hypothesis (H) is fulfilled. Then there are constants $\delta > 0$ and $0 < \varepsilon_1 \le \varepsilon_0$ such that if $\|D_S F_\varepsilon\|_{L(Y^\alpha, Y)} \le \delta$ for any $\varepsilon \in [0, \varepsilon_1]$ then there exists an invariant manifold $\mathfrak{M}_\varepsilon$ for the semi-flow $\mathcal{S}_\varepsilon$ generated by the system of evolutionary equations (1.1),

$$\mathfrak{M}_\varepsilon = \{(U, \Phi_\varepsilon(U)), U \in X\}, \qquad \text{where } \Phi_\varepsilon \in C^1_{bdd}(X, Y^\alpha),$$

$$\Phi_\varepsilon \to \Phi_0 \qquad \text{as } \varepsilon \to 0^+ \text{ in } C^1_{bdd}(B, Y^\alpha)$$

for any bounded open subset $B \subset X$.

If $\dim(X) = \infty$ then $\mathcal{M}_\varepsilon$ is infinite dimensional Banach submanifold of the phase-space $X \times Y^\alpha$. If $\dim(Y) = \infty$ then $\mathrm{codim}(\mathcal{M}_\varepsilon) = \infty$.

If, in addition, the resolvent operator $A^{-1}: Y \to Y$ is compact then the manifold $\mathcal{M}_\varepsilon$, $\varepsilon \in (0, \varepsilon_1]$ is also exponentially attractive, i.e. $\mathcal{M}_\varepsilon$ attracts exponentially any bounded subset of $X \times Y^\alpha$.

Remark 3.10. One may ask whether the assumption that the F_ε and G_ε are globally bounded in the respective topologies is not too much restrictive from the point of view of possible applications of the results obtained in theorem 3.9. In case of dissipative semi-flows one can, however, prepare the nonlinearities F_ε, G_ε in such a way that they are vanishing far from the vicinity of a globally attracting set (see, e.g. [4, 6, 8]). In Section 4 we present an example of such a modification of the governing equations. Let us also emphasize that having modified the nonlinearities in (1.1) by their truncation we are afterwards dealing with local invariant manifolds only.

4. AN APPLICATION TO THE JOHNSON-SEGALMAN-OLDROYD MODEL OF SHEARING MOTIONS OF A PRESSURE DRIVEN NON-NEWTONIAN FLUID

Many striking phenomena like spurt or hysteresis were apparently observed in rheological experiments involving the channel flow of highly elastic and very viscous non-Newtonian fluid like some synthesized polymers. The interested reader is referred to the paper by Vinogradov *et al.* [10] for details. Much effort has been spent to explain such and related phenomena mathematically. In [9, 12, 13] Nohel *et al.* have considered the Johnson–Segalman–Oldroyd model of shearing motions of the planar Poisseule flow within a thin channel. The channel is aligned along the y axis and extends between $x \in [-1, 1]$. The flow is assumed to be symmetric with respect to $x = 0$ and the fluid undergoes the simple shearing. Therefore, we can restrict ourselves to the interval $x \in [0, 1]$. Moreover, the flow variables (velocity and stresses) are independent of y so $\mathbf{v} = (0, v(t, x))$. In order to determine extra stress tensor as a functional of the rate of deformation tensor we consider the Johnson–Segalman–Oldroyd constitutive law (see [9]). In nondimensional units the system of partial differential equations governing the motion of such a fluid is a system of parabolic-hyperbolic equations

$$\begin{aligned} \sigma_t &= -\sigma + (1 + z)v_x \\ z_t &= -z - \sigma v_x \\ \varepsilon v_t &= v_{xx} + \sigma_x + f \end{aligned} \tag{4.1}$$

subject to boundary and initial conditions

$$\begin{gathered} v_x(t, 0) = v(t, 1) = \sigma(t, 0) = 0 \qquad \text{for any } t \geq 0 \\ v(0, x) = v_0(x), \qquad \sigma(0, x) = \sigma_0(x), \qquad z(0, x) = z_0(x) \qquad \text{for } x \in [0, 1]. \end{gathered} \tag{4.2}$$

We omit here the complete derivation of the initial-boundary value problem (4.1)–(4.2) by referring to [9]. We only remind ourselves that σ is the extra stress, z is the difference of normal stresses, $f \in R$ is a constant pressure gradient driving the flow. The parameter $\varepsilon > 0$ is proportional to the ratio of the Reynolds number to the Deborah number and is very small compared to other constants in (4.1), $\varepsilon = O(10^{-12})$ (see [9]). It gives rise to treating $0 < \varepsilon \ll 1$ as a small parameter and investigating the singular limiting behavior of inertial manifolds of system (4.1)–(4.2) when $\varepsilon \to 0^+$.

For the purpose of the analysis, let us introduce the total stress function $S = v_x + \sigma + fx$, $x \in [0, 1]$. Since the flow is assumed to be symmetric about the centerline the extra shear stress function must be an odd function, i.e. $\sigma(t, 0) = 0$. System (4.1)-(4.2) can, therefore, be rewritten as

$$\begin{aligned} \sigma_t &= -\sigma + (1 + z)(S - \sigma - fx) \\ z_t &= -z - \sigma(S - \sigma - fx) \\ \varepsilon S_t - S_{xx} &= \varepsilon(-\sigma + (1 + z)(S - \sigma - fx)) \end{aligned} \tag{4.3$_\varepsilon$}$$

subject to boundary and initial conditions

$$\begin{aligned} &S(t, 0) = S_x(t, 1) = 0 \qquad \text{for any } t \geq 0 \\ &S(0, x) = S_0(x), \qquad \sigma(0, x) = \sigma_0(x), \qquad z(0, x) = z_0(x) \qquad \text{for } x \in [0, 1]. \end{aligned} \tag{4.4}$$

Denote $AS = -S_{xx}$ the selfadjoint operator in $Y = L_2(0, 1)$ its domain being the Sobolev space $D(A) = \{S \in W^{2,2}(0, 1), S(0) = S'(1) = 0\}$. The operator A is sectorial in Y, $\operatorname{Re}\sigma(A) > 1$ and $A^{-1}: Y \to Y$ is compact. Moreover, $Y^{1/2} = W_B^{1,2} = \{S \in W^{1,2}(0, 1), S(0) = 0\}$. Let us consider the Banach space $X = (L_\infty(0, 1))^2$. The problem (4.3)-(4.4) can be viewed as an abstract problem (1.1) where the nonlinear functions $G(U, S)$, $F_\varepsilon(U, S)$, $U = (\sigma, z)$ are defined by the right-hand side of (4.3), i.e. $G(U, S) = [-\sigma + (1 + z)(S - \sigma - fx), -z - \sigma(S - \sigma - fx)]^T$ and $F_\varepsilon(U, S) = \varepsilon\{-\sigma + (1 + z)(S - \sigma - fx)\}$. Nohel *et al.* [12] proved global existence and uniqueness of solutions of the initial-boundary problem (4.3)-(4.4) in the phase-space $X \times Y^{1/2}$ (cf. [12]). The inertialess approximation of system $(4.3)_\varepsilon$ when $\varepsilon = 0$ yields $S \equiv 0$ and, hence, $(4.3)_0$ becomes a system of ordinary differential equations in the Banach space $X = (L_\infty(0, 1))^2$

$$\begin{aligned} \sigma_t &= -\sigma - (1 + z)(\sigma + fx) \\ z_t &= -z + \sigma(\sigma + fx) \end{aligned} \tag{4.3$_0$}$$

extensively studied by Nohel *et al.* [9, 11, 13].

Let us emphasize that nonlinear functions F_ε, G do not satisfy the assumptions of the hypothesis (H). In fact they are not smoothly bounded functions. Nevertheless, as is usual in similar circumstances (see, e.g. [6]) we will smoothly modify the functions F_ε, G far from the vicinity of some globally attracting bounded set. In what follows, we will seek a bounded attracting set in $X \times Y^{1/2}$ independent of $\varepsilon \in [0, \varepsilon_0]$. To do so, let us first multiply the first equation in (4.3) by σ and the second one by $1 + z$. Their summation then leads to the estimate

$$\sup_{x \in [0,1]} (\sigma^2(t, x) + (1 + z(t, x))^2) \leq 1 + e^{-t} \sup_{x \in [0,1]} (\sigma_0^2(x) + (1 + z_0(x))^2).$$

We will let $K_0 = K_0(\|\sigma_0\|_\infty + \|z_0\|_\infty + \|S_{0x}\|_2)$ denote any positive constant increasingly depending on its argument. By $C > 0$ we will denote any generic constant independent of $\varepsilon \in [0, \varepsilon_0]$ and initial conditions. From the above inequality it should be obvious that a ball in X of the radius 2 is an attracting set, i.e. for any $(\sigma_0, z_0, S_0) \in X \times Y^{1/2}$ there is $T = T(\sigma_0, z_0) > 0$ such that $\|\sigma(t, \cdot)\|_\infty + \|z(t, \cdot)\|_\infty \leq 2$ for every $t \geq T$. Now we observe that

$$\|-\sigma + (1 + z)(S - \sigma - fx)\|_2 \leq C(1 + K_0 e^{-t})(1 + \|S\|_2) \qquad \text{for } t \geq 0.$$

Taking the inner product in $L_2(0, 1)$ of the third equation in (4.3) with $-S_{xx}$ we obtain

$$\frac{\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|S_x\|_2^2 + \|S_{xx}\|_2^2 \le \varepsilon\|-\sigma + (1+z)(S - \sigma - fx)\|_2\|S_{xx}\|_2$$
$$\le C\varepsilon(1 + K_0\,\mathrm{e}^{-t})(1 + \|S\|_2)\|S_{xx}\|_2 \le C\varepsilon(1 + \|S_{xx}\|_2^2)$$

for any $t \ge K_0(\sigma_0, z_0)$. Since $\|S\|_2 \le \|S_x\|_2 \le \|S_{xx}\|_2$ for any $S \in D(A)$ we obtain $\varepsilon\mathrm{d}/\mathrm{d}t\|S_x\|_2^2 + \|S_x\|_2^2 \le C\varepsilon$ provided that $\varepsilon \in [0, \varepsilon_0]$ and ε_0 is small enough. Then

$$\|S_x(t, \cdot)\|_2^2 \le \|S_x(T, \cdot)\|_2^2 \exp((T - t)/\varepsilon) + C\varepsilon \qquad \text{for any } t \ge T.$$

Furthermore, as the growth of the third equation in $(4.3)_\varepsilon$ is only linear in S one can easily prove that the time-one map $(\sigma_0, z_0, S_0) \mapsto (\sigma(1, \cdot), z(1, \cdot), S(1, \cdot))$ takes bounded sets into bounded sets of the phase-space $X \times Y^{1/2}$. This and the above estimates yield bounded dissipativity of the semiflow generated by (4.3)–(4.4). More precisely, there is a constant $R_0 > 0$ independent of $\varepsilon \in [0, \varepsilon_0]$ and such that, for any bounded set of initial conditions $\mathcal{B} \subset X \times Y^{1/2}$ there is a $T = T(\varepsilon, \mathcal{B}) > 0$ with the property

$$\|\sigma(t, \cdot)\|_\infty^2 + \|z(t, \cdot)\|_\infty^2 + \|S_x(t, \cdot)\|_2 \le R_0$$

for any $(\sigma_0, z_0, S_0) \in \mathcal{B}$ and $t \ge T$.

Let $\theta \in C^2_{bdd}(R^+, R^+)$ be a smooth cut-off function with the property $\theta \equiv 1$ on $[0, 2R_0]$, $\theta \equiv 0$ on $[3R_0, \infty)$ and define the modified functions G^b, F^b_ε as follows

$$G^b(U, S)(x) := \theta(|\sigma(x)|^2 + |z(x)|^2 + \|S\|^2_{W^{1,2}})G(U, S)(x)$$
$$F^b_\varepsilon(U, S)(x) := \theta(|\sigma(x)|^2 + |z(x)|^2 + \|S\|^2_{W^{1,2}})F_\varepsilon(U, S)(x)$$

for a.e. $x \in [0, 1]$. Here $U = (\sigma, z) \in X = (L_\infty(0, 1))^2$ and $S \in Y^{1/2} = W_B^{1/2}(0, 1)$. Note that $W_B^{1,2}$ is a Hilbert space its norm squared being two times continuously differentiable and $W_B^{1,2} \hookrightarrow L_\infty(0, 1)$. Recall also that the Nemitzky operator is C^2 smooth when considered as a function from $L_\infty(0, 1)$ into itself. Thus

$$F^b_\varepsilon \in C^2_{bdd}(X \times Y^{1/2}, Y) \qquad \text{and} \qquad G^b \in C^2_{bdd}(X \times Y^{1/2}, X).$$

For the norm of $D_S F^b_\varepsilon$ we have an estimate $\|D_S F^b_\varepsilon\| = O(\varepsilon)$ as $\varepsilon \to 0^+$. Since

$$\mathcal{M}_0 = \{(U, S), AS = F_0(U, S) = 0\} = \{(U, 0), U \in X\}$$

we have $\Phi_0 \equiv 0$. Now we can apply theorem 3.9 to obtain the following theorem.

THEOREM 3.11. There exists an $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0]$ the nonlinear system of equations (4.3)–(4.4) governing shearing motions of a Poisseule planar flow of the Johnson-Segalman-Oldroyd fluid:

(i) possesses an infinite dimensional local invariant manifold $\mathcal{M}_\varepsilon$ attracting any solution of $(4.3)_\varepsilon$–(4.4);

(ii) there is an $R_0 > 1$ such that any solution of $(4.3)_\varepsilon$–(4.4) enters a ball of the radius R_0 in the space $(L_\infty(0, 1))^2 \times W_B^{1/2}(0, 1)$;

(iii) $\mathcal{M}_\varepsilon = \{(\sigma, z, \Phi_\varepsilon(\sigma, z)), (\sigma, z) \in B_{R_0}\}$, $\Phi_\varepsilon \in C^1_{bdd}(B_{R_0}, W_B^{1,2}(0, 1))$, where $B_{R_0} = \{(\sigma, z) \in (L_\infty(0, 1))^2, \|\sigma\|_\infty^2 + \|z\|_\infty^2 < R_0\}$;

(iv) $\Phi_\varepsilon \to 0$ as $\varepsilon \to 0^+$ in the topology of the space $C^1_{bdd}(B_{R_0}, W_B^{1,2}(0, 1))$.

REFERENCES

1. HENRY D., *Geometric Theory of Semilinear Parabolic Equations, Lecture Notes in Math.*, Vol. 840. Springer, Berlin (1981).
2. MARION M., Inertial manifolds associated to partly dissipative reaction–diffusion systems, *J. math. Analysis Applic.* **143**, 295–326 (1989).
3. CHOW S.-N. & LU K., Invariant manifolds for flows in Banach spaces, *J. diff. Eqns* **74**, 285–317 (1988).
4. FOIAS C., SELL G. R. & TEMAN R., Inertial manifolds for nonlinear evolutionary equations, *J. diff. Eqns* **73**, 309–353 (1988).
5. SVIRIDYUK G. A. & SUKACHEVA T. G., Cauchy problem for a class of semilinear equations of Sobolev type, *Sib. mat. J.* **31**(5), 120–127 (1990). (English translation.)
6. MORA X. & SOLÀ-MORALES J., The singular limit dynamics of semilinear damped wave equation, *J. diff. Eqns* **78**, 262–307 (1989).
7. MIKLAVČIČ M., A sharp condition for existence of an inertial manifold, *J. Dyn. diff. Eqns* **3**, 437–457 (1991).
8. ŠEVČOVIČ D., Limiting behaviour of invariant manifolds for a system of singularly perturbed evolution equations, *Math. Meth. appl. Sci.* **17**, 643–666 (1994).
9. MALKUS D. S., NOHEL J. A. & PLOHR B. J., Analysis of new phenomena in shear flow of a non-Newtonian fluid, *SIAM J. appl. Math.* **51**, 899–929 (1991).
10. VINOGRADOV G., MALKIN A., YANOVSKII YU., BORISENKOVA E., YARLYKOV B. & BEREZHNAYA G., Viscoelastic properties and flow of narrow distribution polybutadienes and polyisoprenes, *J. Polymer Sci.* **A-2 10**, 1061–1084 (1972).
11. MALKUS D. S., NOHEL J. A. & PLOHR B. J., Approximation of piston-driven flows of a non-Newtonian fluid, in *Control Theory, Dynamical Systems and Geometry of Dynamics* (Edited by ELSWORTHY, EVERITT and LEE), pp. 173–192. Marcel Dekker, New York (1993).
12. NOHEL J. A., PEGO R. L. & TZAVARAS A., Stability of discontinuous steady states in shearing motion of a non-Newtonian fluid, *Proc. R. Soc. Edinb.* **115A**, 39–59 (1990).
13. NOHEL J. A. & PEGO R. L., Nonlinear stability and asymptotic behavior of shearing motions of a non-Newtonian fluid, *SIAM J. math. Analysis* **24**, 911–942 (1993).
14. VANDERBAUWHEDE A. & VAN GILS S. A., Center manifolds and contraction on a scale of Banach spaces, *J. funct. Analysis* **72**, 209–224 (1987).

Príloha 6.1.2

Reprint práce:

D. Ševčovič: *Dissipative feedback synthesis for a singularly perturbed model of a piston driven flow of a non-Newtonian fluid.* Math. Methods in the Appl. Sci. 20 (1997), 79-94

Mathematical Methods in the Applied Sciences, Vol. 20, 79–94 (1997)
MOS subject classification: 35 B 40, 35 B 35, 35 B 25, 76 A 05, 76 D 30

Dissipative Feedback Synthesis for a Singularly Perturbed Model of a Piston Driven Flow of a Non-Newtonian Fluid

Daniel Ševčovič

Institute of Applied Mathematics, Comenius University, Mlynská dolina, 842 15 Bratislava, Slovak Republic

Communicated by G. F. Roach

The limiting behaviour of solutions of a system of singularly perturbed equations is studied. The goal is to construct a dissipative feedback control synthesis that stabilizes the prescribed output functional along trajectories of solutions. The results are applied to a singularly perturbed Johnson–Sagelman–Oldroyd model of shearing motions of a piston driven flow of a non-Newtonian fluid.

1. Introduction

The aim of this paper is to construct a dissipative feedback control synthesis that stabilizes a given output functional along solutions of the following system of singularly perturbed evolution equations

$$\begin{aligned} x_t &= G_\varepsilon(x, y, z), \\ \varepsilon y_t + By &= F_\varepsilon(x, y, z), \end{aligned} \tag{1.1}$$

where $0 \leqslant \varepsilon \ll 1$ is a small parameter, $x \in X$, $y \in Y$, X and Y are Banach spaces, B is a sectorial operator in Y. In this paper we consider a specific feedback control mechanism of the form

$$z = \Xi(x),$$

where Ξ is a smooth function from X into another Banach space Z. In other words, a synthesis $z = \Xi(x)$ should only depend on the slow variable x. It is well-known that the Cauchy problem for the full system of equations, $\varepsilon > 0$,

$$\begin{aligned} x_t &= G_\varepsilon(x, y, \Xi(x)), \\ \varepsilon y_t + By &= F_\varepsilon(x, y, \Xi(x)) \end{aligned} \tag{1.2}$$

generates a globally defined semi-flow $\mathscr{S}_\varepsilon(t)$, $t \geqslant 0$, on a phase-space $\mathscr{X} = X \times Y^\beta$, provided that the nonlinearities G_ε, F_ε and Ξ satisfy certain regularity and growth conditions (cf. [6]). Furthermore, under a suitable assumption on a function F_0,

CCC 0170–4214/97/010079–16 *Received 5 December 1995*

system (1.2) generates a semi-flow $\mathscr{S}_0(t)$, $t \geqslant 0$, on a phase-space $\mathscr{M}_0$ which is a Banach submanifold of $\mathscr{X}$.

Typically, the structure of the reduced system of equations (1.1), $\varepsilon = 0$, allows us to construct a feedback law $z = \Xi_0(x)$ with the property that a prescribed output functional Q_0 asymptotically vanishes along all solutions of (1.2), i.e. $Q_0(\mathscr{S}_0(t)(x_0, y_0)) \to 0$ as $t \to \infty$. We discuss an example of such a reduced dynamics in section 6. Under assumptions made in sections 2 and 3, our goal in this work is to find a feedback synthesis $\Xi = \Xi_\varepsilon$ stabilizing the given output functional Q_ε along trajectories of the full system of equations (1.2) whenever $\varepsilon > 0$ is sufficiently small. It should be noted that an explicit construction of such a synthesis is not obvious, in many cases, and this is why we have to turn to functional analytic methods in order to prove the existence of a stabilizing feedback law and to examine the limiting behavior of Ξ_ε when $\varepsilon \to 0^+$.

Before stating our main result we need several definitions.

Definition 1.1. Let $\mathscr{S}(t)$, $t \geqslant 0$, be a semi-flow on a metric space $(\mathscr{X}, d)$. Let $\mathscr{M}$ be an attracting invariant set for $\mathscr{S}$, i.e. $\mathscr{S}(t)\mathscr{M} = \mathscr{M}$ for any $t \geqslant 0$ and dist $(\mathscr{S}(t)u, \mathscr{M}) \to 0$ as $t \to \infty$ for any $u \in \mathscr{X}$. Let $Q : \mathscr{X} \to E$ be a prescribed output functional, E is a metric space. We say that the semi-flow $\mathscr{S}(t)$ is *asymptotically Q-constrained on $\mathscr{M}$* if $Q(u) = 0$ for any $u \in \mathscr{M}$.

Remark 1.1. Notice that, if $Q : \mathscr{X} \to E$ is continuous then any Q-asymptotically constrained semi-flow $\mathscr{S}(t)$ on the attracting invariant set $\mathscr{M}$, has the property $Q(\mathscr{S}(t)u) \to 0$ as $t \to \infty$ for any $u \in \mathscr{X}$. Clearly, if a functional Q vanishes on $\mathscr{X}$ then any semi-flow on $\mathscr{X}$ is Q-asymptotically constrained on the whole phase-space $\mathscr{X}$.

Definition 1.2. Let $\varepsilon \in [0, \varepsilon_0]$ be fixed. Let $Q_\varepsilon : \mathscr{X} \to E$ be a continuous mapping, $\mathscr{X}$ is the phase-space for (1.1). We say that system of equations (1.1) admits a *dissipative feedback synthesis* $\Xi : X \to Z$ if the semi-flow $\mathscr{S}_\varepsilon(t)$ generated by solutions of (1.2) possesses an attracting invariant manifold $\mathscr{M}_\varepsilon$ and the semi-flow $\mathscr{S}(t)$ is Q_ε-asymptotically constrained on $\mathscr{M}_\varepsilon$.

We also recall the notion of an inertial manifold.

Definition 1.3. Let $\mathscr{S}(t)$, $t \geqslant 0$, be a semi-flow in the Banach space $\mathscr{X}$. We say that a Banach submanifold $\mathscr{M} \subset \mathscr{X}$ is an inertial manifold. for semi-flow $\mathscr{S}$ if:

(a) it is an invariant, i.e. $\mathscr{S}(t)\mathscr{M} = \mathscr{M}$ for any $t \geqslant 0$; and

(b) $\mathscr{M}$ attracts exponentially all solutions, i.e. there is $\mu > 0$ such that dist $(\mathscr{S}(t)u_0, \mathscr{M}) = O(e^{-\mu t})$ as $t \to \infty$ for any $u_0 \in \mathscr{X}$.

In contrast to the classical definition of an inertial manifold due to Foias *et al.* [4], we allow the exponentially attractive invariant manifold to be an infinite-dimensional Banach submanifold of the phase-space $\mathscr{X}$. (see e.g. [8]).

Given a family of output functionals Q_ε, $\varepsilon \geqslant 0$, the main result can be stated as follows:

Theorem 1.1. *Assume hypotheses* (H1)–(H4) *and the structural condition* (5.1) *below. Then, for any* $\varepsilon > 0$ *small enough,*

(a) *system* (1.1) *admits a dissipative feedback synthesis* $\Xi_\varepsilon \in C^1_{bdd}(\mathscr{B}, Z) \cap C^{0,1}(X, Z)$ *and, moreover,*

(b) $\lim_{\varepsilon \to 0^+} \Xi_\varepsilon = \Xi_0$ *in* $C^1_{bdd}(\mathscr{B}, Z)$ *for any* $\mathscr{B}$ *bounded and open subset of* X.

(c) *The feedback law* $z = \Xi_\varepsilon(x)$ *stabilizes the prescribed output functional* Q_ε. *This means that* $\lim_{t \to \infty} Q_\varepsilon\ (x(t), y(t)) = 0$ *for any solution* $(x(.), y(.))$ *of* (1.2).

(d) *The semi-flow $\mathcal{S}_\varepsilon$ generated by solutions of system* (1.2) *is Q_ε-asymptotically constrained on a C^1 smooth inertial manifold $\mathcal{M}_\varepsilon$. The manifold $\mathcal{M}_\varepsilon$ is C^1 close to $\mathcal{M}_0$ for $\varepsilon > 0$ sufficiently small.*

The idea of the proof and the organization of the paper is as follows. In section 3 we find a synthesis $z = \theta_\varepsilon(x, y)$ depending on the both slow and fast variables. Under suitable assumptions (see (H3)) such a function θ_ε can be uniquely determined from the governing equations and the condition that $\varepsilon \mathrm{d}/\mathrm{d}t Q_\varepsilon(x(t), y(t)) + Q_\varepsilon(x(t), y(t)) = 0$, i.e. $\| Q_\varepsilon(x(t), y(t)) \| = O(\mathrm{e}^{-t/\varepsilon})$ as $t \to +\infty$ for any solution of system (1.1) with $z = \theta_\varepsilon(x, y)$. Incorporating the feedback law $z = \theta_\varepsilon(x, y)$ into system (1.1) we then construct an inertial manifold $\mathcal{M}_\varepsilon$ for (1.1) as a smooth graph $\mathcal{M}_\varepsilon = \{(x, \Phi_\varepsilon(x)), x \in X\}$. To this end we make use of the abstract singular perturbation theorem proved in [14]. We recall this result in section 4. Roughly speaking, the existence of such an inertial manifold $\mathcal{M}_\varepsilon$ means that the fast variable y is governed by the slow variable x when restricted on the manifold $\mathcal{M}_\varepsilon$. This enables us to construct Ξ as a composite function $\Xi_\varepsilon(x) = \theta_\varepsilon(x, \Phi_\varepsilon(x))$.

In section 6 we are concerned with the problem of the existence of a feedback control law stabilizing a given output of solutions for a system of singularly perturbed equations arising from the non-Newtonian fluid dynamics. Several authors have considered various constitutive models of a non-Newtonian fluid in order to describe flow instability phenomena like e.g. spurt, hysteresis loop under cyclic load for pressure driven flows of a Johnson–Segalman–Oldroyd (JSO) fluid [9, 11, 5], or KBKZ fluid (see [1, 5]). In this paper we consider the JSO model and research which has been motivated by recent rheological experiments due to Lim and Schowalter [7]. Their experimental data suggests that a nearly periodic regime bifurcates from a steady state when the volumetric flow rate was gradually loaded beyond a critical value. In [10] Malkus *et al.* developed a mathematical theory capable of describing bifurcation phenomena in a piston driven flow of shearing motions of a non-Newtonian fluid. They considered the Johnson–Segalman–Oldroyd model of a shear flow of a non-Newtonian fluid leading to a system of three parabolic–hyperbolic equations.

$$
\begin{aligned}
&\varepsilon v_t - v_{\xi\xi} = \sigma_\xi + f, \\
&\sigma_t + \sigma = (1 + n) v_\xi, \quad (t, \xi) \in [0, \infty) \times [0, 1], \qquad (1.3) \\
&n_t + n = -\sigma v_\xi,
\end{aligned}
$$

where v is directional velocity of a planar shear flow, σ is the extra shear stress and n is the normal stress difference. The dimensionless number $\varepsilon > 0$ is proportional to the ratio of the Reynolds number to Deborah number and, in practice, ε is very small compared to other the terms in (1.3), $\varepsilon = O(10^{-12})$. This gives rise to treating $0 < \varepsilon \ll 1$ as a small parameter and to study a reduced system of equations (1.3) in which $\varepsilon = 0$. The problem to be considered here consists in the construction of a driving pressure gradient f as a function of the flow variables σ, n in such a way that the output of the volumetric flow rate per unit cross-section, $Q(t) = \int_0^1 v(t, \xi)\,\mathrm{d}\xi$ is fixed at the prescribed value Q_{fix}. It turns out that f has the form of a non-local functional of σ, $f = \Xi_0(\sigma) = 3\eta Q_{\mathrm{fix}} - 3\int_0^1 \xi\sigma(\xi)\,\mathrm{d}\xi$ (see, [10, (FB)]). Numerical simulations performed in [10] showed that such a quasi-dynamic approximation of the full system (1.3) is capable of capturing an interesting phenomenon of the existence of nearly periodic oscillations in the pressure gradient f observed recently in rheological experiments due to Lim and Showalter [7].

We apply Theorem 1.1 in order to show that, for small values of $\varepsilon > 0$, there exists a real valued dissipative feedback synthesis $f = f_\varepsilon(\sigma, n)$ for the pressure gradient such that $Q(t) \to Q_{\text{fix}}$ as $t \to \infty$ along solutions of the full system of equations (1.3). Moreover, there exists an infinite-dimensional inertial manifold $\mathscr{M}_\varepsilon$ for system (1.3), $0 < \varepsilon \ll 1$, and the volumetric flow rate Q of a solution belonging to $\mathscr{M}_\varepsilon$ is fixed at the prescribed value Q_{fix}. These results are summarized in Theorem 6.3. The vector field governing the motion on the invariant manifold $\mathscr{M}_\varepsilon$ is compared to that of the reduced problem. It is shown that they are locally C^1 close for small values of the singular parameter.

2. Preliminaries

Let E_1, E_2 be Banach spaces and $\eta \in (0, 1]$. By $L(E_1, E_2)$ we denote the Banach space of all linear bounded operators from E_1 to E_2. For an open subset $\mathscr{B} \subset E_1$, $C^k(\mathscr{B}, E_2)$ denotes the vector space of all k-times continuously Frechet differentiable mappings $F: \mathscr{B} \to E_2$. By $C^{k,1}(\mathscr{B}, E_2)$ we denote the vector space consisting of all $F \in C^k(\mathscr{B}, E_2)$ such that all derivatives $D^i F$, $i = 0, 1, \ldots, k$ are globally Lipschitz continuous. $C^1_{bdd}(\mathscr{B}, E_2)$ denotes the Banach space consisting of the mappings $F \in C^1(\mathscr{B}, E_2)$ which are Frechet differentiable and such that F, DF are bounded and uniformly continuous, the norm being given by $\|F\|_1^2 := (\sup|F|)^2 + (\sup|DF|)^2$. Finally, $C^{1+\eta}_{bdd}(\mathscr{B}, E_2)$ will denote the Banach space consisting of the mappings $F \in C^1_{bdd}(\mathscr{B}, E_2)$ such that DF is η-Hölder continuous, the norm being given by $\|F\|_{1,\eta} := \|F\|_1 + \sup_{x \neq y} \|DF(x) - DF(y)\| \, \|x - y\|^{-\eta}$.

Throughout the paper we will assume that

$$
\text{(H1)} \quad \begin{array}{l} X, Y, Z \text{ are real Banach spaces;} \\ B \text{ is a sectorial operator in } X; \\ \operatorname{Re}\sigma(B) > \omega > 0 \text{ and } B^{-1}: Y \to Y \text{ is compact.} \end{array}
$$

It follows from the theory of sectorial operators that $-B$ generates the exponentially decaying analytic semigroup of linear operators $\exp(-Bt)$, $t \geqslant 0$, on Y. Moreover, there is a constant $M \geqslant 1$ such that

$$
\|\exp(-Bt)\|_{Y^\beta} \leqslant M t^{-\beta} e^{-\omega t} \quad \text{for any } t > 0 \text{ and } \beta \geqslant 0. \tag{2.1}
$$

By Y^β, $\beta \in \mathbb{R}$ we have denoted a fractional power space with respect to the sectorial operator B, $Y^\beta = [D(B^\beta)]$, $\|y\|_{Y^\beta} = \|B^\beta y\|_Y$. Furthermore, $\|B^{\beta-1}\| \leqslant M\omega^{\beta-1}$ (cf [6, chapter 1]).

3. Construction of an (x, y)-dependent dissipative feedback synthesis

In this section we give a partial answer to the problem of the existence of a dissipative feedback synthesis that stabilizes a given output functional $Q_\varepsilon(x, y)$. We present a constructive method on how to obtain a feedback law of the form $z = \theta_\varepsilon(x, y)$ from the governing equations. In contrast to the required form of the synthesis $z = \Xi_\varepsilon(x)$ we allow the variable z to be a functional of both the x and y variables. The idea is rather simple and a function $\theta_\varepsilon : X \times Y^\beta \to Z$ is constructed in such a way that the E-valued functional $t \mapsto Q_\varepsilon(x(t), y(t))$ decays exponentially along any solution $(x(t), y(t))$ of system (1.1). Obviously, such an asymptotic behaviour is justified in the case when

$$\varepsilon \frac{\mathrm{d}}{\mathrm{d}t} Q_\varepsilon(x(t), y(t)) + \kappa Q_\varepsilon(x(t), y(t)) = 0, \quad t > 0 \tag{3.1}$$

for any solution $(x(.), y(.))$ of (1.1). Here $\kappa > 0$ is a fixed positive constant. Let us assume that G_ε and F_ε are X and Y valued functions, respectively. Using the chain rule the equation for $z = \theta_\varepsilon(x, y)$ can be deduced from equation (3.1), i.e.

$$\mathscr{H}_\varepsilon(x, y, z) = 0, \tag{3.2}$$

where $x = x(t)$, $y = y(t)$, $t > 0$, $z = \theta_\varepsilon(x, y)$ and

$$\begin{aligned} \mathscr{H}_\varepsilon(x, y, z) = \varepsilon D_x Q_\varepsilon(x, y) G_\varepsilon(x, y, z) &+ D_y Q_\varepsilon(x, y)[F_\varepsilon(x, y, z) - By] \\ &+ \kappa Q_\varepsilon(x, y). \end{aligned} \tag{3.3}$$

Suppose that there are constants $\beta \in [0, 1)$, $\eta \in (0, 1]$ such that, for any $\varepsilon \in [0, \varepsilon_0]$,

(H2) $Q_\varepsilon \in C^{2,1}(X \times Y^{\beta-1}, E)$, E is a real Banach space.

The function $\mathscr{H}_\varepsilon : X \times Y^\beta \times Z \to E$ is well-defined because $F_\varepsilon(x, y, z) - By \in Y^{\beta-1}$ for any $(x, y, z) \in X \times Y^\beta \times Z$ and $D_y Q_\varepsilon \in L(Y^{\beta-1}, E)$.

(H3) For any bounded and open subset $\mathscr{B} \subset X \times Y^\beta$ there is a function $\theta_\varepsilon \in C^{1,1}_{bdd}(\mathscr{B}, Z) \cap C^{0,1}(X \times Y^\beta, Z)$ such that $\mathscr{H}_\varepsilon(x, y, z) = 0$ iff $z = \theta_\varepsilon(x, y)$ for any $(x, y) \in X \times Y^\beta$, and $\theta_\varepsilon \to \theta_0$ as $\varepsilon \to 0^+$ in $C^{1,1}_{bdd}(\mathscr{B}, Z)$

If, in addition to (H2), hypothesis (H3) is fulfilled then by (3.1) we have

$$\begin{aligned} &Q_\varepsilon(x(t), y(t)) = O(\mathrm{e}^{-\kappa t/\varepsilon}) \text{ as } t \to \infty \quad \text{for } 0 < \varepsilon \leqslant \varepsilon_0 \\ &Q_0(x(t), y(t)) = 0 \quad \text{for any } t \geqslant 0. \end{aligned} \tag{3.4}$$

Henceforth, the property

$$\lim_{\varepsilon \to 0^+} \phi_\varepsilon = \phi_0 \text{ in } C^1_{bdd}(\mathscr{B}, E_2) \text{ for any bounded and open subset } \mathscr{B} \subset E_1$$

will be referred to as local C^1 closeness of ϕ_ε and ϕ_0.

Up to this point we did not make any precise assumptions on smoothness of non-linearities G_ε and F_ε appearing in (1.1) as right-hand sides. Henceforth, we will assume that G_ε and F_ε are such that

(H4) $\mathscr{G}_\varepsilon \in C^1_{bdd}(X \times Y^\beta, X)$, $\quad \mathscr{F}_\varepsilon \in C^{1+\eta}_{bdd}(X \times Y^\beta, Y)$,
$\|\mathscr{G}_\varepsilon - \mathscr{G}_0\|_1 + \|\mathscr{F}_\varepsilon - \mathscr{F}_0\|_1 = O(\varepsilon)$ as $\varepsilon \to 0^+$,
where $\mathscr{G}_\varepsilon(x, y) := G_\varepsilon(x, y, \theta_\varepsilon(x, y))$, $\quad \mathscr{F}_\varepsilon(x, y) := F_\varepsilon(x, y, \theta_\varepsilon(x, y))$.

We remark that $G_\varepsilon(F_\varepsilon)$ need not be necessarily a function from $X \times Y^\beta \times Z$ into $X(Y)$. We only require that the composite function $\mathscr{G}_\varepsilon(\mathscr{F}_\varepsilon)$ takes $X \times Y^\beta$ into $X(Y)$.

According to the theory of abstract parabolic equations due to Henry [6, Theorems 3.3.3, 3.3.4], the initial value problem for the system of equations

$$\begin{aligned} &x_t = \mathscr{G}_\varepsilon(x, y), \\ &\varepsilon y_t + By = \mathscr{F}_\varepsilon(x, y) \end{aligned} \tag{3.5}$$

possesses global-in-time strong solutions and system (3.5) generates a global C^1 semiflow $\bar{\mathscr{S}}_\varepsilon$, $t \geqslant 0$, on the phase-space

$$\mathscr{X} = X \times Y^\beta.$$

By a global strong solution of (3.5) with an initial condition $(x_0, y_0) \in \mathscr{X}$ we mean a function $(x, y) \in C_{\mathrm{loc}}([0, \infty); \mathscr{X}) \cap C^1_{\mathrm{loc}}((0, \infty); \mathscr{X})$ such that $(x(t), y(t)) \in X \times D(B)$ for any $t > 0$, and $(x(\cdot), y(\cdot))$ solves system (3.5) on $(0, \infty)$.

Let us denote

$$\delta(F_0, \theta_0) = \sup_{(x,y)} \| D_y \mathscr{F}_0(x, y) \|, \quad \text{where } \mathscr{F}_0(x, y) := F_0(x, y, \theta_0(x, y)). \tag{3.6}$$

If $\delta(F_0, \theta_0) < \omega^{1-\beta}/M$ then we have

$$\| B^{-1} D_y \mathscr{F}_0(x, y) \|_{L(Y^\beta, Y^\beta)} \leqslant \| B^{\beta-1} \| \sup \| D_y \mathscr{F}_0 \| \leqslant M \omega^{\beta-1} \delta < 1.$$

By the implicit function theorem there exists a C^1_{bdd} function $\Phi_0 : X \to Y^\beta$ such that $By = \mathscr{F}_0(x, y)$ iff $y = \Phi_0(x)$. By a global strong solution of (3.5), $\varepsilon = 0$, with an initial condition $x_0 \in X$ we mean a function $x \in C_{\mathrm{loc}}([0, \infty); X) \cap C^1_{\mathrm{loc}}(0, \infty); X)$ such that $x(\cdot)$ solves the equation $x_t = \mathscr{G}_0(x, \Phi_0(x))$ on $\mathbb{R}^+$. Again due to the above references to Henry's lecture notes this equation generates a global semi-flow $\hat{\mathscr{S}}_0(t)$, $t \geqslant 0$, on X. The semi-flow $\hat{\mathscr{S}}_0$ can be naturally extended to a semi-flow $\bar{\mathscr{S}}_0$ acting on the Banach submanifold

$$\mathscr{M}_0 = \{(x, \Phi_0(x)), \, x \in X\} \subset \mathscr{X} \tag{3.7}$$

by $\bar{\mathscr{S}}_0(t)(x, \Phi_0(x)) := \hat{\mathscr{S}}_0(t)x$ for any $x \in X$. In what follows, we will identify the semi-flow $\hat{\mathscr{S}}_0$ with $\bar{\mathscr{S}}_0$.

4. Abstract singular perturbation theorem

This section is focused on the C^1 singular limiting behaviour of inertial manifolds $\mathscr{M}_\varepsilon$ for semiflows $\bar{\mathscr{S}}_\varepsilon$ generated by solutions of the ε-parameterized system of equations (3.5). We recall an abstract result on limiting behaviour of inertial manifolds for a singularly perturbed system of evolution equations (3.5). The theorem below ensures both the existence of $\mathscr{M}_\varepsilon$ as well as C^1 closeness of $\mathscr{M}_\varepsilon$ and $\mathscr{M}_0$ for $\varepsilon > 0$ small enough.

Theorem 4.1. ([14, Theorem 3.9]). *Assume that hypotheses* (H1) *and* (H4) *hold. Then there are constants* $\delta_0 > 0$ *and* $0 < \varepsilon_1 \leqslant \varepsilon_0$ *such that if* $\sup_{(x,y)} \| D_y \mathscr{F}_\varepsilon(x, y) \|_{L(Y^\beta, Y)} \leqslant \delta_0$ *then, for any* $\varepsilon \in [0, \varepsilon_1]$, *there exists an inertial manifold* $\mathscr{M}_\varepsilon$ *for the semi-flow* $\bar{\mathscr{S}}_\varepsilon$ *generated by the system of evolution equations* (3.5) *and, moreover,*

(a) $\mathscr{M}_\varepsilon = \{(x, \Phi_\varepsilon(x)), \, x \in X\}$, *where* $\Phi_\varepsilon \in C^1_{bdd}(X, Y^\beta)$;

(b) $\Phi_\varepsilon \to \Phi_0$ *as* $\varepsilon \to 0^+$ *in* $C^1_{bdd}(\mathscr{B}, Y^\beta)$ *for any bounded open subset* $\mathscr{B} \subset X$.

If $\dim(X) = \infty$ *then* $\mathscr{M}_\varepsilon$ *is an infinite-dimensional Banach submanifold of the phase-space* $\mathscr{X} = X \times Y^\beta$. *If* $\dim(Y) = \infty$ *then* $\operatorname{codim}(\mathscr{M}_\varepsilon) = \infty$.

5. Construction of an *x*-dependent dissipative feedback synthesis. Proof of Theorem 1.1

Now we are in a position to prove the existence of a dissipative feedback synthesis of the required form $z = \Xi_\varepsilon(x)$. We assume that hypotheses (H1)–(H4) hold and, moreover,

$$\delta(F_0, \theta_0) < \delta_0, \tag{5.1}$$

where $\delta_0 > 0$ is the constant of Theorem 4.1. Then $\sup_{(x,y)} \|D_y \mathscr{F}_\varepsilon(x, y)\| < \delta_0$ for any $\varepsilon \in [0, \varepsilon_0]$, $\varepsilon_0 > 0$ small enough. As an immediate consequence of Theorem 4.1 we obtain the existence of an inertial manifold

$$\mathscr{M}_\varepsilon = \{(x, \Phi_\varepsilon(x)),\ x \in X\} \subset \mathscr{X} \tag{5.2}$$

for the semi-flow $\bar{\mathscr{S}}_\varepsilon$ generated by system (3.5). Moreover, $\Phi_\varepsilon \in C^1_{bdd}(X, Y^\beta)$ and

$$\Phi_\varepsilon \to \Phi_0 \quad \text{as} \quad \varepsilon \to 0^+ \quad \text{in } C^1_{bdd}(\mathscr{B}, Y^\beta) \tag{5.3}$$

for any bounded and open subset $\mathscr{B} \subset X$. Let us define the feedback law $\Xi_\varepsilon : X \to Z$ as follows:

$$\Xi_\varepsilon(x) := \theta_\varepsilon(x, \Phi_\varepsilon(x)) \quad x \in X. \tag{5.4}$$

Since we have assumed $\theta_\varepsilon \in C^{1,1}_{bdd}(\mathscr{B}, Z) \cap C^{0,1}(\mathscr{X}, Z)$ and $\theta_\varepsilon \to \theta_0$ in $C^1_{bdd}(\mathscr{B}, Z)$ as $\varepsilon \to 0^+$ for any bounded and open subset $\mathscr{B} \subset \mathscr{X} = X \times Y^\beta$ we infer from Theorem 4.1 that

$$\Xi_\varepsilon \in C^{0,1}(X, Z) \cap C^1_{bdd}(\mathscr{B}, Z), \qquad \Xi_\varepsilon \to \Xi_0 \text{ in } C^1_{bdd}(\mathscr{B}, Z) \text{ as } \varepsilon \to 0^+, \tag{5.5}$$

where $\mathscr{B}$ is an arbitrary bounded and open subset of X. Again due to Henry's theory the system

$$\begin{aligned} x_t &= G_\varepsilon(x, y, \Xi_\varepsilon(x)), \\ \varepsilon y_t + By &= F_\varepsilon(x, y, \Xi_\varepsilon(x)) \end{aligned} \tag{5.6}$$

generates a global semiflow $\mathscr{S}_\varepsilon$ on $\mathscr{X}$ for $0 < \varepsilon \leqslant \varepsilon_1$ and $\mathscr{S}_0$ on $\mathscr{M}_0$, respectively. Furthermore, we observe that the right-hand side of system (5.6) and that of system (3.5), i.e.

$$\begin{aligned} x_t &= G_\varepsilon(x, y, \theta_\varepsilon(x, y)), \\ \varepsilon y_t + By &= F_\varepsilon(x, y, \theta_\varepsilon(x, y)) \end{aligned} \tag{5.7}$$

coincide on the set $\mathscr{M}_\varepsilon$, $\varepsilon \in [0, \varepsilon_1]$. Thus $\mathscr{S}_\varepsilon(t)(x_0, y_0) = \bar{\mathscr{S}}_\varepsilon(t)(x_0, y_0)$ for any $(x_0, y_0) \in \mathscr{M}_\varepsilon$ and $t \geqslant 0$. Since $\mathscr{M}_\varepsilon$ is invariant for the semi-flow $\bar{\mathscr{S}}_\varepsilon$ we conclude that the set $\mathscr{M}_\varepsilon$ is an invariant manifold for the semi-flow $\mathscr{S}_\varepsilon$ as well. Notice that $\mathscr{S}_0$ and $\bar{\mathscr{S}}_0$ are defined on $\mathscr{M}_0$ and they are equal. Although the set $\mathscr{M}_\varepsilon$ is an attractive invariant manifold (inertial manifold) for $\bar{\mathscr{S}}_\varepsilon$ it should be emphasized that it is not obvious that $\mathscr{M}_\varepsilon$ is an attractive set for $\mathscr{S}_\varepsilon$. The reason is that governing systems (5.6) and (5.7) may differ outside the set $\mathscr{M}_\varepsilon$. Nevertheless, we will show that the semi-flows $\mathscr{S}_\varepsilon$ and $\bar{\mathscr{S}}_\varepsilon$ are exponentially asymptotically equivalent.

Lemma 5.1. *There exists a constant* $\mu > 0$ *such that for any* $(x_0, y_0) \in \mathscr{X}$ *there is* $(x_0^*, y_0^*) \in \mathscr{M}_\varepsilon$ *with the property*

$$\| \mathscr{S}_\varepsilon(t)(x_0, y_0) - \bar{\mathscr{S}}_\varepsilon(t)(x_0^*, y_0^*) \|_{\mathscr{X}} = O(\mathrm{e}^{-\mu t}) \quad \text{as } t \to \infty. \tag{5.8}$$

Proof. This is just the proof of [3, Theorem 5.1] and it follows the lines of the proof of the existence of exponential tracking to a centre-unstable manifold. A slightly modified version of this proof is also contained in [14, Lemma 3.5]. This version utilizes compactness of the operator B^{-1}.

The idea is as follows. Let us fix $0 < \varepsilon \leqslant \varepsilon_1$. Given a solution $(x(\cdot), y(\cdot)) = \mathscr{S}_\varepsilon(\cdot)(x_0, y_0)$ of (5.6) we will prove the existence of an initial condition $(x_0^*, y_0^*) \in \mathscr{M}_\varepsilon$ with the property $(u(\cdot), v(\cdot)) \in C_\mu^+(\mathscr{X})$, where $(u(t), v(t)) = \bar{\mathscr{S}}_\varepsilon(t)(x_0^*, y_0^*) - \mathscr{S}_\varepsilon(t)(x_0, y_0)$ and C_μ^+ is the Banach space

$$C_\mu^+(\mathscr{X}) := \{ f \in C([0, \infty), \mathscr{X}), \ \| f \|_{C_\mu^+} = \sup_{t \geqslant 0} \mathrm{e}^{\mu t} \| f(t) \|_{\mathscr{X}} < \infty \}.$$

Obviously, the existence of such an initial condition (x_0^*, y_0^*) implies statement (5.8).

Let us choose $\mu > 0$. Taking into account the decay estimate (2.1) for the semigroup $\exp(-Bt)$ we have that (u, v) belongs to C_μ^+, if and only if it is a solution of the following pair of integral equations:

$$\begin{aligned} u(t) &= \int_{-\infty}^{t} g(s, u(s), v(s))\, \mathrm{d}s \\ v(t) &= \exp(-Bt/\varepsilon)\xi + \frac{1}{\varepsilon}\int_0^t \exp(-B(t-s)/\varepsilon) f(s, u(s), v(s))\, \mathrm{d}s, \quad t \geqslant 0, \end{aligned} \tag{5.9}$$

for some $\xi \in Y^\beta$, where

$$\begin{aligned} g(s, u, v) &= G_\varepsilon(x^*(s), y^*(s), \theta_\varepsilon(x^*(s), y^*(s))) - G_\varepsilon(x^*(s) - u, y^*(s) \\ &\quad - v, \Xi_\varepsilon(x^*(s) - u)), \\ f(s, u, v) &= F_\varepsilon(x^*(s), y^*(s), \theta_\varepsilon(x^*(s), y^*(s))) - F_\varepsilon(x^*(s) - u, y^*(s) \\ &\quad - v, \Xi_\varepsilon(x^*(s) - u)). \end{aligned}$$

Since $\mathscr{M}_\varepsilon$ is invariant for $\bar{\mathscr{S}}_\varepsilon$ we have $y^*(s) = \Phi_\varepsilon(x^*(s))$ and hence $\theta_\varepsilon(x^*(s), y^*(s)) = \Xi_\varepsilon(x^*(s))$ for any $s \geqslant 0$. Thus, $\| \zeta(s, u, v) \|_X \leqslant C(\| u \|_X + \| v \|_{Y^\beta})$ where ζ stands either for g or f and $C > 0$ is a positive constant depending only on the Lipschitz constants of the mappings $G_\varepsilon, F_\varepsilon, \theta_\varepsilon, \Phi_\varepsilon$. Notice that the constant $C > 0$ can be chosen to be independent of $\varepsilon \in (0, \varepsilon_1]$. The rest of the proof is essentially the same as that of [3, Theorem 5.1] or [14, Lemma 3.5] and therefore is omitted. We only remind ourselves that, using the integral equations (5.9), the main idea is to set-up a suitable fixed point equation for $\xi \in Y^\beta$ by requiring that $(x_0^*, y_0^*) = (x_0 - u(0), y_0 - \xi)$ must be an element of the manifold $\mathscr{M}_\varepsilon = \mathrm{Graph}(\Phi_\varepsilon)$. To solve such a fixed point equation $\mu > 0$ must be chosen large enough. □

Lemma 5.2. *The output functional* Q_ε *vanishes on* $\mathscr{M}_\varepsilon$, *i.e.* $Q_\varepsilon(x_0, y_0) = 0$ *for any* $(x_0, y_0) \in \mathscr{M}_\varepsilon$.

Proof. The proof utilizes a simple invariance argument. Let $(x_0, y_0) \in \mathscr{M}_\varepsilon$ be fixed. Since $\mathscr{M}_\varepsilon$ is invariant for the semi-flow $\bar{\mathscr{S}}_\varepsilon$, for any $t \geqslant 0$, there is $(x_{-t}, y_{-t}) \in \mathscr{M}_\varepsilon$ such that $\bar{\mathscr{S}}_\varepsilon(t)(x_{-t}, y_{-t}) = (x_0, y_0)$. Clearly, $x_0 = x_{-t} + \int_{-t}^{0} \mathscr{G}_\varepsilon(\bar{\mathscr{S}}_\varepsilon(s)(x_{-t}, y_{-t}))\,ds$. Hence, $\| x_0 - x_{-t} \| \leqslant \| \mathscr{G}_\varepsilon \|_0 t$. Furthermore, as $(x_{-t}, y_{-t}) \in \mathscr{M}_\varepsilon$ we have $y_{-t} = \Phi_\varepsilon(x_{-t})$ and so $\| y_{-t} \| \leqslant \| \Phi_\varepsilon \|_0$. Solving the linear homogeneous equation (3.1) we obtain $Q_\varepsilon(x_0, y_0) = Q_\varepsilon(\bar{\mathscr{S}}_\varepsilon(t)(x_{-t}, y_{-t})) = Q_\varepsilon(x_{-t}, y_{-t})e^{-\kappa t/\varepsilon}$, $t \geqslant 0$. We remind ourselves that the output functional is assumed to be globally Lipschitz continuous and this is why

$$\begin{aligned} \| Q_\varepsilon(x_0, y_0) \| (e^{\kappa t/\varepsilon} - 1) &= \| Q_\varepsilon(x_{-t}, y_{-t}) - Q_\varepsilon(x_0, y_0) \| \\ &\leqslant \mathrm{lip}(Q_\varepsilon)(\| x_{-t} - x_0 \|_X + \| y_{-t} - y_0 \|_{Y^\beta}) \leqslant \mathrm{lip}(Q_\varepsilon)(2\| \Phi_\varepsilon \|_0 + \| \mathscr{G}_\varepsilon \|_0 t). \end{aligned}$$

Comparing the growth in $t \geqslant 0$ of the left- and right-hand sides of the above inequality we conclude $Q_\varepsilon(x_0, y_0) = 0$. Since $(x_0, y_0) \in \mathscr{M}_\varepsilon$ was arbitrary the proof of the lemma follows. □

Proof of Theorem 1.1. Under hypotheses (H1)–(H4) and assumption (5.1) we have established the existence of a dissipative feedback synthesis Ξ_ε (see (5.4) and Lemma 5.2). The regularity and convergence properties of Ξ_ε were shown in (5.5). Since, Q_ε is globally Lipschitz continuous the statement c) of Theorem 1.1 follows from Lemmas 5.1 and 5.2. Again with regard to Lemma 5.1, the manifold $\mathscr{M}_\varepsilon$ is an inertial manifold for the semi-flow $\mathscr{S}_\varepsilon$ generated by system (5.6). By (5.2) $\mathscr{M}_\varepsilon$ is a C^1 graph over the space X and the convergence property $\Phi_\varepsilon \to \Phi_0$ as $\varepsilon \to 0^+$ follows from (5.3). Hence, the statement (d) also holds. □

6. An application to the Johnson–Segalman–Oldroyd model of shearing motions of a piston driven non-Newtonian fluid

6.1. Governing equations

In order to examine the behaviour of a piston driven flow of a non-Newtonian fluid we consider the Johnson–Segalman–Oldroyd constitutive model of shearing motions of a planar Poiseuille flow within a thin channel. The channel is aligned along the y-axis and extends between $x \in [-1, 1]$. The flow is assumed to be symmetric with respect to $x = 0$ and the fluid undergoes simple shearing. Therefore, we can restrict ourselves to the interval $x \in [0, 1]$. Moreover, the flow variables (velocity and stresses) are independent of y so $\vec{v} = (0, v(t, x))$. To determine the extra stress tensor as a functional of the rate of a deformation tensor we consider the Johnson–Segalman–Oldroyd constitutive law (see [9] for details). In non-dimensional units the system of partial differential equations governing the motion of such a fluid is a system of parabolic–hyperbolic equations:

$$\begin{aligned} \sigma_t &= -\sigma + (1 + n)v_x, \\ n_t &= -n - \sigma v_x, \\ \varepsilon v_t &= \eta v_{xx} + \sigma_x + f, \end{aligned} \tag{6.1}$$

$(t, x) \in [0, \infty) \times [0, 1]$, subject to boundary and initial conditions

$$\begin{aligned} &v_x(t, 0) = v(t, 1) = \sigma(t, 0) = 0 \quad \text{for any } t \geqslant 0 \\ &v(0, x) = v_0(x),\ \sigma(0, x) = \sigma_0(x),\ n(0, x) = n_0(x) \quad \text{for } x \in [0, 1]. \end{aligned} \tag{6.2}$$

Here σ is the extra shear stress, n is the normal stress difference. It should be noted that in the case of a pressure driven flow studied in [9, 11, 15] the pressure gradient $f \in R$ is fixed. On the other hand, in the case of a piston driven flow (see [10] or [5, chapter 3]) the pressure gradient f is assumed to vary with respect to time. The parameters $\varepsilon > 0$ and $\eta > 0$ are proportional to the ratio of the Reynolds number to the Deborah number and the Newtonian viscosity to shear viscosity, respectively. In rheological experiments the number ε is very small compared to other terms in (6.1), $\varepsilon = O(10^{-12})$ (see [9]). This gives rise to treating $0 < \varepsilon \ll 1$ as a small parameter and investigate the singular limiting behavior of system (6.1)–(6.2) when $\varepsilon \to 0^+$. We refer to [9] for the complete derivation of a system of governing equations.

For the purpose of this analysis, let us introduce the following change of variables:

$$(\sigma, n, v) \leftrightarrow (\Sigma, n, u), \quad \Sigma(x) := -\int_x^1 \sigma(\xi)\,\mathrm{d}\xi, \; u := \eta v + \Sigma. \tag{6.3}$$

In terms of the new variables (Σ, n, u) system (6.1) has the form

$$\begin{aligned} &\Sigma_t = G^{(\Sigma)}, \\ &n_t = G^{(n)}, \\ &\varepsilon u_t - \eta u_{xx} = \eta f + \varepsilon G^{(\Sigma)}, \end{aligned} \tag{6.4}$$

where the non-linear functions $G^{(\Sigma)}$, $G^{(n)}$ are defined as

$$\begin{aligned} G^{(\Sigma)} &= G^{(\Sigma)}(\Sigma, n, u) = -\Sigma - \frac{1}{\eta}\int_x^1 (1 + \eta(\xi))[u_x(\xi) - \Sigma_x(\xi)]\,\mathrm{d}\xi, \\ G^{(n)} &= G^{(n)}(\Sigma, n, u) = -n - \frac{1}{\eta}\Sigma_x[u_x - \Sigma_x]. \end{aligned} \tag{6.5}$$

The corresponding boundary conditions are

$$u_x(t, 0) = u(t, 1) = \Sigma_x(t, 0) = \Sigma(t, 1) = 0 \quad \text{for any } t \geqslant 0. \tag{6.6}$$

Let $Q_{\mathrm{fix}} \in R$ be a prescribed value of the volumetric flow rate. If Q denotes the variation in the volumetric flow rate of a planar flow per unit cross-section, i.e. $Q = \int_0^1 v(\xi)\,\mathrm{d}\xi - Q_{\mathrm{fix}}$ then Q can be rewritten in terms of Σ and u as

$$Q(\Sigma, u) = \frac{1}{\eta}\int_0^1 [u(\xi) - \Sigma(\xi)]\,\mathrm{d}\xi - Q_{\mathrm{fix}}. \tag{6.7}$$

The feedback law $f = \theta_\varepsilon((\Sigma, n), u)$ can be then readily deduced from equation (3.2). In our application (3.1) and (3.2) become

$$\begin{aligned} &\varepsilon D_\Sigma Q \circ G^{(\Sigma)} + D_u Q \circ [\varepsilon G^{(\Sigma)} + \eta f + \eta u_{xx}] \\ &\quad = -\frac{\varepsilon}{\eta}\int_0^1 G^{(\Sigma)} + \frac{1}{\eta}\int_0^1 [\eta u_{xx}(\xi) + \eta f + \varepsilon G^{(\Sigma)}]\,\mathrm{d}\xi + \frac{\kappa}{\eta}\int_0^1 [u(\xi) - \Sigma(\xi)]\,\mathrm{d}\xi \\ &\qquad - \kappa Q_{\mathrm{fix}} = 0. \end{aligned}$$

Thus, for any $\varepsilon \geqslant 0$, we obtain

$$f = \theta(\Sigma, u) = -u_x(1) - \frac{\kappa}{\eta}\int_0^1 [u(\xi) - \Sigma(\xi)]\,\mathrm{d}\xi + \kappa Q_{\mathrm{fix}}. \tag{6.8}$$

Remark 6.1. It should be noted that in the case of the reduced problem ($\varepsilon = 0$) one can calculate that $u(x) = (1 - x^2)f/2$. Taking into account (6.8) one has $f = 3\eta Q_{\mathrm{fix}} + 3\int_0^1 \Sigma(\xi)\,\mathrm{d}\xi$. In terms of the flow variable σ it means that

$$f = 3\eta Q_{\mathrm{fix}} - 3\int_0^1 \xi\sigma(\xi)\,\mathrm{d}\xi$$

which is, up to rescaling, the same formula for the driving pressure gradient as that obtained in [10], formulae (FB).

Incorporating the feedback law $f = \theta(\Sigma, u)$ into system (6.4) we can rewrite the system of governing equations (6.4) in an abstract form

$$\begin{aligned} \Sigma_t &= G^{(\Sigma)}(\Sigma, n, u), \\ n_t &= G^{(n)}(\Sigma, n, u), \\ \varepsilon u_t + Bu &= \mathscr{F}_\varepsilon(\Sigma, n, u), \end{aligned} \tag{6.9}$$

where B is a linear operator, $Bu(x) = -\eta u_{xx}(x) + \eta u_x(1) + \kappa\int_0^1 u(\xi)\,\mathrm{d}\xi$, $x \in [0, 1]$, and

$$\mathscr{F}_\varepsilon(\Sigma, n, u) = \kappa\int_0^1 \Sigma + \kappa\eta Q_{\mathrm{fix}} + \varepsilon G^{(\Sigma)}(\Sigma, n, u) \tag{6.10}$$

and the non-linearities $G^{(\Sigma)}$, $G^{(n)}$ are as defined in (6.5). Notice that the derivative $D_u\mathscr{F}_\varepsilon$ vanishes for $\varepsilon = 0$.

6.2. Function space and operator setting

Let Y denote the real Hilbert space $L^2(0, 1)$ of square integrable functions; $\|u\|_Y^2 = \int_0^1 |u|^2$. For fixed positive real numbers η, $\kappa > 0$, we denote by B the linear operator $Bu = -\eta u_{xx} + \eta u_x(1) + \kappa\int_0^1 u(\xi)\,\mathrm{d}\xi$ its domain being the Sobolev space $D(B) = \{u \in H^2(0, 1),\ u_x(0) = u(1) = 0\}$. B is a non self-adjoint nonlocal operator. In what follows, we will show that B is a sectorial operator in Y, and, moreover, $\operatorname{Re}\sigma(B) > 0$. To this end, we decompose the operator B as $B = \mathscr{B} + \mathscr{L}$ where $\mathscr{L}u = \eta u_x(1) + \kappa\int_0^1 u(\xi)\,\mathrm{d}\xi$ and $\mathscr{B}$ is a self-adjoint operator in Y, $\mathscr{B}u = -\eta u_{xx}$ for any $u \in D(\mathscr{B}) = D(B)$. The operator $\mathscr{B}$ is sectorial in Y and $\operatorname{Re}\sigma(\mathscr{B}) \geqslant \eta\pi^2/4 > 0$ (see [6, chapter 1]). Since the embedding $[D(\mathscr{B}^\beta)] \hookrightarrow C^1_{bdd}(0, 1)$ is continuous for any $\beta > 3/4$ we have $\|\mathscr{L}u\|_Y \leqslant C\|\mathscr{B}^\beta u\|$ for any $u \in D(\mathscr{B})$ and $\beta > \frac{3}{4}$. According to [6, Corollary 1.4.5 and Example 11, p. 28] we conclude that the sum $B = \mathscr{B} + \mathscr{L}$ is a sectorial operator in Y as well. Moreover, the norm in the fractional power space $[D(B^\beta)]$ is equivalent to that of $[D(\mathscr{B}^\beta)]$. It remains to estimate the spectrum of B from below. First we notice that the operator $B^{-1}\colon Y \to Y$ exists and is given by $B^{-1}g = \int_0^1 K(.,\xi)g(\xi)\,\mathrm{d}\xi$, where K is a Green function.

$$K(x,\xi) = \begin{cases} \dfrac{1-x}{\eta} + \dfrac{3}{2\kappa}(1-x^2) - \dfrac{3}{4\eta}(1-x^2)(1-\xi^2), & 0 \leqslant \xi \leqslant x \leqslant 1, \\[2ex] \dfrac{1-\xi}{\eta} + \dfrac{3}{2\kappa}(1-x^2) - \dfrac{3}{4\eta}(1-x^2)(1-\xi^2), & 0 \leqslant x < \xi \leqslant 1. \end{cases}$$

Since, the kernel K is bounded the operator B^{-1} is compact and therefore the spectrum $\sigma(B)$ consists of eigenvalues, i.e. $\sigma(B) = \sigma_P(B)$. Let $\lambda \in \sigma(B)$ be an eigenvalue and $u \neq 0$ be the corresponding eigenfunction. Then $-\eta u_{xx}(x) + \eta u_x(1) + \kappa\int_0^1 u(\xi)\,\mathrm{d}\xi = \lambda u(x)$, $x \in [0, 1]$. Integrating this equation over $[0, 1]$ and taking into account the boundary condition $u_x(0) = 0$ we obtain $(\kappa - \lambda)\int_0^1 u = 0$. Then either $\lambda = \kappa > 0$ or $\int_0^1 u = 0$. The latter implies $-\eta u_{xx}(x) + \eta u_x(1) = \lambda u(x)$. By taking the inner product in a complexification of Y with $\bar{u}$ we obtain $\eta\int_0^1|u_x|^2 = \eta\int_0^1|u_x|^2 + \eta u_x(1)\int_0^1\bar{u} = \lambda\int_0^1|u|^2$. Hence λ is a real number and, moreover, $\lambda \geqslant \inf_{u\neq 0}\eta\|u_x\|^2/\|u\|^2 = \eta\pi^2/4$. Summarizing we have shown the following proposition.

Lemma 6.2. *Let η, κ be any positive constants. Then the linear operator $Bu = -\eta u_{xx} + \eta u_x(1) + \kappa\int_0^1 u(\xi)\,\mathrm{d}\xi$, $D(B) = \{u \in H^2(0,1),\ u_x(0) = u(1) = 0\}$, is sectorial in $Y = L^2(0,1)$. Furthermore, $\sigma(B) \subset [\omega, \infty)$ where $\omega = \min\{\kappa, \eta\pi^2/4\} > 0$. The fractional power space $Y^\beta =: [D(\mathscr{B}^\beta)]$ is imbeded into the Sobolev–Slobodeckii space $H^{2\beta}(0,1)$ for $1 > \beta > 3/4$. The resolvent operator $B^{-1}: Y \to Y$ is compact.*

Let X be the Banach space $X := \{(\Sigma, n) \in C^1_{bdd}(0,1) \times C^0_{bdd}(0,1),\ \Sigma_x(0) = \Sigma(1) = 0\}$. With regard to the continuity of the imbedding $Y^\beta \subsetneq C^1_{bdd}(0,1)$ for $\beta > \frac{3}{4}$, we conclude that the nonlinearities $G := (G^{(\Sigma)}, G^{(n)}): X \times Y^\beta \to X$ and $\mathscr{F}_\varepsilon: X \times Y^\beta \to Y$ are locally Lipschitz continuous. Thus local solvability in $\mathscr{X} = X \times Y^\beta$, $\frac{3}{4} < \beta < 1$, of system (6.9) follows from [6, Theorem 3.3.3]. To prove global-in-time solvability of solutions we have to find *a priori* estimates of any solution of (6.9).

6.3. A priori estimates of solutions, dissipativeness of a semi-flow, modification of governing equations

If (Σ, n, u) is a local solution of (6.9) in the phase space $\mathscr{X}$ then (σ, n, v), $\sigma = \Sigma_x$, $v = (u - \Sigma)/\eta$ is a local solution of (6.1) in $C_{bdd}(0,1) \cap \{\sigma, \sigma(0) = 0\} \times C_{bdd}(0,1) \times Y^\beta$. Let us multiply the first equation in (6.1) by σ and the second one by $(1+n)$. Their summation leads to the identity $(\mathrm{d}/\mathrm{d}t)(\sigma^2 + (1+n)^2) + 2(\sigma^2 + (1+n)n) = 0$. As $\sigma^2 + (1+n)^2 \leqslant 2(\sigma^2 + n(1+n)) + 1$ we obtain for Σ and n the estimate

$$\|\Sigma(t,\cdot)\|_1^2 + \|1 + n(t,\cdot)\|_0^2 \leqslant 2 + 2e^{-t}(\|\Sigma_0\|_1^2 + \|1 + n_0\|_0^2). \tag{6.11}$$

To obtain a bound of a solution u we take the inner product in $Y = L^2(0,1)$ of the equation

$$\varepsilon u_t - \eta u_{xx} + \eta u_x(1) + \kappa\int_0^1 u = \mathscr{F}_\varepsilon \tag{6.12}$$

with $3\kappa u - \eta u_{xx}$. Since $u_x(1) = \int_0^1 u_{xx}$ for any $u \in D(B)$ we have

$$\frac{\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}(3\kappa\|u\|^2 + \eta\|u_x\|^2) + \eta(3\kappa\|u_x\|^2 + \eta\|u_{xx}\|^2)$$
$$+\left(\sqrt{3\kappa}\int_0^1 u + \eta u_x(1)/\sqrt{3}\right)^2 = \frac{4}{3}\eta^2|u_x(1)|^2 + (\mathscr{F}_\varepsilon, 3\kappa u - \eta u_{xx})_Y.$$

Clearly, $\frac{4}{3}\eta^2|u_x(1)|^2 = \frac{8}{3}\eta^2\int_0^1 u_{xx}u_x \leqslant \frac{4}{3}\sqrt{\frac{\eta}{3\kappa}}\eta(3\kappa\|u_x\|^2 + \eta\|u_{xx}\|^2)$. Notice that $\frac{4}{3}\sqrt{\frac{\eta}{3\kappa}} < 1$ iff $\kappa > \frac{16}{27}\eta$. Furthermore, as $\|u\|_Y \leqslant \|u_x\|_Y \leqslant \|u_{xx}\|_Y$ for any $u \in D(B)$, we have $\|3\kappa u - \eta u_{xx}\|_Y^2 \leqslant \max\{6\kappa, 2\eta\}(3\kappa\|u_x\|^2 + \eta\|u_{xx}\|^2)$. Assuming $\kappa > \frac{16}{27}\eta$ and applying Schwartz's inequality to the inner product $(\mathscr{F}_\varepsilon, 3\kappa u - \eta u_{xx})_Y$ one can show the existence of positive constants δ, $C > 0$ independent of $\varepsilon \geqslant 0$, such that the following Lyapunov-type inequality is satisfied

$$\frac{\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}(3\kappa\|u\|^2 + \eta\|u_x\|^2) + \delta(3\kappa\|u_x\|^2 + \eta\|u_{xx}\|^2) \leqslant C\|\mathscr{F}_\varepsilon\|_Y^2. \tag{6.13}$$

Henceforth, C, δ will denote any generic positive constant independent of $\varepsilon \geqslant 0$ and initial conditions. Now, it follows from the definition of $G^{(\Sigma)}$ and $\mathscr{F}_\varepsilon$ that

$$\|\mathscr{F}_\varepsilon\|_Y \leqslant \|\mathscr{F}_\varepsilon\|_0 \leqslant C(1 + \|\Sigma\|_1^2 + \|n\|_0^2)(1 + \varepsilon\|u_x\|_Y). \tag{6.14}$$

Then differential inequality (6.13) implies that

$$\varepsilon\frac{\mathrm{d}U}{\mathrm{d}t} + \delta U \leqslant C(1 + \|\Sigma\|_1^4 + \|n\|_0^4)(1 + \varepsilon U), \tag{6.15}$$

where $U(t) := 3\kappa\|u(t,\cdot)\|_Y^2 + \eta\|u_x(t,\cdot)\|_Y^2$. To obtain a bound for $\|u_t\|_Y$ we differentiate equation (6.12) with respect to time. Denoting $w = u_t$, w is a solution of

$$\varepsilon w_t - \eta w_{xx} + \eta w_x(1) + \kappa\int_0^1 w = \frac{\mathrm{d}}{\mathrm{d}t}\mathscr{F}_\varepsilon \tag{6.16}$$

subject to the boundary conditions $w_x(t, 0) = w(t, 1) = 0$. Since,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{F}_\varepsilon = \kappa\int_0^1 \Sigma_t + \varepsilon\left(-\Sigma_t - \frac{1}{\eta}\int_x^1 [(1+n)(w_x - \Sigma_{tx}) + n_t(u_x - \Sigma_x)]\right)$$

and

$$\|\Sigma_t\|_0 \leqslant C(1 + \|\Sigma\|_1^2 + \|n\|_0^2 + \|u_x\|_Y^2)$$
$$|\Sigma_{tx}(\cdot, x)| \leqslant C(1 + \|\Sigma\|_1^2 + \|n\|_0^2 + \|1 + n\|_0|u_x(\cdot, x)|)$$
$$|n_t(\cdot, x)| \leqslant C(1 + \|\Sigma\|_1^2 + \|n\|_0^2 + \|\Sigma\|_1|u_x(\cdot, x)|)$$

for a.e. $x \in [0, 1]$, we have

$$\left\|\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{F}_\varepsilon\right\|_Y \leqslant \left\|\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{F}_\varepsilon\right\|_0 \leqslant C(1 + \|\Sigma\|_1^4 + \|n\|_0^4)(1 + \|u_x\|_Y^2 + \varepsilon\|w_x\|_Y). \tag{6.17}$$

Now one can proceed similarly as in the proof of inequality (6.15). By taking the inner product in Y of (6.16) with $3\kappa w - \eta w_{xx}$ we obtain a differential inequality

$$\varepsilon\frac{\mathrm{d}W}{\mathrm{d}t} + \delta W \leqslant C(1 + \|\Sigma\|_1^8 + \|n\|_0^8)(1 + U^2 + \varepsilon W), \tag{6.18}$$

where $W(t) := 3\kappa\|w(t,\cdot)\|_Y^2 + \eta\|w_x(t,\cdot)\|_Y^2$. Now it follows from the evolution equation for u that $\|u\|_{Y^1} = \|Bu\|_Y \leqslant \varepsilon\|u_t\|_Y + \|\mathscr{F}_\varepsilon\|_Y$. Since $\mathrm{Re}\,\sigma(B) > 0$ the norm $\|u\|_{Y^\beta}$, $3/4 < \beta < 1$ is dominated by $\|Bu\|_Y$. Taking into account estimates (6.11), (6.14), (6.15) and (6.18) and using a simple Gronwall's lemma argument we obtain *a priori* estimate

$$\|\Sigma(t,\cdot)\|_1 + \|n(t,\cdot)\|_0 + \|u(t,\cdot)\|_{Y^\beta} \leqslant \text{const} \quad \text{for any } t \in (0, T_{\max}),$$

where $T_{\max}$ is the maximal time of existence of a solution $(\Sigma(t,\cdot), n(t,\cdot), u(t,\cdot))$. Hence, $T_{\max} = \infty$ and the global-in-time existence of solutions in the phase space $\mathscr{X} = X \times Y^\beta$, $3/4 < \beta < 1$, is established.

In what follows, we will prove the existence of a ball in the phase-space $\mathscr{X}$ that dissipates any solution of (6.9). Let $(\Sigma_0, n_0, u_0) \in \mathscr{X}$ be an initial condition. With regard to (6.11) there exists time $T_1 = T_1(\Sigma_0, n_0) > 0$ such that

$$1 + \|\Sigma\|_1^p + \|n\|_0^p \leqslant 1995 \quad \text{for any } t \geqslant T_1 \;\; p = 4, 8.$$

One can choose $0 < \varepsilon_0 \ll 1$ such that $1995C\varepsilon_0 < \delta$ where constants $C, \delta > 0$ appear in inequalities (6.15) and (6.18). Then

$$\varepsilon \frac{\mathrm{d}U(t)}{\mathrm{d}t} + \delta U(t) \leqslant C,$$

$$\varepsilon \frac{\mathrm{d}W(t)}{\mathrm{d}t} + \delta W(t) \leqslant C(1 + U^2(t)) \quad \text{for any } t \geqslant T_1,$$

where $C, \delta > 0$ are constants independent of $\varepsilon \in [0, \varepsilon_0]$ and the initial condition (Σ_0, n_0, u_0). It should be noted that the first differential inequality does not involve W. Then solving the above differential inequalities one can show the existence of a time $T = T(\Sigma_0, n_0, u_0) \geqslant T_1$ such that $U(t) + W(t) \leqslant C$ for any $t \geqslant T$. Recall that $\|u_t(t, \cdot)\|_Y^2 \leqslant W(t)$ and $\|\mathscr{F}_\varepsilon\|_Y$ can be estimated in terms of $U(t)$ for $t \geqslant T$ (see (6.15)). Thus, $\|Bu(t, \cdot)\|_Y \leqslant C$ for $t \geqslant T$. In summary, we have shown the existence of a constant $\varrho_0 > 0$ independent of $\varepsilon \in [0, \varepsilon_0]$ and initial data, such that

$$\|u(t, \cdot)\|_{Y^\beta}^2 + \|(\Sigma(t, \cdot), n(t, \cdot))\|_X^2 \leqslant \varrho_0 \quad \text{for any } t \geqslant T(\Sigma_0, n_0, u_0). \tag{6.19}$$

This means that the ball in $X \times Y^\beta$ of radius $\varrho_0^{1/2}$ is a dissipative set for solutions of (6.9), i.e. any solution enters this ball after a certain amount of time. In other words, the long-time behavior of solutions takes place inside this ball.

As is usual, we will modify the governing equation outside the ball of radius $\varrho_0^{1/2}$. Let $\zeta \in C_{bdd}^2(\mathbb{R}^+, \mathbb{R}^+)$ by any smooth cut-off function with the property $\zeta \equiv 1$ on $[0, 2\varrho_0]$, $\zeta \equiv 0$ on $[3\varrho_0, \infty)$. We define the modified functions $\bar{G} = \bar{G}^{(\Sigma)}, \bar{G}^{(n)} : X \times Y^\beta \to X$ and $\mathscr{F}_\varepsilon : X \times Y^\beta \to Y$ as follows:

$$\bar{G}^{(i)}(\Sigma, n, u)(x) := \zeta(|\Sigma(x)|^2 + |\Sigma_x(x)|^2 + |n(x)|^2 + \|u\|_{Y^\beta}^2) G^{(i)}(\Sigma, n, u)(x),$$

$$\bar{\mathscr{F}}_\varepsilon(\Sigma, n, u)(x) := \zeta(|\Sigma(x)|^2 + |\Sigma_x(x)|^2 + |n(x)|^2 + \|u\|_{Y^\beta}^2) \mathscr{F}_\varepsilon(\Sigma, n, u)(x)$$

for $x \in [0, 1]$, i stands either for Σ or n. We remind ourselves that the mapping $u \mapsto \|u\|_{Y^\beta}^2$ is a twice continuously Frechet differentiable function from Y^β to R. The modified functions $\mathscr{G}$ and $\bar{\mathscr{F}}_\varepsilon$ obey hypothesis (H4). With regard to the definitions of Q and θ (see (6.7), (6.8)) it is easy to verify that hypotheses (H2) and (H3) are also fulfilled. Since $\mathscr{F}_0$ does not depend on u, the structural condition (5.1) is satisfied for any $\delta_0 > 0$. Taking into account Lemma 6.2 and (6.3) we have shown that all the conclusions of Theorem 1.1 hold for system (6.9) except for the statement that $\mathscr{M}_\varepsilon$ is an invariant manifold for the semi-flow generated by solutions of (6.9). This is due to the fact that we have modified the governing equations far from the vicinity of a dissipative ball of the radius $\varrho_0^{1/2}$. Hence, $\mathscr{M}_\varepsilon$ need not be invariant outside this ball. On the other hand, it should be emphasized that the long-time behaviour of solutions of (6.9) takes place inside this ball as it was shown in (6.19). Henceforth, we will therefore refer to $\mathscr{M}_\varepsilon$ as a local invariant manifold for solutions of (6.9).

Now we can rewrite the feedback law in terms of the flow variables σ, n, v as follows: $f = f_\varepsilon(\sigma, n)$ where $f_\varepsilon(\sigma, n) = \Xi_\varepsilon(\Sigma, n)$. For the velocity field on the manifold $\mathscr{M}_\varepsilon$ we obtain the expression $v = \Psi_\varepsilon(\sigma, n) = (u - \Sigma)/\eta = (\Phi_\varepsilon(\Sigma, n) - \Sigma)/\eta$ where $\Sigma(x) = -\int_x^1 \sigma(\xi)\,\mathrm{d}\xi$. We infer from the continuity of the imbedding $Y^\beta \subsetneq C^1_{bdd}(0, 1)$, $\frac{3}{4} < \beta$, (see Lemma 6.2) that

$$\Psi_\varepsilon : C^0_{bdd}(0, 1) \cap \{\sigma, \sigma(0) = 0\} \times C^0_{bdd}(0, 1) \to C^1_{bdd}(0, 1)$$

is C^1 smooth and Ψ_ε is locally C^1 close to Ψ_0. Similarly, one has

$$f_\varepsilon : C^0_{bdd}(0, 1) \cap \{\sigma, \sigma(0) = 0\} \times C^0_{bdd}(0, 1) \to \mathbb{R}$$

is C^1 smooth and f_ε is locally C^1 close to Ψ_0. Furthermore, with regard to Remark 6.1 we have an explicit formula for f_0 and Ψ_0,

$$f_0 = 3\eta Q_{\mathrm{fix}} + 3\int_0^1 \Sigma(\xi)\,\mathrm{d}\xi$$

$$v(x) = \Psi_0(\sigma, n)(x) = \left((1 - x^2) f_0/2 + \int_x^1 \sigma(\xi)\,\mathrm{d}\xi\right).$$

Summarizing the results of section 6 we can state the following theorem.

Theorem 6.3. *There exists* $0 < \varepsilon_0 \ll 1$ *such that, for any* $\varepsilon \in [0, \varepsilon_0]$, *the system of equations governing the Poiseuille flow of the Johnson–Segalman–Oldroyd fluid* (6.1)–(6.2) *admits a dissipative feedback synthesis of the pressure gradient*

$$f = f_\varepsilon(\sigma, n), \quad \sigma, n \in C^0_{bdd}(0, 1)$$

that stabilizes the volumetric flow rate at the prescribed value Q_{fix}. *The mapping* $f_\varepsilon : C^0_{bdd}(0, 1) \cap \{\sigma, \sigma(0) = 0\} \times C^0_{bdd}(0, 1) \to \mathbb{R}$ *is* C^1*-smooth and* f_ε *is locally* C^1 *close to* f_0 *whenever* $\varepsilon > 0$ *is small enough. The feedback law* f_0 *for the reduced system of equations has the form*

$$f_0(\sigma, n) = 3\eta Q_{\mathrm{fix}} - 3\int_0^1 \xi\sigma(\xi)\,\mathrm{d}\xi.$$

The initial-value problem (6.1)–(6.2) *with* $f = f_\varepsilon(\sigma, n)$ *possesses an infinite dimensional locally invariant attractive manifold* $\mathscr{M}_\varepsilon$. *The volumetric flow rate for solutions belonging to* $\mathscr{M}_\varepsilon$ *is fixed at the prescribed value* Q_{fix}. *The manifold* $\mathscr{M}_\varepsilon$ *is a* C^1 *smooth graph,*

$$\mathscr{M}_\varepsilon = \{(\sigma, n, v),\ v = \Psi_\varepsilon(\sigma, n),\ \sigma, n \in C^0_{bdd}(0, 1),\ \|\sigma\|_0^2 + \|n\|_0^2 < \varrho_0\},$$

where $\Psi_\varepsilon : C^0_{bdd}(0, 1) \cap \{\sigma, \sigma(0) = 0\} \times C^0_{bdd}(0, 1) \to C^1_{bdd}(0, 1)$ *is a* C^1 *function which is locally* C^1 *close to* Ψ_0,

$$\Psi_0(\sigma, n)(x) = \frac{1}{\eta}\left((1 - x^2) f_0(\sigma, n)/2 + \int_x^1 \sigma(\xi)\,\mathrm{d}\xi\right), \quad x \in [0, 1].$$

Finally, the flow when restricted to the manifold $\mathscr{M}_\varepsilon$ *is governed by the following system of functional differential equations*:

$$\begin{aligned} \sigma_t &= -\sigma + (1 + n)\Psi_\varepsilon(\sigma, n)_x, \\ n_t &= -n - \sigma\Psi_\varepsilon(\sigma, n)_x, \end{aligned} \tag{FDE}$$

$(t, x) \in [0, \infty) \times [0, 1]$, *subject to boundary and initial conditions* (6.2). *For small values of* $\varepsilon > 0$, *the vector field defined by the right-hand side of* (*FDE*) *is locally* C^1 *close to that of the reduced system of equations*

$$\begin{aligned} \sigma_t &= -\sigma + (1 + n)(T - \sigma)/\eta, \\ n_t &= -n - \sigma(T - \sigma)/\eta, \end{aligned} \tag{QFDE}$$

where $T = -f_0(\sigma, n)x$.

Acknowledgement

The author wishes to thank Prof. J. Nohel for stimulating discussions concerning non-Newtonian fluid mechanics.

References

1. Aarts, A. and Van De Ven, A., 'Transient behaviour and instability points of the Poiseuille flow of a KBKZ fluid', *J. Eng. Math.*, **29,** 371–392 (1995).
2. Brunovský, P. and Ševčovič, D., 'Explanation of spurt for a non-Newtonian fluid by a diffusion term', *Quart. Appl. Math.*, **52,** 401–426 (1994).
3. Chow, S.-N. and Lu, K., 'Invariant manifolds for flows in Banach spaces,' *J. Differential Equations*, **74,** 285–317 (1988).
4. Foias, C., Sell, G. R. and Temam, R., 'Inertial manifolds for nonlinear evolutionary equations', *J. Differential Equations*, **73,** 309–353 (1988).
5. Grob, M., 'Flow instabilities in a polymer extrusion process' *Ph.D. thesis*, Eidhoven Institute of Technology, Nov. 1994.
6. Henry, D., 'Geometric theory of semilinear parabolic equations.' Lecture Notes in Math. 840, Springer, Berlin, 1981.
7. Lim, F. J. and Schowalter, W. R., Wall slip of narrow molecular weight distribution polybutadienes, *J. Rheology*, **33,** 1359–1382 (1989).
8. Marion, M., 'Inertial manifolds associated to partly dissipative reaction – diffusion systems', *J. Math. Anal. Appl.*, **143,** 295–326 (1989).
9. Malkus, D. S., Nohel, J. A. and Plohr, B. J., 'Analysis of new phenomena in shear flow of non-Newtonian fluids', *SIAM J. Appl. Math.*, **51,** 899–929 (1991).
10. Malkus, D. S., Nohel, J. A. and Plohr, B. J., 'Approximation of piston-driven flows of a non-Newtonian fluid', in: *Control Theory, Dynamical Systems and Geometry of Dynamics*, 1993 (Elworthy, Everitt, and Lee, eds.), Marcel Dekker, New York, 1993; pp. 173–192.
11. Nohel, J. A., Pego, R. L. and Tzavaras, A. E., Stability of discontinuous steady states in shearing motion of a non-newtonian fluid', *Proc. of the Royal Society of Edinburgh*, **115A,** 39–59 (1990).
12. Nohel, J. A. and Pego, R. L., 'Nonlinear stability and asymptotic behavior of shearing motions of a non-Newtonian fluid, *SIAM J. Math. Anal.*, **24,** 911–942 (1993).
13. Ševčovič, D., 'Limiting behaviour of invariant manifolds for a system of singularly perturbed evolution equations', *Math. Methods in the Appl. Sci.*, **17,** 643–666 (1994).
14. Ševčovič, D., 'The C^1 stability of slow manifolds for a system of singularly perturbed evolution equations', *Comment. Math. Univ. Carolinae*, **36,** 89–107 (1995).
15. Ševčovič, D., 'Smoothness of the singular limit of inertial manifolds of singularly perturbed evolution equations', *Nonlinear Anal. TMA.*, **28,** (1996).

Príloha 6.2.1

Reprint práce:

K. Mikula, D. Ševčovič: *Evolution of plane curves driven by a nonlinear function of curvature and anisotropy.* To appear in: SIAM Journal of Applied Mathematics.

SIAM J. Appl. Math.
Vol. 61, No. 5, pp. 1473–1501

EVOLUTION OF PLANE CURVES DRIVEN BY A NONLINEAR FUNCTION OF CURVATURE AND ANISOTROPY*

KAROL MIKULA† AND DANIEL ŠEVČOVIČ‡

Abstract. In this paper we study evolution of plane curves satisfying a geometric equation $v = \beta(k, \nu)$, where v is the normal velocity and k and ν are the curvature and tangential angle of a plane curve Γ. We follow the direct approach and we analyze the so-called intrinsic heat equation governing the motion of plane curves obeying such a geometric equation. The intrinsic heat equation is modified to include an appropriate nontrivial tangential velocity functional α. We show how the presence of a nontrivial tangential velocity can prevent numerical solutions from forming various instabilities. From an analytical point of view we present some new results on short time existence of a regular family of evolving curves in the degenerate case when $\beta(k, \nu) = \gamma(\nu)k^m$, $0 < m \le 2$, and the governing system of equations includes a nontrivial tangential velocity functional.

Key words. nonlinear curve evolution, intrinsic heat equation, degenerate parabolic equations, tangential velocity, numerical solution

AMS subject classifications. 35K65, 65N40, 53C80

PII. S0036139999359288

1. Introduction. The goal of this paper is to study curvature-driven evolution of a family of closed smooth plane curves. We consider the case when the normal velocity v of an evolving family of plane curves $\Gamma^t : S^1 \to \mathbb{R}^2, t \in (0, T)$, is a function of the curvature k and the tangential angle ν:

$$v = \beta(k, \nu). \tag{1.1}$$

In past years, geometric equations of the form (1.1) have attracted a lot of attention from both the theoretical and the practical point of view. There is a wide range of possible applications of geometric equations of the form (1.1). They arise from various applied problems in mathematical modeling and scientific computing, and they can be investigated in a purely mathematical context.

In the theory of phase interfaces separating solid and liquid phases, (1.1) corresponds to the so-called Gibbs–Thomson law governing the crystal growth in an undercooled liquid [25, 39, 13]. In the series of papers [9, 10, 11] Angenent and Gurtin studied motion of phase interfaces. They proposed to study the equation of the form $\mu(\nu, v)v = h(\nu)k - g$, where μ is the kinetic coefficient and quantities h, g arise from constitutive description of the phase boundary. The dependence of the normal velocity v on the curvature k is related to surface tension effects on the interface, whereas the dependence on ν (orientation of interface) introduces anisotropic effects into the model. In general, the kinetic coefficient μ may also depend on the velocity v itself giving rise to a nonlinear dependence of the function $v = \beta(k, \nu)$ on k and ν. If the motion of an interface is very slow, then $\beta(k, \nu)$ is linear in k (cf. [9]) and (1.1) corresponds to the classical mean curvature flow studied extensively from both

*Received by the editors July 16, 1999; accepted for publication (in revised form) June 14, 2000; published electronically January 19, 2001. This research was supported by grants 1/7132/20 and 1/7677/20 from the Slovak Scientific Grant Agency VEGA.

http://www.siam.org/journals/siap/61-5/35928.html

†Department of Mathematics, Slovak University of Technology, Radlinského 11, 813 68 Bratislava, Slovak Republic (mikula@vox.svf.stuba.sk).

‡Institute of Applied Mathematics, Faculty of Mathematics and Physics, Comenius University, 842 48 Bratislava, Slovak Republic (sevcovic@fmph.uniba.sk).

the mathematical (see, e.g., [21, 1, 5, 24]) and the numerical point of view (see, e.g., [18, 16, 31, 35, 36]).

In the image processing the so-called morphological image and shape multiscale analysis is often used because of its contrast and affine invariance properties. It has been introduced by Sapiro and Tannenbaum [38] and Alvarez et al. [2, 3]. Analysis of image silhouettes (boundaries of distinguished shapes) leads to an equation of the form (1.1) without anisotropic part. Among various choices of a function $\beta(k)$ the so-called affine invariant scale space has special conceptual meaning and importance. In this case the velocity v is given by $v = \beta(k) = k^{1/3}$ [2, 38, 12]. In the context of image segmentation, various anisotropic models with $v = \beta(k, \nu)$ have been studied just recently [27, 30, 15]. For a comprehensive overview of applications of (1.1) in other applied problems, we refer to [42].

The analytical methods for mathematical treatment of (1.1) are strongly related to numerical techniques for computing curve evolutions. In the *direct approach* one seeks for a parameterization of the evolving family of curves. By solving the so-called *intrinsic heat equation* one can directly find a position vector of a curve (see, e.g., [17, 18, 19, 33, 39, 40]). There are also other direct methods based on solution of a porous medium–like equation for curvature of a curve [31, 32], a crystalline curvature approximation [22, 23, 44], special finite difference schemes [28, 29], and a method based on erosion of polygons in the affine invariant scale case [34]. By contrast to the direct approach, *level set methods* are based on introducing an auxiliary function whose zero level sets represent an evolving family of planar curves undergoing the geometric equation (1.1) (see, e.g., [36, 41, 42, 43, 26]). The other indirect method is based on the phase-field formulations (see, e.g., [14, 35, 20, 13]). The level set approach handles implicitly the curvature-driven motion, passing the problem to higher dimensional space. One can deal with splitting and/or merging of evolving curves in a robust way. However, from the computational point of view, level set methods are much more expensive than methods based on the direct approach.

In this paper we are concerned with the direct approach only. We consider the power-like function $\beta(k, \nu) = \gamma(\nu)|k|^{m-1}k$, where $\gamma(\nu) > 0$ is a given anisotropy function and $m > 0$. From the analytical perspective, the main purpose is to establish short time existence of a family of regular smooth plane curves satisfying the geometric equation (1.1). A short time existence result was obtained for the singular case in which $0 < m < 1$ as well as for the degenerate case in which $1 < m \leq 2$. Let us emphasize that we needed additional geometric assumptions made on an initial curve in the degenerate case $1 < m \leq 2$. Cases with higher powers of m do not seem to be treatable by our techniques. On the other hand, recent results due to Andrews [4] show that the value $m = 2$ is critical in the sense that, for higher powers of m, a solution need not necessarily be classical in points where the curvature vanishes.

In our approach, a family of evolving curves is represented by their position vector x satisfying the geometric equation

$$\partial_t x = \beta(k, \nu)\vec{N} + \alpha\vec{T}. \tag{1.2}$$

Notice that the presence of an arbitrary tangential velocity functional α has no effect on the shape of evolving curves. The usual choice is therefore $\alpha = 0$. From the numerical point of view, such a choice of α may lead to computational instabilities caused by merging of numerical grid points representing a discrete curve or by formation of the so-called swallow tails. In this paper we present an appropriate choice of a nontrivial tangential velocity α. It turns out that if α is a solution of the nonlocal equation

$$\frac{\partial \alpha}{\partial s} = k\beta(k,\nu) - \frac{1}{|\Gamma|}\int_\Gamma k\beta(k,\nu)\, ds, \tag{1.3}$$

then material points are uniformly redistributed along the evolved curve. This choice of α results in a powerful numerical scheme having the property of uniform-in-time redistribution of grid points and preventing the computed numerical solution from forming the above-mentioned instabilities. Note that (1.2) can be transformed into a one-dimensional intrinsic heat equation (see (2.2)), and the functional α can be easily resolved from (1.3). In each time step we have to solve several linear tridiagonal systems in order to obtain a new position of the curve.

The outline of the paper is as follows. In section 2 we present the governing system of PDEs. Evolution of plane curves is parameterized by solutions of an intrinsic heat equation. We discuss the effect of a nontrivial tangential velocity on numerically computed solutions. Section 3 is focused on the analysis of the system of governing equations. The aim is to set up a closed system of parabolic equations solutions which include the curvature, the tangent angle, and the local length of a plane curve. The basic theory on short time existence of classical solutions is given in section 4. Here we consider only the case when β'_k is nondegenerate. We follow the abstract theory due to Angenent slightly modified for the case when a nontrivial tangential velocity functional is involved in the system of governing PDEs. Section 5 is devoted to the study of the singular case when $\beta(k,\nu) = \gamma(\nu)|k|^{m-1}k$, $m \neq 1$. We extend the result due to Angenent, Sapiro, and Tannenbaum obtained for the power $m = 1/3$ to the general fast diffusion powers $0 < m < 1$ as well as for degenerate slow diffusion cases where $1 < m \leq 2$. In section 6 we present a suitable choice of a tangential velocity leading to a powerful numerical scheme. We show how to construct a nontrivial tangential velocity as a nonlocal curve functional in such a way that relative local length (defined as the ratio of the local length to the total length of a curve) is constant along the evolution. A numerical scheme for full space-time discretization of the governing intrinsic heat equation is presented in section 7. We derive this scheme by using the method of so-called flowing finite volumes. In section 8 we show several numerical solutions of the governing system of equations and we make a comparison between results obtained by considering the trivial and nontrivial tangential velocities, respectively. One can observe the importance of the presence of a suitable nontrivial tangential velocity functional in the governing system of equations for stability of numerical computations.

2. Preliminaries.

Consider an embedded regular plane curve Γ that can be parameterized by a C^2 smooth function $x : S^1 \to \mathbb{R}^2$ such that $\Gamma = \text{Image}(x) = \{x(u), u \in [0,1]\}$ and $|\partial_u x| > 0$. One can define the unit tangent vector $\vec{T} = \partial_u x/|\partial_u x|$ and the unit normal vector $\vec{N}$ in such a way that $\vec{T} \wedge \vec{N} = 1$, where $\vec{a} \wedge \vec{b}$ is the determinant of the 2×2 matrix with column vectors $\vec{a}, \vec{b}$. Henceforth, we will denote $\vec{a} \,.\, \vec{b}$ as the Euclidean inner product of two vectors. By $|\vec{a}| = (\vec{a} \,.\, \vec{a})^{1/2}$ we denote the Euclidean norm of a vector $\vec{a}$. The derivative of a function $f = f(\xi)$ with respect to ξ will be denoted by $\partial_\xi f$. The arc-length parameterization will be denoted by s. Clearly, $ds = |\partial_u x| du$. By k we denote the signed curvature of the curve $\Gamma = \text{Image}(x)$ defined as

$$k = \partial_s x \wedge \partial_s^2 x = \frac{\partial_u x \wedge \partial_u^2 x}{|\partial_u x|^3}; \tag{2.1}$$

then Frenet's formulae read as follows: $\partial_s \vec{T} = k\vec{N}, \partial_s \vec{N} = -k\vec{T}$. The angle ν of the tangential vector is given by $\nu = \arg(\vec{T})$, i.e., $(\cos\nu, \sin\nu) = \partial_s x$. To describe the time evolution $\{\Gamma^t\}, t \in [0,T)$ of an initial curve $\Gamma = \Gamma^0 = \text{Image}(x^0)$, we adopt the notation $\Gamma^t = \{x(u,t), u \in [0,1]\}, \quad t \in [0,T)$, where $x \in C^2(Q_T, \mathbb{R}^2)$ and $Q_T = S^1 \times [0,T)$. We will frequently identify Q_T with $[0,1] \times [0,T)$ and the space $C^l(Q_T, \mathbb{R}^2)$ with the space of C^l differentiable functions defined on $[0,1]$ and satisfying periodic boundary conditions. The main idea in describing a family of evolving plane curves $\Gamma^t, t > 0$, satisfying the geometric equation (1.1) is to parameterize Γ^t by a solution $x \in C^2(Q_T, \mathbb{R}^2)$ of the so-called *intrinsic heat equation*

$$\frac{\partial x}{\partial t} = \frac{1}{\theta_1}\frac{\partial}{\partial s}\left(\frac{1}{\theta_2}\frac{\partial x}{\partial s}\right), \quad x(.,0) = x^0(.), \tag{2.2}$$

where θ_1, θ_2 are geometric quantities for the curve $\Gamma^t = \text{Image}(x(.,t))$, i.e., functions whose definition is independent of particular parameterization of Γ^t and such that

$$\theta_1\theta_2 = \frac{k}{\beta(k,\nu)}. \tag{2.3}$$

By using (2.3) and Frenet's formulae, (2.2) can be rewritten in the following equivalent form:

$$\frac{\partial x}{\partial t} = \beta\vec{N} + \alpha\vec{T}, \quad x(.,0) = x^0(.), \tag{2.4}$$

where $\beta = \beta(k,\nu)$ is the normal velocity of the evolving curve and α is the tangential velocity given by

$$\alpha = \frac{1}{\theta_1}\frac{\partial}{\partial s}\left(\frac{1}{\theta_2}\right). \tag{2.5}$$

The normal component v of the velocity $\partial_t x$ is therefore equal to $\beta(k,\nu)$. By [12, Lemma 4.1] the family $\Gamma^t = \text{Image}(x(.,t))$ parameterized by a solution x of the geometric equation (2.4) can be converted into a solution of $\partial_t x = \beta\vec{N} + \bar{\alpha}\vec{T}$ for any continuous function $\bar{\alpha}$ by changing the space parameterization of the original curve. In particular, it means that one can take $\bar{\alpha} = 0$ without changing the shape of evolving curves. On the other hand, as can be observed from our numerical simulations, the presence of a suitable tangential velocity term $\alpha\vec{T}$ is necessary for construction of a numerical scheme capable of suitable redistribution of numerical grid points along a computed curve.

In [33] the authors studied the intrinsic heat equation (2.2) with $\theta_1 = \theta_2 = (k/\beta(k))^{1/2}$. In this case, (2.2) has the form $\partial_t x = \partial^2_{\bar{s}} x$, where $d\bar{s} = \theta_1 ds$. Using this particular choice of θ_1, θ_2 we were able to simulate the evolution of plane convex and nonconvex curves for the case where $v = |k|^{m-1}k$. Satisfactory results were obtained only for $0 < m \le 1$, whereas various numerical instabilities appeared for the case $m > 1$. The mathematical explanation for such a behavior is very simple. If $\theta_1 = \theta_2 = |k|^{\frac{m-1}{2}}$, then, by (2.5), $\alpha = \frac{m-1}{2}|k|^{m-3}k\partial_s k = \frac{1}{2}\partial_s(|k|^{m-1})$. In the case $m > 1$ numerical grid points were driven by the tangential velocity $\alpha\vec{T}$ toward pieces of the curve with the increasing curvature. It may lead to serious computational troubles. The effect of α is just the opposite when $0 < m < 1$.

Another possible choice of a nontrivial tangential velocity was studied by Deckelnick in [16] for the case $\beta(k) = k$. He proposed a governing PDE in the form

$\partial_t x = \partial_u^2 x/|\partial_u x|^2$. In this case $\alpha = -\partial_u(|\partial_u x|^{-1})$, and thus $\theta_2 = \theta_1^{-1}, \theta_1 = |\partial_u x|$. This algorithm also has the property of a suitable redistribution of grid points along the computed curve. Notice that θ_1, θ_2 are not geometric quantities because of their dependence on a particular parameterization.

Note that the arc-length parameterization s occurring in the intrinsic equation (2.2) depends on time t and its initial position u at $t = 0$ via $ds = |\partial_u x| du$. We can therefore rewrite (2.2) into the following Eulerian form:

$$\frac{\partial x}{\partial t} = \frac{1}{\theta_1 |\partial_u x|} \frac{\partial}{\partial u} \left(\frac{1}{\theta_2 |\partial_u x|} \frac{\partial x}{\partial u} \right), \qquad x(.,0) = x^0(.), \quad (u,t) \in Q_T. \tag{2.6}$$

(2.6) seems to be a parabolic PDE for $x = x(u,t)$. However, as θ_2 may depend on the curvature, the right-hand side of (2.6) may eventually contain the third-order derivative term $\partial_u^3 x$. In the next section we will show how to overcome this difficulty by embedding (2.6) into a complete system of nonlinear parabolic equations.

3. Equations for geometric quantities. The goal of this section is to derive a system of PDEs governing the evolution of the curvature k of $\Gamma^t = \text{Image}(x(.,t)), t \in [0,T)$, and some other geometric quantities where the family of regular plane curves where $x = x(u,t)$ is a solution to the intrinsic heat equation (2.2). These equations will be used in order to derive a priori estimates of solutions. Notice that such an equation for the curvature is well known for the case when $\alpha = 0$, and it reads as follows: $\partial_t k = \partial_s^2 \beta + k^2 \beta$, where $\beta = \beta(k, \nu)$ (cf. [21, 9]). Here we present a brief sketch of the derivation of the corresponding equations for the case of a nontrivial tangential velocity α.

Let us denote $\vec{p} = \partial_u x$. Then, by using Frenet's formulae, one has

$$\begin{aligned} \partial_t \vec{p} &= |\partial_u x|((\partial_s \beta + \alpha k)\vec{N} + (-\beta k + \partial_s \alpha)\vec{T}), \\ \vec{p}\,.\,\partial_t \vec{p} &= |\partial_u x|\; \vec{T}\,.\,\partial_t \vec{p} = |\partial_u x|^2(-\beta k + \partial_s \alpha), \\ \vec{p} \wedge \partial_t \vec{p} &= |\partial_u x|\; \vec{T} \wedge \partial_t \vec{p} = |\partial_u x|^2\; (\partial_s \beta + \alpha k), \\ \partial_t \vec{p} \wedge \partial_u \vec{p} &= -|\partial_u x| \partial_u |\partial_u x|(\partial_s \beta + \alpha k) + |\partial_u x|^3\; (-\beta k + \partial_s \alpha), \end{aligned} \tag{3.1}$$

because $p_u = \partial_u^2 x = \partial_u(|\partial_u x|\; \vec{T}) = \partial_u |\partial_u x|\; \vec{T} + k |\partial_u x|^2\; \vec{N}$. Since $\partial_u(\vec{p} \wedge \partial_t \vec{p}) = \partial_u \vec{p} \wedge \partial_t \vec{p} + \vec{p} \wedge \partial_u \partial_t \vec{p}$, we have $\vec{p} \wedge \partial_u \partial_t \vec{p} = \partial_u(\vec{p} \wedge \partial_t \vec{p}) + \partial_t \vec{p} \wedge \partial_u \vec{p}$. As $k = (\vec{p} \wedge \partial_u \vec{p})\, |\vec{p}|^{-3}$ (see (2.1)), we obtain

$$\begin{aligned} \partial_t k &= -3|p|^{-5}(\vec{p}\,.\,\partial_t \vec{p})(\vec{p} \wedge \partial_u \vec{p}) + |\vec{p}|^{-3}\, ((\partial_t \vec{p} \wedge \partial_u \vec{p}) + (\vec{p} \wedge \partial_u \partial_t \vec{p})) \\ &= -3k|\vec{p}|^{-2}(\vec{p}\,.\,\partial_t \vec{p}) + 2|\vec{p}|^{-3}(\partial_t \vec{p} \wedge \partial_u \vec{p}) + |\vec{p}|^{-3} \partial_u(\vec{p} \wedge \partial_t \vec{p}). \end{aligned}$$

Finally, by applying identities (3.1), we end up with the second-order nonlinear parabolic PDE, the *equation for the curvature*:

$$\partial_t k = \partial_s^2 \beta + \alpha \partial_s k + k^2 \beta, \quad k(.,0) = k^0(.). \tag{3.2}$$

Similarly, as in (2.6), the above equation can be rewritten into the Eulerian form

$$\begin{gathered} \frac{\partial k}{\partial t} = \frac{1}{|\partial_u x|} \frac{\partial}{\partial u} \left(\frac{1}{|\partial_u x|} \frac{\partial}{\partial u} \beta(k, \nu) \right) + \alpha \frac{1}{|\partial_u x|} \frac{\partial k}{\partial u} + k^2 \beta(k, \nu), \\ k(.,0) = k^0(.), \end{gathered} \tag{3.3}$$

where $(u,t) \in Q_T$. The identities (3.1) can be used in order to derive an evolutionary equation for the local length $|\partial_u x|$. Indeed, $|\partial_u x|_t = (\partial_u x \,.\, \partial_u \partial_t x)/|\partial_u x| = (\vec{p} \,.\, \partial_t \vec{p})/|\partial_u x|$. By (3.1) we have the *local length equation*

$$\frac{\partial}{\partial t}|\partial_u x| = -|\partial_u x|\, k\beta \; + \; \frac{\partial \alpha}{\partial u}, \qquad |\partial_u x(.,0)| = |\partial_u x^0(.)|, \tag{3.4}$$

where $(u,t) \in Q_T$. In other words, $\partial_t ds = (-k\beta + \partial_s \alpha)ds$. By integrating (3.4) over the interval $[0,1]$ and taking into account that α satisfies periodic boundary conditions, we obtain the *total length equation*

$$\frac{d}{dt}L^t + \int_{\Gamma^t} k\beta(k,\nu)ds = 0, \tag{3.5}$$

where $L^t = L(\Gamma^t)$ is the total length of the curve Γ^t, $L^t = \int_{\Gamma^t} ds = \int_0^1 |\partial_u x(u,t)|\, du$. If $k\beta(k,\nu) \geq 0$, then the evolution of plane curves parameterized by a solution of (2.2) represents a curve shortening flow, i.e., $L^{t_2} \leq L^{t_1} \leq L^0$ for any $0 \leq t_1 \leq t_2 \leq T$. The condition $k\beta(k,\nu) \geq 0$ is obviously satisfied in the case $\beta(k,\nu) = \gamma(\nu)|k|^{m-1}k$, where $m > 0$ and γ is a nonnegative anisotropy function.

The area enclosed by an embedded non-self-intersecting curve $\Gamma = \mathrm{Image}(x)$ can be computed as $A = \frac{1}{2}\int_0^1 x \wedge \partial_u x\, du$. Applying the identities (3.1) and taking into account that $0 = \int_0^1 \partial_u (x \wedge \partial_t x)\, du = \int_0^1 (\vec{p} \wedge \partial_t x + x \wedge \partial_t \vec{p})\, du$, where $\vec{p} = \partial_u x$, we obtain the *area equation*

$$\frac{d}{dt}A^t + \int_{\Gamma^t} \beta(k,\nu)ds = 0. \tag{3.6}$$

If $\beta(k,\nu)$ is nonnegative along the evolution, then the area is a nonincreasing function of the time.

Denote by $[\partial_t, \partial_s]$ the commutator of the differential operators ∂_t and ∂_s, i.e., $[\partial_t, \partial_s] = \partial_t \partial_s - \partial_s \partial_t$. Since $ds = |\partial_u x| du$ it follows from the local length equation (3.4) that the *commutation relation*

$$[\partial_t, \partial_s] = (\beta k - \partial_s \alpha)\partial_s. \tag{3.7}$$

Recall that the tangential vector ν to a curve $\Gamma = \mathrm{Image}(x)$ is given by $\nu = \arg(\vec{T})$, i.e., $(\cos\nu, \sin\nu) = \partial_s x$. From (3.7) we obtain $\partial_t \nu = \partial_s x \wedge \partial_t \partial_s x = \partial_s x \wedge \partial_s \partial_t x + (\beta k - \partial_s \alpha)(\partial_s x \wedge \partial_s x)$. Applying Frenet's formulae and (2.4), we obtain the *tangential vector equation*

$$\partial_t \nu = \partial_s \beta + \alpha k, \quad \nu(.,0) = \nu^0(.). \tag{3.8}$$

Clearly,

$$\partial_s \nu = \partial_s x \wedge \partial_s^2 x = k. \tag{3.9}$$

Differentiating the curvature equation (3.2) with respect to t and taking into account (3.8) yield an equation for the normal velocity $v = \beta(k,\nu)$, i.e., the *normal velocity equation*

$$\begin{aligned} \partial_t v &= \beta'_k \left(\partial_s^2 v + \alpha \partial_s k + k^2 v\right) + \beta'_\nu \left(\partial_s v + \alpha k\right), \\ v(.,0) &= v^0(.) = \beta(k^0(.), \nu^0(.)), \end{aligned} \tag{3.10}$$

where β'_k and β'_ν are partial derivatives of the function $\beta = \beta(k, \nu)$ with respect to k and ν, respectively. Next we derive an equation for the gradient of the normal velocity $w = \partial_s v = \partial_s \beta(k, \nu)$. Using the commutation relation (3.7) we have

$$\begin{aligned}\partial_t w &= \partial_t \partial_s v = \partial_s \partial_t v + (vk - \partial_s \alpha)\partial_s v \\ &= \partial_s \left(\beta'_k(\partial_s w + \alpha \partial_s k + k^2 v)\right) + \beta'_\nu (w + \alpha k) + (vk - \partial_s \alpha) w.\end{aligned}$$

Since

$$w = \partial_s \beta(k, \nu) = \beta'_k \partial_s k + \beta'_\nu \partial_s \nu = \beta'_k \partial_s k + \beta'_\nu k, \tag{3.11}$$

we end up with an *equation for the gradient w of the velocity v*:

$$\begin{aligned}\partial_t w &= \partial_s \left(\beta'_k \partial_s w\right) + \alpha \partial_s w + \partial_s \left(\beta'_k k^2 v + \beta'_\nu w\right) + kvw, \\ w(.,0) &= w^0(.) = \partial_s v^0(.).\end{aligned} \tag{3.12}$$

Now we are in a position to derive a closed system of governing equations for the geometric motion satisfying (1.2). It follows from (3.9) and (3.11) that $\partial_t \nu = \beta'_k \partial_s^2 \nu + k(\alpha + \beta'_\nu)$. Denoting $g = |\partial_u x|$, we can rewrite (3.3), (3.4), and (3.8) into the following closed form *governing equations*

$$\begin{aligned}\frac{\partial k}{\partial t} &= \frac{1}{g}\frac{\partial}{\partial u}\left(\frac{1}{g}\frac{\partial}{\partial u}\beta(k, \nu)\right) + \frac{\alpha}{g}\frac{\partial k}{\partial u} + k^2 \beta(k, \nu), \\ \frac{\partial \nu}{\partial t} &= \frac{\beta'_k(k, \nu)}{g}\frac{\partial}{\partial u}\left(\frac{1}{g}\frac{\partial \nu}{\partial u}\right) + k(\alpha + \beta'_\nu(k, \nu)), \\ \frac{\partial g}{\partial t} &= -gk\beta(k, \nu) + \frac{\partial \alpha}{\partial u},\end{aligned} \tag{3.13}$$

$(u, t) \in [0, 1] \times (0, T)$. A solution to (3.13) is subject to the initial conditions

$$k(.,0) = k^0, \quad \nu(.,0) = \nu^0, \quad g(.,0) = g^0 \tag{3.14}$$

and periodic boundary conditions. Notice that the initial conditions for k^0, ν^0, g^0 are related through the identity

$$\partial_u \nu^0 = g^0 k^0. \tag{3.15}$$

In general, the function $\alpha = \alpha(k, \nu, g)$ is a nonlinear function that will be determined later. In section 6 of this paper we present the choice of α leading to a powerful numerical algorithm preserving relative local length between numerical grid points.

4. Short time existence of solutions in the nondegenerate case. In this section we prove short time existence of a classical solution of the governing system of equations (3.13) by using the abstract result due to Angenent (cf. [8]).

Denote $\Phi = (k, \nu, g)^T$. Then (3.13) can be rewritten as a fully nonlinear PDE of the form

$$\partial_t \Phi = f(\Phi), \qquad \Phi(0) = \Phi^0, \tag{4.1}$$

where $f(\Phi) = F(\Phi, \alpha(\Phi))$ and $F(\Phi, \alpha)$ is the right-hand side of (3.13). Suppose that $\beta = \beta(k, \nu)$ is a C^2 smooth function such that

$$0 < \lambda_- \le \beta'_k(k, \nu) \le \lambda_+ < \infty \quad \text{for any } k, \nu, \tag{4.2}$$

where $\lambda_\pm > 0$ are constants and β'_k is a partial derivative of β with respect to k. Given $0 < \sigma < 1$, we denote by $E_0, E_{1/2}, E_1$ the following Banach spaces:

$$
\begin{aligned}
E_0 &= c^\sigma(S^1) \times c^\sigma(S^1) \times c^{1+\sigma}(S^1),\\
E_{1/2} &= c^{1+\sigma}(S^1) \times c^{1+\sigma}(S^1) \times c^{1+\sigma}(S^1),\\
E_1 &= c^{2+\sigma}(S^1) \times c^{2+\sigma}(S^1) \times c^{1+\sigma}(S^1),
\end{aligned}
\tag{4.3}
$$

where $c^{k+\sigma}, k = 0, 1, 2$, is the little Hölder space, i.e., the closure of $C^\infty(S^1)$ in the topology of the Hölder space $C^{k+\sigma}(S^1)$ (see [6]). Let $\mathcal{O}_i \subset E_i$ be an open subset in E_i such that $g > 0$ for any $(k, \nu, g)^T \in \mathcal{O}_i$, $i = \frac{1}{2}, 1$. If we assume

$$
\alpha \in C^1(\mathcal{O}_{1/2}, c^{2+\sigma}(S^1)), \tag{4.4}
$$

then the mapping f is a smooth mapping from $\mathcal{O}_1 \subset E_1$ into E_0.

If the Fréchet derivative $df(\bar\Phi) \in \mathcal{L}(E_1, E_0)$ belongs to the maximal regularity class $\mathcal{M}_1(E_0, E_1)$ for any $\bar\Phi \in \mathcal{O}_1$, then by [8, Theorem 2.7], (4.1) has a unique solution $\Phi \in Y^T = C([0,T], E_1) \cap C^1([0,T], E_0)$ on some small enough interval $[0,T]$. Recall that the class $\mathcal{M}_1(E_0, E_1) \subset \mathcal{L}(E_1, E_0)$ consists of those generators of analytic semigroups $A : D(A) = E_1 \subset E_0 \to E_0$ for which the linear equation $\partial_t \Phi = A\Phi + h(t)$, $0 < t \le 1$, $\Phi(0) = \Phi^0$, has a unique solution $\Phi \in Y^1$ for any $h \in C([0,1], E_0)$ and $\Phi^0 \in E_1$. In other words, (E_0, E_1) is a maximal parabolic regularity pair.

THEOREM 4.1. *Assume that* $(k^0, \nu^0, g^0)^T \in \mathcal{O}_1 \subset E_1$, *where* k^0 *is the curvature,* ν^0 *is the tangential vector, and* $g^0 = |\partial_u x^0| > 0$ *is the local length element of the initial regular curve* $\Gamma^0 = \mathrm{Image}(x^0)$. *If* $\beta = \beta(k, \nu)$ *is a* C^3 *smooth function satisfying* (4.2) *and* α *obeys* (4.4), *then there exists a unique classical solution* $\Phi = (k, \nu, g)^T \in C([0,T], E_1) \cap C^1([0,T], E_0)$ *of the governing system of equations* (3.13) *defined on some small time interval* $[0,T]$. *Moreover, if* Φ *is a maximal solution defined on* $[0, T_{\max})$ *and* $T_{\max} < \infty$, *then* $\max |k(.,t)| \to \infty$ *as* $t \to T_{\max}$.

Proof. Let $\Phi^0 \in \mathcal{O}_1$ where $\mathcal{O}_1 \subset E_1$ is an open and bounded subset of E_1, $g > 0$, for any $(k, \nu, g)^T \in \mathcal{O}_1$. The linearization of f at $\bar\Phi = (\bar k, \bar\nu, \bar g)^T \in \mathcal{O}_1$ has the form $df(\bar\Phi) = d_\Phi F(\bar\Phi, \bar\alpha) + d_\alpha F(\bar\Phi, \bar\alpha)\, d_\Phi \alpha(\bar\Phi)$, where $\bar\alpha = \alpha(\bar\Phi)$ and

$$
\begin{aligned}
d_\Phi F(\bar\Phi, \bar\alpha)\Phi &= \partial_u(\bar D \partial_u \Phi) + \bar B \partial_u \Phi + \bar C \Phi,\\
d_\alpha F(\bar\Phi, \bar\alpha)\alpha &= \left(\alpha \bar g^{-1} \partial_u \bar k,\ \alpha \bar k,\ \partial_u \alpha\right)^T;
\end{aligned}
$$

$\bar D = \mathrm{diag}(\bar D_{11}, \bar D_{22}, 0)$, $\bar D_{11} = \bar D_{22} = \beta'_k(\bar k, \bar\nu)\bar g^{-2} \in C^{1+\sigma}(S^1)$, and $\bar B, \bar C$ are 3×3 matrices with $C^\sigma(S^1)$ smooth coefficients, $\bar B_{3j} = 0, \bar C_{3j} \in C^{1+\sigma}$. By (3.9) we have $g^{-1}\partial_u \beta(k, \nu) = g^{-1}\beta'_k \partial_u k + \beta'_\nu k$, and so the principal part is indeed a diagonal one. The linear operator $A_1 = \partial_u(\bar D \partial_u \Phi)$, $D(A_1) = E_1$, is a generator of an analytic semigroup on E_0, and moreover $A_1 \in \mathcal{M}_1(E_0, E_1)$ (cf. [8]). Notice that $d_\alpha F(\bar\Phi, \bar\alpha)$ belongs to $\mathcal{L}(C^{2+\sigma}(S^1), E_{1/2})$ and this is why we can write $d_\Phi f(\bar\Phi)$ as the sum $A_1 + A_2$ where $A_2 \in L(E_{1/2}, E_0)$, $\|A_2\Phi\|_{E_0} \le C\|\Phi\|_{E_{1/2}} \le C\|\Phi\|_{E_0}^{1/2}\|\Phi\|_{E_1}^{1/2}$ is a relatively bounded linear perturbation of A_1 with zero relative bound (cf. [8]). Since the class $\mathcal{M}_1$ is closed with respect to such perturbations (see [8, Lemma 2.5]), we have $d_\Phi f(\bar\Phi) \in \mathcal{M}_1(E_0, E_1)$. The proof of the short time existence of a solution Φ now follows from [8, Theorem 2.7].

Finally, we will show that the maximal curvature becomes unbounded as $t \to T_{\max} < \infty$. Suppose to the contrary that $\max_{\Gamma^t} |k(.,t)| \le M < \infty$ for any $t \in [0, T_{\max})$. According to [6, Theorem 3.1], there exists a unique maximal solution $\Gamma : [0, T'_{\max}) \to \Omega(\mathbb{R}^2)$ satisfying $\Gamma(0) = \Gamma^0$ and the geometric equation (1.1). Recall that $\Omega(\mathbb{R}^2)$ is the space of C^1 regular curves in the plane (cf. [6]). Moreover, $\Gamma(t)$ is a C^∞ smooth curve for any $t \in (0, T'_{\max})$ and the maximum of the absolute value of the curvature tends to infinity as $t \to T'_{\max}$. Thus $T_{\max} < T'_{\max}$ and therefore the curvature and, subsequently, ν remain bounded in the $C^{2+\sigma'}$ norm on the interval $[0, T_{\max}]$ for any $\sigma' \in (\sigma, 1)$. Applying compactness arguments one sees that the limit $\lim_{t\to T_{\max}} \Phi(., t)$ exists and remains bounded in the space E_1 and one can continue the solution Φ beyond $T_{\max}$, which is a contradiction. □

Next we will show how to construct a classical solution $x = x(u, t)$ of the intrinsic heat equation (2.2). Suppose that $\tilde{\Phi} = (\tilde{k}, \tilde{\nu}, \tilde{g})^T$ is a classical solution of the system (3.13) existing on the time interval $[0, T]$. Let us construct a flow of plane curves $\Gamma^t = \mathrm{Image}(x(., t)), t \in [0, T]$, as follows:

$$x(u,t) = x^0(u) + \int_0^t (\tilde{\beta}\tilde{\vec{N}} + \tilde{\alpha}\tilde{\vec{T}})\, d\tau, \tag{4.5}$$

where $\tilde{\vec{N}} = (-\sin\tilde{\nu}, \cos\tilde{\nu})^T$, $\tilde{\vec{T}} = (\cos\tilde{\nu}, \sin\tilde{\nu})^T$, $\tilde{\beta} = \beta(\tilde{k}, \tilde{\nu})$, and $\tilde{\alpha} = \alpha(\tilde{k}, \tilde{\nu}, \tilde{g})$. We claim that $x(u, t)$ is a classical solution of (2.2).

THEOREM 4.2. *Assume β and α satisfy assumptions of Theorem* 4.1. *Let $\tilde{\Phi} = (\tilde{k}, \tilde{\nu}, \tilde{g})^T$ be a classical solution of* (3.13) *such that the quantities $\tilde{k}, \tilde{\beta}$, and $\tilde{g}^{-1}\partial_u\tilde{\alpha}$ are bounded. Then $x = x(u, t)$ given by* (4.5) *satisfies $|\partial_u x| = \tilde{g}$, $k = \tilde{k}$, $\nu = \tilde{\nu}$, $\vec{N} = \tilde{\vec{N}}$, $\vec{T} = \tilde{\vec{T}}$, where $k, \nu, \vec{N}, \vec{T}$ represent the curvature, the tangent angle, and the unit normal and tangent vectors of the curve $\Gamma^t = \mathrm{Image}(x(., t))$. Moreover, $x \in C([0, T]; (C^{2+\sigma}(S^1))^2) \cap C^1([0, T]; (C^\sigma(S^1))^2)$ is a classical solution of the intrinsic heat equation* (2.2).

Proof. First we prove that $\partial_u\tilde{\nu} = \tilde{g}\tilde{k}$ for any classical solution of (3.13). Indeed, if we denote $K = \tilde{k} - \tilde{g}^{-1}\partial_u\tilde{\nu}$, then it is easy calculus to verify that K satisfies the linear parabolic equation

$$\frac{\partial K}{\partial t} = \frac{1}{\tilde{g}}\frac{\partial}{\partial u}\left(\frac{\beta'_k(\tilde{k}, \tilde{\nu})}{\tilde{g}}\frac{\partial K}{\partial u}\right) - \frac{1}{\tilde{g}}\frac{\partial}{\partial u}\left(\beta'_\nu(\tilde{k}, \tilde{\nu})K\right) + \left(\tilde{k}\tilde{\beta} - \frac{1}{\tilde{g}}\frac{\partial\tilde{\alpha}}{\partial u}\right)K.$$

Moreover, $K(u, 0) = 0$ because $\partial_u\tilde{\nu}^0 = \tilde{g}^0\tilde{k}^0$ (see (3.15)). The term $\tilde{k}\tilde{\beta} - \tilde{g}^{-1}\partial_u\tilde{\alpha}$ is assumed to be bounded and therefore we may conclude that $K(u, t) = 0$ for any $u \in [0, 1], t \in [0, T]$. As $\tilde{g}\tilde{k} = \partial_u\tilde{\nu}$ we end up with Frenet's formulae $\partial_u\tilde{\vec{T}} = \tilde{g}\tilde{k}\tilde{\vec{N}}$ and $\partial_u\tilde{\vec{N}} = -\tilde{g}\tilde{k}\tilde{\vec{T}}$. Similarly as in the proof of the identities (3.1), the equation $\partial_t x = \tilde{\beta}\tilde{\vec{N}} + \tilde{\alpha}\tilde{\vec{T}}$ yields $p.\partial_t p = \tilde{g}^2(-k\tilde{\beta} - \tilde{g}^{-1}\partial_u\tilde{\alpha})$, where $p = \partial_u x$. Thus $\partial_t(|p|^2) = 2p.\partial_t p = 2\partial_t(\tilde{g}^2)$ and therefore $|\partial_u x| = |p| = \tilde{g}$ because $|\partial_u x^0| = g^0$. Again, using the last two equations in (3.1) we obtain $k = (\partial_u x \wedge \partial_u^2 x)/|\partial_u x|^3 = \tilde{k}$ and subsequently $\nu = \tilde{\nu}$, which gives us $\vec{N} = \tilde{\vec{N}}, \vec{T} = \tilde{\vec{T}}$. Hence $x = x(u, t)$ obeys (2.4), i.e., $\partial_t x = \beta(k, \nu)\vec{N} + \alpha(k, \nu, g)\vec{T}$. Therefore x is a solution of the intrinsic heat equation (2.2). The regularity properties of x follow directly from the regularity of the solution $\tilde{\Phi}$ and (4.5) (see Theorem 4.1). □

5. Analysis of the equations for geometric quantities and short time existence of solutions in the degenerate case. The aim of this section is to prove the short time existence of smooth solutions of the curve shortening flow governed by the intrinsic heat equation (2.2). Throughout the rest of the paper we will assume that the normal velocity function $v = \beta(k, \nu)$ has the form

$$\beta(k, \nu) = \gamma(\nu)|k|^{m-1}k,$$

where $m > 0$ and $\gamma : \mathbb{R} \to \mathbb{R}^+$ is a given C^∞ smooth anisotropy function satisfying

$$0 < C_1^{-1} \le \gamma(\nu) \le C_1, \quad |\gamma'_\nu(\nu)| \le C_1 \qquad \text{for any } \nu \in \mathbb{R}, \tag{5.1}$$

where $C_1 > 0$ is a constant.

The assumptions guaranteeing the local existence of classical solutions of a nonlinear curve shortening flow developed by Angenent in [6, 7] as well as those of Theorem 4.1 do not directly apply to the case $\beta(k,\nu) = \gamma(\nu)|k|^{m-1}k$ for $m \neq 1$. Recall that these theories require β to satisfy (4.2). To use the result established in Theorem 4.1 we must go through a regularization argument. A similar technique was applied in the paper by Angenent, Sapiro, and Tannenbaum [12] for the case of an isotropic function $\beta(k) = k^{1/3}$. In what follows, we will slightly modify their approach for the more general anisotropic power-like function $\beta(k,\nu)$ and for the case when the curvature equation involves a nontrivial tangential velocity term α.

Henceforth, we denote by C_i, M_i any generic positive constant independent on the regularization parameter $0 < \varepsilon \le 1$. Constants M_i may also depend on the initial curve Γ^0. We make the following *regularization assumption on the function* β:

There is a family of nondecreasing C^∞ functions $\beta^\varepsilon, 0 < \varepsilon \le 1$, such that

(i) $\beta^\varepsilon(k,\nu) \to \beta^0(k,\nu) = \beta(k,\nu)$ as $\varepsilon \to 0^+$ locally uniformly with respect to $(k,\nu) \in \mathbb{R}^2$;

(ii) $|\beta^\varepsilon(k,\nu)| \le C_2(1 + |k|^m)$ for any $k,\nu \in \mathbb{R}$;

(iii) there exist constants $\lambda^\varepsilon_\pm = \lambda^\varepsilon_\pm(M_1) > 0$ such that $\lambda^\varepsilon_- \le \beta^{\varepsilon}_k{}'(k,\nu) \le \lambda^\varepsilon_+$;

(iv) $|\beta^{\varepsilon}_k{}'(k,\nu)k^4(\beta^\varepsilon(k,\nu))^2| + \frac{|\beta^{\varepsilon}_\nu{}'(k,\nu)|^2}{\beta^{\varepsilon}_k{}'(k,\nu)} \le C_3(M_1)$ for any $|k| \le M_1$ and $\nu \in \mathbb{R}$.

It is easy to verify that the regularization family (β^ε) defined as

$$\beta^\varepsilon(k,\nu) = m\gamma(\nu)\int_0^k (\varepsilon^2 + \xi^2)^{\frac{m-1}{2}}\, d\xi \qquad \text{if } \ 0 < m \le 1,$$
$$\beta^\varepsilon(k,\nu) = \beta(k,\nu) + \varepsilon k \qquad \text{if } \ m > 1$$

satisfies the above assumptions (i)–(iv) with constants $\lambda^\varepsilon_\pm > 0$ given by

$$\begin{aligned} &\lambda^\varepsilon_- = \delta, \quad \lambda^\varepsilon_+ = C_1 m\varepsilon^{m-1} \qquad \text{if } \ 0 < m \le 1,\\ &\lambda^\varepsilon_- = \varepsilon, \quad \lambda^\varepsilon_+ = 1 + \max_{|k|\le M_1} \beta'_k(k,\nu) \qquad \text{if } \ m > 1, \end{aligned} \tag{5.2}$$

where $\delta > 0$ is a constant independent of $0 < \varepsilon \le 1$. Furthermore,

$$0 \le \frac{\beta^\varepsilon(k,\nu)}{k} \le \max(1, m^{-1})\, \beta^{\varepsilon}_k{}' \quad \text{for any } k,\nu \in \mathbb{R} \ \text{ and } 0 < \varepsilon \le 1. \tag{5.3}$$

Let us emphasize the fact that the tangential velocity α may also depend on the regularization parameter ε, i.e., $\alpha = \alpha^\varepsilon$. For instance, α^ε may depend on k and $\beta^\varepsilon = \beta^\varepsilon(k,\nu)$. Concerning the structural properties of α^ε we make the following hypotheses:

$$\sup_{\Phi \in B_{1/2}} \{|\alpha^\varepsilon| + |\partial_s \alpha^\varepsilon|; \quad \alpha^\varepsilon = \alpha^\varepsilon(\Phi),\ 0 \le \varepsilon \le 1\} < \infty \tag{5.4}$$

for any set $B_{1/2} = \{(k,\nu,g)^T \in \mathcal{O}_{1/2},\ |k| \le M_1\}$ and

$$\|\alpha^\varepsilon(k,\nu,g)\|_{C^2(S^1)} \le C\left(1 + \|g\|_{C^1(S^1)} + \|\beta^\varepsilon(k,\nu)\|^q_{C^1(S^1)}\right) \tag{5.5}$$

for any $\Phi = (k, \nu, g)^T \in B_{1/2}$, where $C = C(M_1) > 0$ is a constant and $1 \le q < \frac{4}{3}$. In section 6 we will show how to construct a so-called tangential velocity preserving the relative local length satisfying the above hypotheses.

Let Γ^0 be a smooth initial curve such that $\Phi^0 = (k^0, \nu^0, g^0)^T \in \mathcal{O}_1 \subset E_1$. By $\Phi_\varepsilon = (k_\varepsilon, \nu_\varepsilon, g_\varepsilon)^T$ we denote the classical solution of the governing system of equations (3.13) with $\beta = \beta^\varepsilon$ and $\alpha = \alpha^\varepsilon$. The short time existence of Φ_ε has been justified by Theorem 4.1 for any $0 < \varepsilon \le 1$. From (4.5) and Theorem 4.2 we furthermore know that the function

$$x_\varepsilon(u,t) = x^0(u) + \int_0^t \left(\beta^\varepsilon \vec{N}_\varepsilon + \alpha^\varepsilon \vec{T}_\varepsilon\right) d\tau$$

is a classical solution of the intrinsic heat equation (2.2) for any $0 < \varepsilon \le 1$.

First we will show that the maximum of $|k|$ remains bounded in a short time interval $[0,T]$ and the parameterization of the curve Γ^t is regular.

LEMMA 5.1. *Suppose that the regularization assumptions* (i), (ii) *are satisfied. Then there exist constants $T > 0$ and $M_1 > 0$ such that*

$$\max_{\Gamma^t} |k_\varepsilon(.,t)| \le M_1 \quad \textit{for any } t \in [0,T] \textit{ and } \varepsilon \in (0,1].$$

If α^ε satisfies (5.4), *then there are constants $g_\pm > 0$ such that*

$$0 < g_- < g_\varepsilon(u,t) < g_+ < \infty \quad \textit{for any } (u,t) \in Q_T \textit{ and } \varepsilon \in (0,1].$$

Proof. The proof of the first part is essentially the same as that of [12, Theorem 6.2]. Indeed, as $\partial_t k_\varepsilon = \partial_s^2 \beta^\varepsilon + \alpha^\varepsilon \partial_s k_\varepsilon + k_\varepsilon^2 \beta^\varepsilon$, then by applying a maximum principle argument we get $\partial_t(\max_{\Gamma^t_\varepsilon} |k_\varepsilon(.,t)|) \le F^\varepsilon(\max_{\Gamma^t_\varepsilon} |k_\varepsilon(.,t)|)$, where $F^\varepsilon(k) = \max_\nu k^2 |\beta^\varepsilon(k,\nu)| \le C_2 k^2 (1+|k|^m)$ for any $0 < \varepsilon \le 1$. Solving this differential inequality we conclude the proof of the bound for the total variation of the curvature. To prove estimates on g we integrate the third equation in (3.14) with respect to time. We obtain $g_\varepsilon(u,t) = g^0(u)\exp(\int_0^t (-k_\varepsilon \beta^\varepsilon + g_\varepsilon^{-1} \partial_u \alpha^\varepsilon)\, d\tau)$, where $\beta^\varepsilon = \beta^\varepsilon(k_\varepsilon, \nu_\varepsilon)$. The proof now follows from the fact that both $k_\varepsilon \beta^\varepsilon$ and $g_\varepsilon^{-1} \partial_u \alpha^\varepsilon = \partial_s \alpha^\varepsilon$ are bounded for $|k| \le M_1$ and $0 < g^0 < \infty$ uniformly with respect to $\varepsilon \in (0,1]$. ◻

In the next lemma we analyze the degenerate case when $1 < m \le 2$. It is a key technical tool in order to establish some a priori estimates needed in the proof of short time existence of a solution in this degenerate case. Interestingly enough, a new geometric assumption on the initial curve is needed.

LEMMA 5.2. *Assume $1 < m \le 2$. Suppose that the initial curve Γ^0 satisfies*

$$\int_{\Gamma^0} \frac{k^0}{\beta(k^0, \nu^0)}\, ds < \infty. \tag{5.6}$$

Then there exists a constant $M_2 > 0$ such that

$$\max_{t \in [0,T]} \int_{\Gamma^t} \frac{k_\varepsilon}{\beta^\varepsilon(k_\varepsilon, \nu_\varepsilon)}\, ds + \int_0^T \int_{\Gamma^t} |\partial_s k_\varepsilon|^2\, ds \le M_2 \quad \textit{for any} \;\; 0 < \varepsilon \le 1. \tag{5.7}$$

Proof. Denote $v = \beta^\varepsilon(k_\varepsilon, \nu_\varepsilon)$, $k = k_\varepsilon$, and $\nu = \nu_\varepsilon$. By using (3.4), the curvature equation (3.2) and the velocity equation (3.10), we obtain

$$\frac{d}{dt} \int_{\Gamma^t} \frac{k}{v}\, ds = \int_{\Gamma^t} \frac{\partial}{\partial t}\left(\frac{k}{v}\right) + \frac{k}{v}(-kv + \partial_s \alpha)\, ds$$

$$
\begin{aligned}
(5.8) \qquad &= \int_{\Gamma^t} \left(\frac{1}{v} - \frac{k\beta_k^{\varepsilon}{}'}{v^2}\right)\left(\partial_s^2 v + \alpha\partial_s k + k^2 v\right) \\
&\qquad - \frac{k\beta_\nu^{\varepsilon}{}'}{v^2}(\partial_s v + \alpha k) - k^2 + \frac{k}{v}\partial_s\alpha\, ds \\
&= \int_{\Gamma^t} \left(\frac{1}{v} - \frac{k\beta_k^{\varepsilon}{}'}{v^2}\right)\partial_s^2 v - \frac{k^3\beta_k^{\varepsilon}{}'}{v} - \frac{k\beta_\nu^{\varepsilon}{}'}{v^2}\partial_s v\, ds
\end{aligned}
$$

because of the identity $0 = \int_{\Gamma^t}\partial_s\left(\frac{\alpha k}{v}\right)ds = \int_{\Gamma^t}\frac{k}{v}\partial_s\alpha + \frac{\alpha}{v}\partial_s k - \frac{\alpha k}{v^2}\partial_s v\, ds$, (3.3), and (3.11). Recall that $\beta_k^{\varepsilon}{}' = \beta_k' + \varepsilon = \frac{m\beta}{k} + \varepsilon$ and therefore

$$
\frac{k\beta_k^{\varepsilon}{}'}{v} = m + \varepsilon(1-m)\frac{k}{v}.
$$

Plugging the above expression into (5.8) and integration by parts yield the identity

$$
\begin{aligned}
&\frac{d}{dt}\int_{\Gamma^t}\frac{k}{v}\,ds + (m-1)\int_{\Gamma^t}\frac{1}{v^2}\left(|\partial_s v|^2 + \varepsilon\partial_s v\partial_s k - 2\varepsilon\frac{k}{v}|\partial_s v|^2\right)ds \\
&\qquad = -m\int_{\Gamma^t}k^2\,ds + \varepsilon(m-1)\int_{\Gamma^t}\frac{k^3}{v}\,ds - \int_{\Gamma^t}\frac{k\beta_\nu^{\varepsilon}{}'}{v^2}\partial_s v\,ds.
\end{aligned}
$$

It follows from (3.11) that $\partial_s k = (\partial_s v - \beta_\nu^{\varepsilon}{}' k)/\beta_k^{\varepsilon}{}'$. Thus

$$
\begin{aligned}
&\frac{d}{dt}\int_{\Gamma^t}\frac{k}{v}\,ds + (m-1)\int_{\Gamma^t}\left|\frac{1}{v}\partial_s v\right|^2\left(1 + \frac{\varepsilon}{\beta_k^{\varepsilon}{}'} - 2\varepsilon\frac{k}{v}\right)ds \\
&\quad = -m\int_{\Gamma^t}k^2\left(1 - \frac{\varepsilon k}{v}\right)ds - \varepsilon\int_{\Gamma^t}\frac{k^3}{v}\,ds + \int_{\Gamma^t}\frac{k\beta_\nu^{\varepsilon}{}'}{v^2}\partial_s v\left(\frac{\varepsilon(m-1)}{\beta_k^{\varepsilon}{}'} - 1\right)ds \\
&\quad \le \int_{\Gamma^t}\frac{\gamma'(\nu)}{\gamma(\nu)}\frac{k}{v}\partial_s v\left(\frac{\varepsilon(m-1)}{\beta_k^{\varepsilon}{}'} - 1\right)ds \le mC_1^2\int_{\Gamma^t}\left|\frac{k}{v}\partial_s v\right|ds
\end{aligned}
$$

because of the inequalities $|\frac{\beta_\nu^{\varepsilon}{}'}{v}| = |\frac{\gamma'(\nu)}{\gamma(\nu)}| \le C_1^2$, $0 \le \frac{\varepsilon k}{v} \le 1$, and $0 < \frac{\varepsilon}{\beta_k^{\varepsilon}{}'} \le 1$. Let us consider the auxiliary function ϕ defined as follows:

$$
\phi(k) = \frac{1}{k^2}\left(1 + \frac{\varepsilon}{\beta_k^{\varepsilon}{}'} - \frac{2\varepsilon k}{v}\right) = \frac{1}{k^2}\left(1 + \frac{\varepsilon}{m\gamma(\nu)|k|^{m-1} + \varepsilon} - \frac{2\varepsilon}{\gamma(\nu)|k|^{m-1} + \varepsilon}\right).
$$

It is easy calculus to verify that if $1 < m \le 2$ then there exists a constant $M_3 > 0$ independent of $0 < \varepsilon \le 1$ and such that $\inf_{|k|\le M_1}\phi(k) \ge M_3$. Using the Cauchy–Schwarz inequality we get

$$
\begin{aligned}
&\frac{d}{dt}\int_{\Gamma^t}\frac{k}{v}\,ds + M_3(m-1)\int_{\Gamma^t}\left|\frac{k}{v}\partial_s v\right|^2 ds \le mC_1^2\int_{\Gamma^t}\left|\frac{k}{v}\partial_s v\right|ds \\
&\qquad \le \frac{m^2C_1^4}{2(m-1)M_3}L^t + \frac{M_3(m-1)}{2}\int_{\Gamma^t}\left|\frac{k}{v}\partial_s v\right|^2 ds
\end{aligned}
$$

and so

$$
\frac{d}{dt}\int_{\Gamma^t}\frac{k}{v}\,ds + \frac{M_3(m-1)}{2}\int_{\Gamma^t}\left|\frac{k}{v}\partial_s v\right|^2 ds \le \frac{m^2C_1^4}{2(m-1)M_3}L^0.
$$

Integrating the above inequality over the interval $[0,T]$, taking into account the inequality

$$\left|\frac{k}{v}\partial_s v\right| = \left|\left(m+(1-m)\frac{\varepsilon k}{v}\right)\partial_s k + \frac{\gamma'(\nu)}{\gamma(\nu)}k^2\right| \geq |\partial_s k| - C_1^2 M_1^2$$

and the initial time inequality $\int_{\Gamma^0}\frac{k}{v}\,ds = \int_{\Gamma^0}\frac{k^0}{\beta^\varepsilon(k^0,\nu^0)}\,ds \leq \int_{\Gamma^0}\frac{k^0}{\beta(k^0,\nu^0)}\,ds < \infty$ we finally obtain the estimate (5.7). □

Remark 5.1. The assumption (5.6) seems to be quite restrictive. Note that it is fulfilled in the case when the initial curve Γ^0 is strictly convex or in the case of a nonconvex smooth curve whose inflection points have at most $(2+\frac{1}{m-1})$-order contact with their tangents. As an example one can consider the Bernoulli lemniscate $(x^2+y^2)^2 = 4xy$ having the third-order contact with its tangents at the origin. In this example the assumption (5.6) is satisfied iff $1 < m < 2$.

Remark 5.2. It would be of interest to know whether the power $m = 2$ is an optimal value. It follows from recent results due to Andrews [4] that for higher powers of m the curve Γ^t need not be sufficiently smooth in the vicinity of a point where the curvature vanishes.

LEMMA 5.3. *For any $t \in (0,T)$ we have*

$$\frac{d}{dt}X_p(t) \leq -\int_{\Gamma^t}\beta_k^{\varepsilon}{}'|\partial_s(w^{p/2})|^2\,ds + M_4 p^2(1+X_p(t)), \tag{5.9}$$

where $X_p(t) = \int_{\Gamma^t}|w|^p ds = \int_0^1 |w|^p|\partial_u x_\varepsilon|\,du$ and $w = \partial_s\beta^\varepsilon(k_\varepsilon,\nu_\varepsilon)$, $p \geq 1$.

Proof. Denote $k = k_\varepsilon, \nu = \nu^\varepsilon$. Applying the local length equation (3.4) and the equation for the gradient of velocity (3.12) we obtain

$$\begin{aligned}
\frac{d}{dt}X_p(t) &= \int_0^1 \left(\partial_t(|w|^p)|\partial_u x| + |w|^p(-|\partial_u x|kv + \partial_u\alpha^\varepsilon)\right)du \\
&= \int_{\Gamma^t}\left(p|w|^{p-2}w\partial_t w + |w|^p(-kv+\partial_s\alpha^\varepsilon)\right)ds \\
&= \int_{\Gamma^t}\Bigg(p|w|^{p-2}w\left[\partial_s\left(\beta_k^{\varepsilon}{}'\partial_s w + \beta_k^{\varepsilon}{}'k^2 v + \beta_\nu^{\varepsilon}{}'w\right) + \alpha^\varepsilon\partial_s w + kvw\right] \\
&\qquad\qquad + |w|^p(-kv+\partial_s\alpha^\varepsilon)\Bigg)ds \\
&= -p(p-1)\int_{\Gamma^t}|w|^{p-2}\left[\beta_k^{\varepsilon}{}'|\partial_s w|^2 + \beta_k^{\varepsilon}{}'k^2 v\partial_s w + \beta_\nu^{\varepsilon}{}'w\partial_s w\right]ds \\
&\qquad + (p-1)\int_{\Gamma^t}|w|^p kv\,ds
\end{aligned}$$

because $0 = \int_{\Gamma^t}\partial_s\left(|w|^p\alpha^\varepsilon\right) = \int_{\Gamma^t}p|w|^{p-2}w\partial_s w\alpha^\varepsilon + |w|^p\partial_s\alpha^\varepsilon\,ds$. Notice that the tangential velocity term α^ε is involved neither in the expression for X_p nor in $\frac{d}{dt}X_p$.

Applying the Cauchy–Schwarz inequality we get

$$|\beta_k^{\varepsilon}{}'k^2 v\partial_s w + \beta_\nu^{\varepsilon}{}'w\partial_s w| \leq \beta_k^{\varepsilon}{}'k^4v^2 + \beta_k^{\varepsilon}{}'\frac{|\partial_s w|^2}{4} + \frac{|\beta_\nu^{\varepsilon}{}'|^2}{\beta_k^{\varepsilon}{}'}|w|^2 + \beta_k^{\varepsilon}{}'\frac{|\partial_s w|^2}{4}$$

and therefore

$$\frac{d}{dt}X_p(t) = -\frac{p(p-1)}{2}\int_{\Gamma^t}\beta_k^{\varepsilon}{}'|w|^{p-2}|\partial_s w|^2\,ds$$

$$
\begin{aligned}
&+ p(p-1)\int_{\Gamma^t} |w|^{p-2}\beta_k^{\varepsilon\prime} k^4 v^2 + |w|^p \frac{|\beta_\nu^{\varepsilon\prime}|^2}{\beta_k^{\varepsilon\prime}}\, ds + (p-1)\int_{\Gamma^t} |w|^p k v\, ds\\
\le& -\int_{\Gamma^t} \beta_k^{\varepsilon\prime} |\partial_s (w^{p/2})|^2 ds\\
&+ C_3 p(p-1)(X_{p-2}(t) + X_p(t)) + (p-1)M_1^2 X_p(t)
\end{aligned}
$$

because $|w|^{p-2}|\partial_s w|^2 = \frac{4}{p^2}|\partial_s(w^{p/2})|^2$ and $2p(p-1)/p^2 \ge 1$ for any $p \ge 2$. Since $X_{p-2}(t) = \int_{\Gamma^t} |w|^{p-2} \le \int_{\Gamma^t}(1+|w|^p) \le L^t + X_p(t)$ for any $p \ge 2$ and $L^t \le L^0$ (see (3.5)) we finally obtain the inequality (5.9) with a constant $M_4 > 0$ independent of $0 < \varepsilon \le 1$ and $p \ge 2$. □

Lemma 5.4. *Suppose that* $0 < m \le 2$. *If* $1 < m \le 2$ *we additionally suppose that the initial curve* Γ^0 *satisfies the condition* (5.6). *Then there is a constant* $M_7 > 0$ *such that*

if $0 < m \le 1$ *then* $\max_{\Gamma^t} |\partial_s \beta^\varepsilon(k_\varepsilon, \nu_\varepsilon)| \le M_7 t^{-\frac{3}{4}}$;

if $1 < m \le 2$ *then* $\max_{\Gamma^t} |\partial_s \beta^\varepsilon(k_\varepsilon, \nu_\varepsilon)| \le M_7 t^{-\frac{1}{2}}$

for any $0 < \varepsilon \le 1$ *and* $0 < t \le T$.

Proof. The key idea behind the proof of this estimate is a modification of the well-known Nash–Moser iterative technique adopted to the flow of plane curves. It is similar, in spirit and technique, to that of [12, Chapter 6], which has been applied in the case of the affine scaling parameterization, i.e., $\beta(k) = k^{1/3}$. By using the differential inequality (5.9) we will show that $\|w\|_p = X_p^{1/p}(t)$ is bounded uniformly with respect to $p \ge 2$ and $0 < \varepsilon \le 1$, yielding the desired L^∞ estimate on $\partial_s \beta^\varepsilon(k_\varepsilon, \nu_\varepsilon)$.

Let us consider the case $0 < m \le 1$. First, we will prove an estimate for $X_2(t)$. By (5.2) we have $\beta_k^{\varepsilon\prime} \ge \delta > 0$ and $\beta^\varepsilon \le M_5$. Then

$$
\begin{aligned}
X_2 &= \int_{\Gamma^t} |w|^2 = \int_{\Gamma^t} (\partial_s v)^2 = -\int_{\Gamma^t} v \partial_s^2 v = -\int_{\Gamma^t} \beta^\varepsilon \partial_s w\\
&\le \frac{M_5}{\delta}\int_{\Gamma^t} \sqrt{\beta_k^{\varepsilon\prime}} |\partial_s w| \le \frac{M_5 (L^t)^{\frac{1}{2}}}{\delta}\left(\int_{\Gamma^t} \beta_k^{\varepsilon\prime} |\partial_s w|^2\right)^{\frac{1}{2}}.
\end{aligned}
$$

According to (5.9), X_2 is a solution of the differential inequality

$$
\frac{dX_2}{dt} \le -\frac{1}{M^2} X_2^2 + 4M_4(1 + X_2),
$$

where $M = M_5 (L^0)^{\frac{1}{2}}/\delta$. By solving the above differential inequality we obtain $X_2(t) \le A_2^2 t^{-1}$, where A_2 is a constant depending only on m, Γ^0, and T.

Let $p \ge 2$. As $w = \partial_s v$ there must be a point at Γ^t where $w^{p/2}$ vanishes. From the interpolation inequality [12, Proposition 6.1, (25)] we infer

$$
X_p = \int_{\Gamma^t} |w|^p \le \left(\int_{\Gamma^t} |w|^{\frac{p}{2}}\right)^{\frac{4}{3}} \left(\int_{\Gamma^t} |\partial_s (w^{\frac{p}{2}})|^2\right)^{\frac{1}{3}} \le X_{\frac{p}{2}}^{\frac{4}{3}} \left(\frac{1}{\delta}\int_{\Gamma^t} \beta_k^{\varepsilon\prime} |\partial_s (w^{\frac{p}{2}})|^2\right)^{\frac{1}{3}}.
$$

Let us consider the case $1 < m \le 2$. Again, we begin with an estimate for $X_2(t)$. For $1 < m$ the derivative $\beta_k^{\varepsilon\prime}$ is bounded uniformly with respect to $0 < \varepsilon \le 1$. Since $|k| \le M_1$ we have $X_2(0) = \int_{\Gamma^0} |\partial_s \beta^\varepsilon|^2 \le M \int_{\Gamma^0}(1 + |\partial_s k^0|^2) < \infty$. Integrating the differential inequality (5.9) we obtain $X_2(t) \le 1 + X_2(t) \le (1 + X_2(0))\exp(4M_4 t) \le A_2^2$ for any $0 \le t \le T$, where $A_2 > 0$ is a constant. By using the Cauchy–Schwarz inequality we obtain

$$
\sup_{\Gamma^t} |w|^{\frac{p}{2}} \le \int_{\Gamma^t} |\partial_s (w^{\frac{p}{2}})| \le \left(\int_{\Gamma^t} \frac{1}{\beta_k^{\varepsilon\prime}}\right)^{\frac{1}{2}} \left(\int_{\Gamma^t} \beta_k^{\varepsilon\prime} |\partial_s (w^{\frac{p}{2}})|^2\right)^{\frac{1}{2}}.
$$

According to (5.3) and Lemma 5.2 we have $(\int_{\Gamma^t} \frac{1}{\beta_k^{\varepsilon\prime}})^{\frac{1}{2}} \le M_6$ and hence

$$X_p = \int_{\Gamma^t} |w|^p \le \sup_{\Gamma^t} |w|^{\frac{p}{2}} X_{\frac{p}{2}} \le M_6 X_{\frac{p}{2}} \left(\int_{\Gamma^t} \beta_k^{\varepsilon\prime} |\partial_s(w^{\frac{p}{2}})|^2 \right)^{\frac{1}{2}}.$$

In both cases, taking into account (5.9) we end up with a differential inequality

$$\frac{d}{dt} X_p(t) \le -\hat{\delta} \frac{X_p^L(t)}{X_{p/2}^K(t)} + Mp^2(1 + X_p(t)) \tag{5.10}$$

for any $t \in (0, T]$, $p \ge 2$, and $0 < \varepsilon \le 1$, where $(L, K) = (3, 4)$ if $0 < m \le 1$, $(L, K) = (2, 2)$ if $1 < m \le 2$, and $\hat{\delta}, M > 0$ are constants independent of $0 < \varepsilon \le 1$ and $p \ge 2$. In the case $(L, K) = (3, 4)$ this is exactly the same differential inequality as that of [12, eq. (2.4)]. Following the iterative method of supersolutions to the differential inequality (5.10) presented in [12, Chapter 6], given a couple (L, K) such that $K = 2(L - 1)$, $L > 1$, one can prove the existence of a bounded sequence (A_k), $0 < A_k \le M_7$, such that

$$X_{p_k}(t) \le A_k^{p_k} t^{-\alpha_k p_k}$$

for any $p_k = 2^{k+1}, k \ge 0$, where $\alpha_0 = \frac{1}{2}$ for $0 < m \le 1$, $\alpha_0 = 0$ for $1 < m \le 2$ and

$$\alpha_{k+1} = \alpha_k + \frac{1}{(L-1)} 2^{-k-2} = \alpha_0 + \frac{1}{L-1} \sum_{l=0}^{k} 2^{-l-2} \to \alpha_0 + \frac{1}{2(L-1)}$$

as $k \to \infty$. This yields the estimate

$$\sup_{\Gamma^t} |\partial_s \beta^\varepsilon| = \lim_{k\to\infty} X_{p_k}^{\frac{1}{p_k}}(t) \le M_7 t^{-\left(\alpha_0 + \frac{1}{2(L-1)}\right)}$$

for any $t \in (0, T]$ and $0 < \varepsilon \le 1$. Since $\alpha_0 = \frac{1}{2}, L = 3$ for $0 < m \le 1$ and $\alpha_0 = 0, L = 2$ for $1 < m \le 2$, the proof of the lemma follows. ∎

Summarizing all the previous results we conclude the following a priori estimates.

LEMMA 5.5. *Assume* $0 < m \le 2$. *Let* $\Phi_\varepsilon = (k_\varepsilon, \nu_\varepsilon, g_\varepsilon)^T$ *be a classical solution of* (3.13) *existing on the interval* $I = [0, T]$ *and satisfying the initial condition* $\Phi^0 \in \mathcal{O}_1 \subset E_1$. *If* $1 < m \le 2$ *we furthermore assume that the initial curve* Γ^0 *satisfies the condition* (5.6). *If the tangential velocity* α^ε *satisfies the condition* (5.4), *then*

(1) $k_\varepsilon,\ \beta^\varepsilon,\ t^{\frac{3}{4}} \partial_u \beta^\varepsilon \in L^\infty(Q_T)$;
(2) $g_\varepsilon,\ g_\varepsilon^{-1} \in W^{1,\infty}(I, L^\infty(S^1))$;
(3) $\partial_u \nu_\varepsilon,\ t^{\frac{3}{4}} \partial_t \nu_\varepsilon \in L^\infty(Q_T)$;
(4) $x_\varepsilon \in (W^{1,\infty}(Q_T))^2$;

and if, in addition, α^ε *satisfies the condition* (5.5), *then*

(5) $g_\varepsilon, g_\varepsilon^{-1} \in W^{1,\infty}(Q_T)$ *and* $\partial_u x_\varepsilon \in (W^{1,\infty}(Q_T))^2$

and their corresponding norms are bounded independently of $0 < \varepsilon \le 1$.

Proof. The statement (1) is an immediate consequence of Lemmas 5.1 and 5.4 and the assumption (ii) made on the regularization β^ε. Since $\partial_t g_\varepsilon = -g_\varepsilon k_\varepsilon \beta^\varepsilon + \partial_u \alpha^\varepsilon$ and $\partial_t g_\varepsilon^{-1} = -g_\varepsilon^{-2} \partial_t g_\varepsilon$ the statement (2) follows from (1), Lemma 5.1, and the assumption (5.4). The bounds for ν_ε follow from the identities $\partial_u \nu_\varepsilon = g_\varepsilon k_\varepsilon, \partial_t \nu_\varepsilon = \partial_s \beta^\varepsilon + \alpha^\varepsilon k_\varepsilon$ (see (3.8), (3.9)). As $\partial_t x_\varepsilon = \beta^\varepsilon \vec{N}_\varepsilon + \alpha^\varepsilon \vec{T}_\varepsilon$, $\partial_u x_\varepsilon = g_\varepsilon \vec{T}_\varepsilon$, and $\beta^\varepsilon, \alpha^\varepsilon, g_\varepsilon \in L^\infty(S^1)$ we conclude the statement (4). Let us assume α^ε satisfies the condition (5.5). By integrating the third equation in (3.13) we obtain $g_\varepsilon(.,t) = g^0(.)\exp(\int_0^t(-k_\varepsilon\beta^\varepsilon + g_\varepsilon^{-1}\partial_u\alpha^\varepsilon)d\tau)$. Furthermore,

$$\partial_u(k_\varepsilon\beta^\varepsilon) = k_\varepsilon\partial_u\beta^\varepsilon + \beta^\varepsilon\partial_u k_\varepsilon = \left(k_\varepsilon + \frac{\beta^\varepsilon}{\beta_k^{\varepsilon\prime}}\right)\partial_u\beta^\varepsilon - \frac{\beta^\varepsilon\beta_\nu^{\varepsilon\prime}}{\beta_k^{\varepsilon\prime}}g_\varepsilon k_\varepsilon.$$

With regard to (1) and the regularization assumption made on β^ε, we can conclude $\|k_\varepsilon\beta^\varepsilon\|_{C^1} \le M\|\beta^\varepsilon\|_{C^1}$. Taking into account the condition (5.5) and Lemma 5.4 we obtain the estimate

$$\begin{aligned}\|g_\varepsilon(.,t)\|_{C^1} &\le M\left(1 + \int_0^t(\|k_\varepsilon\beta^\varepsilon\|_{C^1} + \|\alpha^\varepsilon\|_{C^2} + \|g_\varepsilon(.,\tau)\|_{C^1})d\tau\right)\\ &\le M\left(1 + \int_0^t\|\beta^\varepsilon\|_{C^1}d\tau + \int_0^t\|g_\varepsilon(.,\tau)\|_{C^1}d\tau\right)\\ &\le M\left(1 + \int_0^t\|g_\varepsilon(.,\tau)\|_{C^1}d\tau\right)\end{aligned}$$

for $t \in [0,T]$ and $0 < \varepsilon \le 1$. Hence the L^∞ bounds for $\partial_u g_\varepsilon$ and $\partial_u g_\varepsilon^{-1} = -g_\varepsilon^{-2}\partial_u g_\varepsilon$ follow from Gronwall's lemma. The L^∞ bounds for $\partial_u^2 x_\varepsilon$ and $\partial_t\partial_u x_\varepsilon$ now follow from the identities $\partial_u^2 x_\varepsilon = \partial_u(g_\varepsilon T_\varepsilon) = \partial_u g_\varepsilon\vec{T}_\varepsilon + g_\varepsilon^2 k_\varepsilon\vec{N}_\varepsilon$ and $\partial_t\partial_u x_\varepsilon = \partial_t(g_\varepsilon T_\varepsilon) = \partial_t g_\varepsilon\vec{T}_\varepsilon + g_\varepsilon\partial_t\nu_\varepsilon\vec{N}_\varepsilon$ and parts (2) and (3). ☐

Now we are in a position to state the main result of this paper.

THEOREM 5.6. *Suppose that* $\beta(k,\nu) = \gamma(\nu)|k|^{m-1}k$, *where* $0 < m \le 2$ *and* γ *satisfies* (5.1). *Let* Γ^0 *be a smooth regular plane curve such that* $(k^0,\nu^0,g^0)^T \in \mathcal{O}_1 \subset E_1$. *If* $1 < m \le 2$, *we also suppose that* Γ^0 *satisfies the condition* (5.6). *If the tangential velocity* α^ε *obeys the conditions* (5.4) *and* (5.5), *then there exists* $T > 0$ *and a family of regular plane curves* $\Gamma^t = \mathrm{Image}(x(.,t)), t \in [0,T]$, $x : \overline{Q_T} = [0,1]\times[0,T] \to \mathbb{R}^2$ *such that*

(1) $x, \partial_u x \in (C(\overline{Q_T}))^2$, $\partial_u^2 x$, $\partial_t x$, $\partial_u\partial_t x \in (L^\infty(Q_T))^2$;

(2) $\partial_t x.\vec{N} = \beta(k,\nu)$ *for any* $t \in [0,T]$ *and a.e.* $u \in [0,1]$, *where* k, ν, *and* $\vec{N}$ *are the curvatures, the tangent angle, and the unit normal vector of the curve* Γ^t.

Proof. It follows from Lemma 5.5, part (4), and the Ascoli–Arzelà theorem that there exists a subsequence of (x_ε) converging uniformly, i.e., $x_\varepsilon \to x$ in $(C(\overline{Q_T}))^2$ as $\varepsilon \to 0^+$. By part (5) we also have $\partial_u x_\varepsilon \to \partial_u x$ in $(C(\overline{Q_T}))^2$ and $\partial_t x, \partial_u\partial_t x, \partial_u^2 x \in (L^\infty(Q_T))^2$. Again, by (4) and (5) we furthermore have $\nu_\varepsilon \rightrightarrows \nu, g_\varepsilon \rightrightarrows g$ in $C(\overline{Q_T})$ and $g > 0$. Hence $\vec{T}_\varepsilon = g_\varepsilon\partial_u x_\varepsilon \rightrightarrows g\partial_u x = \vec{T}$ and $\vec{N}_\varepsilon \rightrightarrows \vec{N}$, where $\vec{T}$ and $\vec{N}$ are the unit tangent and normal vectors to the curve $\Gamma^t = \mathrm{Image}(x(.,t)), t \in [0,T]$. Moreover, $\arg(\vec{T}) = \nu$.

Let $t \in [0,T]$ be a fixed time instant. By (1) we have $|\partial_u\beta^\varepsilon| \le M$ and, as a consequence, one has $\beta^\varepsilon \rightrightarrows \tilde{\beta}$ in $C(S^1)$. Denote by $b^\varepsilon : \mathbb{R} \to \mathbb{R}$ the inverse function to the increasing function $k \mapsto \beta^\varepsilon(k,\nu)/\gamma(\nu)$, $\varepsilon \in [0,1], \beta^0 = \beta$. Notice that the term $\beta^\varepsilon(k,\nu)/\gamma(\nu)$ does not depend on ν. With regard to the regularization assumptions made on β^ε we have $b^\varepsilon \to b^0 = b$ locally uniformly in $\mathbb{R}$. Then for the curvature $k_\varepsilon = b^\varepsilon(\beta^\varepsilon(k_\varepsilon,\nu_\varepsilon)/\gamma(\nu_\varepsilon)) = b^\varepsilon(\beta^\varepsilon/\gamma(\nu_\varepsilon))$ we have the convergence $k_\varepsilon \rightrightarrows k = b(\tilde{\beta})/\gamma(\nu)$ in $C(S^1)$. Thus $\beta(k,\nu) = \tilde{\beta}$. As $(\partial_u^2 x_\varepsilon)$ is bounded in $(L^\infty(S^1))^2$ we have $\partial_u^2 x_\varepsilon \overset{*}{\rightharpoonup} \partial_u^2 x$ weak star in $(L^\infty(S^1))^2$. On the other hand, $k_\varepsilon = g_\varepsilon^{-3}(\partial_u x_\varepsilon \wedge \partial_u^2 x_\varepsilon) \rightrightarrows k, g_\varepsilon^{-3} \rightrightarrows g^{-3}, \partial_u x_\varepsilon \rightrightarrows \partial_u x$. Thus $k(u,t)$ is the curvature of the curve Γ^t at the point $x = x(u,t)$ for every $t \in [0,T]$ and a.e. $u \in [0,1]$. Finally, as $(\partial_t x_\varepsilon)$ is bounded in $(L^\infty(S^1))^2$ we have $\partial_t x_\varepsilon \overset{*}{\rightharpoonup} \partial_t x$ weak star in $(L^\infty(S^1))^2$. Therefore, $\beta^\varepsilon = \partial_t x_\varepsilon.\vec{N}_\varepsilon \overset{*}{\rightharpoonup} \partial_t x.\vec{N}$

as $\varepsilon \to 0^+$. Since $\beta^\varepsilon \rightrightarrows \tilde{\beta} = \beta(k,\nu)$ in $C(S^1)$ we conclude $\partial_t x.\vec{N} = \beta(k,\nu)$, as claimed. ∎

6. Tangential velocity preserving the relative local length. As was already mentioned in section 2, the presence of a nontrivial tangential velocity term $\alpha\vec{T}$ in the governing equation (2.4) can prevent the numerically computed solution of (2.2) from forming numerical singularities like, e.g., collapsing of grid points or formation of the so-called swallow tails. The goal of this section is to propose a suitable choice of the functional $\alpha = \alpha(k,\nu,g)$ in such a way that a numerical scheme based on this choice of α will be capable of uniform redistribution of grid points along the computed curve. The main idea behind the construction of α is to analyze the relative local length function defined as the ratio $\frac{|\partial_u x(u,t)|}{L^t}$, where L^t is the total length of Γ^t and $|\partial_u x(u,t)|$ represents the local length of Γ^t. The idea is to keep this ratio constant with respect to time, i.e., *preservation of the relative local length*:

$$\frac{d}{dt}\left(\frac{|\partial_u x(u,t)|}{L^t}\right) = 0 \tag{6.1}$$

for any $u \in [0,1]$ and $t \in I = (0,T)$. Taking into account (3.4) and (3.5) one sees that (6.1) is fulfilled iff

$$\frac{\partial \alpha}{\partial s} = k\beta(k,\nu) - \frac{1}{L}\int_\Gamma k\beta(k,\nu)\,ds, \tag{6.2}$$

where $\Gamma = \Gamma^t$, $L = L(\Gamma)$, k is the curvature of Γ, and β is the given normal velocity function.

In what follows, we will show that there exist geometric quantities θ_1, θ_2 such that the tangential velocity function α given by $\alpha = \frac{1}{\theta_1}\frac{\partial}{\partial s}(\frac{1}{\theta_2})$ (see (2.5)) obeys (6.2). We will furthermore prove some a priori estimates for α and $\theta_i, i = 1,2$, considered as nonlocal operators from the Banach space $E_{1/2}$ (see (4.3)) into $C^{2+\sigma}(S^1)$. First we need the following simple lemma.

LEMMA 6.1. *Let β^ε be a regularization of β satisfying regularization assumptions* (i)–(iv) *from section* 5. *Let $\Gamma = \text{Image}(x)$ be a C^2 smooth regular plane curve. Then there exists a unique weak solution $\vartheta \in C^1(S^1)$, $\vartheta(0) = \vartheta(1) = 0$, of the equation*

$$-\frac{\partial}{\partial s}\left(\frac{\beta^\varepsilon(k,\nu)}{k}\frac{\partial\vartheta}{\partial s}\right) = k\beta^\varepsilon(k,\nu) - \frac{1}{L}\int_\Gamma k\beta^\varepsilon(k,\nu)\,ds. \tag{6.3}$$

Furthermore, there exists a constant $C_4 = C_4(M_1) > 0$ such that

$$\max_\Gamma |\vartheta| \le C_4 \int_\Gamma \frac{k}{\beta^\varepsilon(k,\nu)}\,ds \quad \textit{and} \quad |\partial_s\vartheta| \le C_4 L(\Gamma)\,\frac{k}{\beta^\varepsilon(k,\nu)}$$

for any $|k| \le M_1$.

Proof. Denote $a = \frac{k}{\beta^\varepsilon}$, $g = |\partial_u x|$, and $f = \frac{1}{L}\int_\Gamma k\beta^\varepsilon(k,\nu)\,ds - k\beta^\varepsilon(k,\nu)$. Then $0 < a < \infty$, $g > 0$, and $a, g, f \in C(S^1)$. Hence

$$\partial_u\vartheta(u) = a(u)g(u)\left(A + \int_0^u f(v)g(v)\,dv\right) \tag{6.4}$$

for some constant A. With regard to the condition $\vartheta(0) = \vartheta(1) = 0$ we obtain the existence of a unique weak solution $\vartheta \in C^1(S^1)$ and

$$\vartheta(u) = A\int_0^u a(\xi)g(\xi)\,d\xi + \int_0^u a(\xi)g(\xi)\int_0^\xi f(v)g(v)\,dv\,d\xi,$$

where $A = -(\int_0^1 a(\xi)g(\xi)\int_0^\xi f(v)g(v)\,dv\,d\xi)(\int_0^1 a(\xi)g(\xi)\,d\xi)^{-1}$. Since $ag \geq 0$ we have $|A| + |\int_0^u fg| \leq 2\max_\xi \int_0^\xi |f|g \leq 2\int_0^1 |f|g = 2\int_\Gamma |f|\,ds \leq 4\int_\Gamma k\beta^\varepsilon(k,\nu)\,ds \leq 4L(\Gamma)M_1C_2(1+M_1^m) = C_4L(\Gamma)$. This, together with (6.4), yields the pointwise estimate for $|\partial_s\vartheta| = |g^{-1}\partial_u\vartheta|$. The bound for $\max|\vartheta|$ now easily follows from the boundary condition $\vartheta(0) = 0$. ∎

LEMMA 6.2. *Let* $\Gamma = \mathrm{Image}(x)$ *be a smooth regular plane curve such that* $\Phi = (k,\nu,g)^T \in \mathcal{O}_{1/2} \subset E_{1/2}$, $|k| \leq M_1$. *Let* β^ε *be any regularization of* β *satisfying the regularization assumptions* (i)–(iv) *from section* 5. *Then there exist geometric quantities* $\theta_i^\varepsilon > 0$, $\theta_i^\varepsilon : \mathcal{O}_{1/2} \to C^1(S^1), i = 1,2$, *such that*

$$\theta_1^\varepsilon\theta_2^\varepsilon = \frac{k}{\beta^\varepsilon(k,\nu)} \quad \textit{and} \quad \alpha^\varepsilon = \frac{1}{\theta_1^\varepsilon}\frac{\partial}{\partial s}\left(\frac{1}{\theta_2^\varepsilon}\right), \quad \theta_2^\varepsilon(0) = \theta_2^\varepsilon(1) = 1,$$

where $\alpha^\varepsilon \in C^1(\mathcal{O}_{1/2}, C^{2+\sigma}(S^1))$ *is the tangential velocity preserving the relative local length satisfying* (6.2). *Moreover,*

$$\max_\Gamma |\theta_2^\varepsilon(\Phi)| + \max_\Gamma |\theta_2^\varepsilon(\Phi)^{-1}| \leq \exp\left(M_6\int_\Gamma \frac{k}{\beta^\varepsilon}ds\right),$$
$$|\partial_s\alpha^\varepsilon(\Phi)| \leq M_7, \qquad \|\alpha^\varepsilon(\Phi)\|_{C^2} \leq M_7(1 + \|\beta^\varepsilon(k,\nu)\|_{C^1} + \|g\|_{C^1});$$

i.e., α^ε *satisfies the hypotheses* (5.4) *and* (5.5).

Proof. Let ϑ be a solution of (6.3). Define $\theta_2^\varepsilon = \exp(\vartheta)$ and $\theta_1^\varepsilon = k/(\beta^\varepsilon\theta_2^\varepsilon)$. The maximum bounds for θ_2^ε and $(\theta_2^\varepsilon)^{-1}$ follow from Lemma 6.1. With regard to Lemma 6.1 we obtain that

$$\alpha^\varepsilon = \frac{1}{\theta_1^\varepsilon}\frac{\partial}{\partial s}\left(\frac{1}{\theta_2^\varepsilon}\right) = -\frac{\beta^\varepsilon}{k}\frac{\partial}{\partial s}\ln\theta_2^\varepsilon = -\frac{\beta^\varepsilon}{k}\frac{\partial\vartheta}{\partial s}$$

is a solution of (6.2). Since β^ε satisfies the regularization assumption we have $\alpha^\varepsilon \in C^1(\mathcal{O}_{1/2}, C^{2+\sigma}(S^1))$. Notice that the estimate for the $C^{2+\sigma}$ norm of α^ε may depend on $0 < \varepsilon \leq 1$. It furthermore follows from Lemma 6.1 that $\|\alpha^\varepsilon(\Phi)\|_{C^0} \leq M_6L(\Gamma)$. With regard to (6.2) we have $\partial_u\alpha^\varepsilon = (k\beta^\varepsilon - \mathrm{const})g$, where $\mathrm{const} = \frac{1}{L}\int_\Gamma k\beta^\varepsilon ds$ is a constant. Hence $|\partial_s\alpha^\varepsilon| = g^{-1}|\partial_u\alpha^\varepsilon| \leq 2\max_\Gamma |k\beta^\varepsilon(k,\nu)| \leq M_7$. Furthermore, as $\|k\beta^\varepsilon(k,\nu)\|_{C^1} \leq M\|\beta^\varepsilon(k,\nu)\|_{C^1}$ and $|\mathrm{const}| \leq \max_\Gamma |k\beta^\varepsilon(k,\nu)|$, we have $|\partial_u^2\alpha^\varepsilon| \leq |\mathrm{const}||\partial_u g| + |\partial_u(gk\beta^\varepsilon(k,\nu))| \leq M_7(1 + \|\beta^\varepsilon(k,\nu)\|_{C^1} + \|g\|_{C^1})$, and the bound for $\|\alpha^\varepsilon\|_{C^2}$ follows. This is why α^ε satisfies the assumptions (5.4) and (5.5). ∎

THEOREM 6.3. *Suppose that* $\beta(k,\nu) = \gamma(\nu)|k|^{m-1}k$, *where* $0 < m \leq 2$ *and* γ *satisfies* (5.1). *Let* $\Gamma^0 = \mathrm{Image}(x^0)$ *be a smooth regular plane curve as in Theorem* 5.6. *Then there exists* $T > 0$ *and a family of regular plane curves* $\Gamma^t = \mathrm{Image}(x(.,t)), t \in [0,T]$ *such that*

(1) $x, \partial_u x \in (C(\overline{Q_T}))^2$, $\partial_u^2 x$, $\partial_t x$, $\partial_u\partial_t x \in (L^\infty(Q_T))^2$;

(2) *the flow* $\Gamma^t = \mathrm{Image}(x(.,t)), t \in [0,T]$ *of regular plane curves satisfies the geometric equation*

$$\partial_t x = \beta\vec{N} + \alpha\vec{T},$$

where $\beta = \beta(k,\nu)$ *and* α *is the tangential velocity preserving the relative local length, i.e.,* α *satisfies* (6.2) *and*

$$\frac{|\partial_u x(u,t)|}{L^t} = \frac{|\partial_u x^0(u)|}{L^0}$$

for any $t \in [0,T]$ *and* $u \in [0,1]$.

Proof. Let us consider the tangential velocity function α^ε, $0 < \varepsilon \le 1$, satisfying $\partial_s \alpha^\varepsilon = k_\varepsilon \beta^\varepsilon(k_\varepsilon, \nu_\varepsilon) - \frac{1}{L_\varepsilon}\int_{\Gamma_\varepsilon} k_\varepsilon \beta^\varepsilon ds$ whose existence has been verified in Lemma 6.2. Moreover, α^ε is a C^1 mapping from $\mathcal{O}_{1/2} \subset E_{1/2}$ into $c^{2+\sigma}(S^1)$ and α^ε satisfies the structural conditions (5.4) and (5.5). By Theorem 5.6 there exists a family of regular plane curves $\Gamma^t = \text{Image}(x(.,t))$ with the properties as in part (1). To prove (2), we put $\alpha = \partial_t x.\vec{T}$ and recall that $g_\varepsilon = |\partial_u x_\varepsilon| \rightrightarrows g = |\partial_u x|$ as $\varepsilon \to 0^+$. Therefore, $L^t_\varepsilon = \int_0^1 |\partial_u x_\varepsilon(u,t)|\, du \to L^t = \int_0^1 |\partial_u x(u,t)|\, du$ as $\varepsilon \to 0^+$. Thus $\frac{|\partial_u x_\varepsilon(u,t)|}{L^t_\varepsilon} \to \frac{|\partial_u x(u,t)|}{L^t}$ as $\varepsilon \to 0^+$. On the other hand, since α^ε is the tangential velocity preserving the relative local length we have $\frac{d}{dt}\frac{|\partial_u x_\varepsilon(u,t)|}{L^t_\varepsilon} = 0$. Hence $\frac{|\partial_u x^0(u)|}{L^0} = \frac{|\partial_u x(u,t)|}{L^t}$. Therefore, α is the tangential velocity preserving the relative local length and from (6.1), (3.4), and (3.6), we may conclude that α satisfies (6.2). ◻

7. Numerical scheme. In this section we describe a numerical procedure that can be used for computing the curve evolution satisfying the geometric equation (1.1). To this end, we will propose a scheme solving the coupled system of intrinsic heat equation (2.2) for the position vector x and (6.2) for the tangential velocity α. A smooth solution x is approximated by discrete plane points x_i^j, $i = 1, \dots, n$, $j = 0, \dots, m$, where index i represents space discretization and index j a discrete time stepping. The approximation of a curve in time $j\tau$ (with uniform time step $\tau = \frac{T}{m}$) is given by a polygon with vertices x_i^j, $i = 1, \dots, n$. In order to obtain such an approximation of an evolving curve in the jth time step, we use the following fully discrete semi-implicit scheme:

$$\frac{1}{2}(g_i^{j-1} + g_{i+1}^{j-1})\frac{x_i^j - x_i^{j-1}}{\tau} = \frac{x_{i+1}^j - x_i^j}{h_{i+1}^{j-1}} - \frac{x_i^j - x_{i-1}^j}{h_i^{j-1}}, \tag{7.1}$$

$i = 1, \dots, n$, for every $j = 1, \dots, m$. The coefficients in (7.1) (for simplicity we omit upper index $j-1$) are given by the following expressions:

$$\begin{aligned}
&g_i = |r_i|\theta_{1,i}, \quad h_i = |r_i|\theta_{2,i}, \quad r_i = x_i - x_{i-1}, \quad \theta_{1,i} = \frac{k_i}{\beta_i \theta_{2,i}},\\
&k_i = \frac{1}{2|r_i|}\text{sgn}(r_{i-1} \wedge r_{i+1}) \arccos\left(\frac{r_{i+1}.r_{i-1}}{|r_{i+1}||r_{i-1}|}\right),\\
&\nu_i = \arccos(r_{i_1}/|r_i|) \text{ if } r_{i_2} \ge 0, \quad \nu_i = 2\pi - \arccos(r_{i_1}/|r_i|) \text{ if } r_{i_2} < 0,\\
&\beta_i = \beta^\varepsilon(k_i, \nu_i), \quad \theta_{2,i} = \exp(\vartheta_i),
\end{aligned} \tag{7.2}$$

and the system (7.1) is subject to the periodic boundary conditions $x_{i+n}^j = x_i^j$ $(i = 0, 1)$. In order to compute ϑ_i, $i = 1, \dots, n$, governing tangential redistribution of flowing points, we solve

$$\begin{aligned}
&-\frac{\frac{\beta_i}{k_i} + \frac{\beta_{i+1}}{k_{i+1}}}{|r_i| + |r_{i+1}|}(\vartheta_{i+1} - \vartheta_i) + \frac{\frac{\beta_i}{k_i} + \frac{\beta_{i-1}}{k_{i-1}}}{|r_i| + |r_{i-1}|}(\vartheta_i - \vartheta_{i-1})\\
&\qquad = |r_i|\left(k_i\beta_i - \left(\sum_{l=1}^n |r_l| k_l \beta_l\right)\left(\sum_{l=1}^n |r_l|\right)^{-1}\right)
\end{aligned} \tag{7.3}$$

for $i = 1, \dots, n$, accompanied by the periodic boundary conditions $\vartheta_{i+n} = \vartheta_i$ $(i = 0, 1)$.

The system (7.3) can be represented by a symmetric positive semidefinite tridiagonal matrix with kernel containing n-dimensional vector each component of which is equal 1. Since $\sum_{i=1}^{n} b_i = 0$, where b_i are the components of the right-hand side of (7.3), we have assured the existence of a solution that is also unique up to an additive constant. We choose the unique solution by imposing the constraint condition $\vartheta_0 = \vartheta_n = 0$.

Then, the linear systems (7.1) can be represented by two symmetric positive definite tridiagonal matrices for which we have the existence and uniqueness of a solution. In each discrete computational time step $j\tau$ the scheme (7.1)–(7.3) leads to solving three tridiagonal systems, namely, one for the redistribution of points along the curve and two for finding the new curve position.

Remark 7.1. The approximation (7.1) can be considered as a full time-space discretization analogy to the backward Euler time semidiscretization scheme

$$\frac{x^j - x^{j-1}}{\tau} = \frac{1}{\theta_1^{j-1}} \frac{\partial}{\partial s^{j-1}} \left(\frac{1}{\theta_2^{j-1}} \frac{x^j}{\partial s^{j-1}} \right), \quad j = 1, 2, \ldots, m, \tag{7.4}$$

of (2.2), where the terms θ_1, θ_2 as well as arclength parameterization s are taken from the previous time step x^{j-1}, and $\Gamma^j = \mathrm{Image}(x^j)$ is considered as a smooth approximation of the evolution in discrete time $j\tau$. Denoting $\delta_\tau(x^j) = (x^j - x^{j-1})/\tau$ and $ds^{j-1} = |\partial_u x^{j-1}| du$, $\theta_1^{j-1}\theta_2^{j-1} = k^{j-1}/\beta^\varepsilon(k^{j-1}, \nu^{j-1})$, we easily obtain

$$\delta_\tau(x^j) = \tilde{\beta}\vec{N}^j + \tilde{\alpha}\vec{T}^j, \tag{7.5}$$

where

$$\tilde{\beta} = \frac{|\partial_u x^j|^2 k^j}{|\partial_u x^{j-1}|^2 k^{j-1}} \beta^\varepsilon(k^{j-1}, \nu^{j-1}), \quad \tilde{\alpha} = \frac{1}{\theta_1^{j-1}} \frac{\partial}{\partial s^{j-1}} \left(\frac{1}{\theta_2^{j-1}} \frac{|\partial_u x^j|}{|\partial_u x^{j-1}|} \right).$$

In the next proposition we show that the backward Euler time discretization scheme generates a discrete curve shortening sequence of plane curves. This result can be considered just as an indication and not a rigorous proof that the sequence of numerically computed discrete polygonal curves is stable in the sense that their length decreases during evolution. The detailed analysis of the stability of the scheme (7.1)–(7.3) is a work in progress and we hope it will be discussed in the forthcoming paper.

PROPOSITION 7.1. *Assume $x^{j-1} \in C^1(S^1; \mathbb{R}^2)$, $\theta_1^{j-1}, \theta_2^{j-1} \in C^1(S^1; \mathbb{R}^2)$ are such that $|\partial_u x^{j-1}| > 0$, $\theta_1^{j-1} > 0$, $\theta_2^{j-1} > 0$. Then there exists a unique solution $x^j \in C^2(S^1; \mathbb{R}^2)$ of (7.4). Moreover,*

$$L^j + \tau \int_{\Gamma^j} \tilde{\beta} k^j ds^j \le L^{j-1}, \tag{7.6}$$

where $L^j = \int_0^1 |\partial_u x^j| du$ represents the length of the curve Γ^j. The sequence $\Gamma^j = \mathrm{Image}(x^j)$ represents a curve shortening discrete flow.

Proof. The existence and uniqueness of a solution x^j can be achieved in the same way as was done in [33, Lemma 4.1] in the case $\theta_1 = \theta_2 = k/\beta$. To prove the estimate (7.6) we proceed in a similar way as in the continuous case (see (3.5)). Indeed, the time-discrete analogy of the first equation in (3.1) is given by

$$\delta_\tau(\partial_u x^j) = |\partial_u x^j| \left((\partial_{s^j}\tilde{\beta} + \tilde{\alpha}k^j)\vec{N}^j + (-\tilde{\beta}k^j + \partial_{s^j}\tilde{\alpha})\vec{T}^j \right)$$

and therefore

$$|\partial_u x^j| = \tau\delta_\tau(\partial_u x^j).\vec{T}^j + \partial_u x^{j-1}.\vec{T}^j = \tau\delta_\tau(\partial_u x^j).\vec{T}^j + |\partial_u x^{j-1}|\vec{T}^{j-1}.\vec{T}^j$$
$$\leq |\partial_u x^{j-1}| + \tau(-\tilde{\beta}k^j + \partial_{s^j}\tilde{\alpha})|\partial_u x^j| = |\partial_u x^{j-1}| - \tau\tilde{\beta}k^j|\partial_u x^j| + \tau\partial_u\tilde{\alpha}.$$

Integrating the above inequality over the interval $[0,1]$ yields the bound (7.6). Since $\beta^\varepsilon k \geq 0$ we have $\tilde{\beta}k^j \geq 0$ and therefore $L^j \leq L^{j-1}$; i.e., Γ^j represents a curve-shortening discrete flow. □

Remark 7.2. The scheme (7.1)–(7.3) can be derived by using the flowing control volume method (cf. [37]). Let us consider points x_i, $i = 1,\dots,n$, belonging to a smooth curve $\Gamma^t = \text{Image}(x(.,t))$, where x is a solution of (2.2) at time t. By $[x_{i-1}, x_i]$ we denote the arc of the curve between the points x_{i-1} and x_i. Let us consider a control volume V_i around x_i consisting of part of the arc connecting centers c_i, c_{i+1} of arcs $[x_{i-1}, x_i]$, $[x_i, x_{i+1}]$, respectively. Such a centered control volume is flowing and changing a length during the evolution respecting the new positions of the points x_i along the curve. Let us integrate intrinsic diffusion equation (2.2) along the finite volume V_i. We obtain

$$\int_{V_i} \theta_1 \frac{\partial x}{\partial t} ds = \left[\frac{1}{\theta_2}\frac{\partial x}{\partial s}\right]_{c_i}^{c_{i+1}}. \tag{7.7}$$

Let us consider piecewise linear approximation of x, i.e., a polygon connecting points x_i, $i = 1,\dots,n$. From (7.2) we can compute constant geometrical quantities k_i, ν_i, β_i for each line segment $[x_{i-1}, x_i]$. The quantity ϑ_i can be computed numerically by solving control volume approximation of the intrinsic equation (6.2). Integrating (6.2) along $[x_{i-1}, x_i]$ (a dual volume to V_i) yields

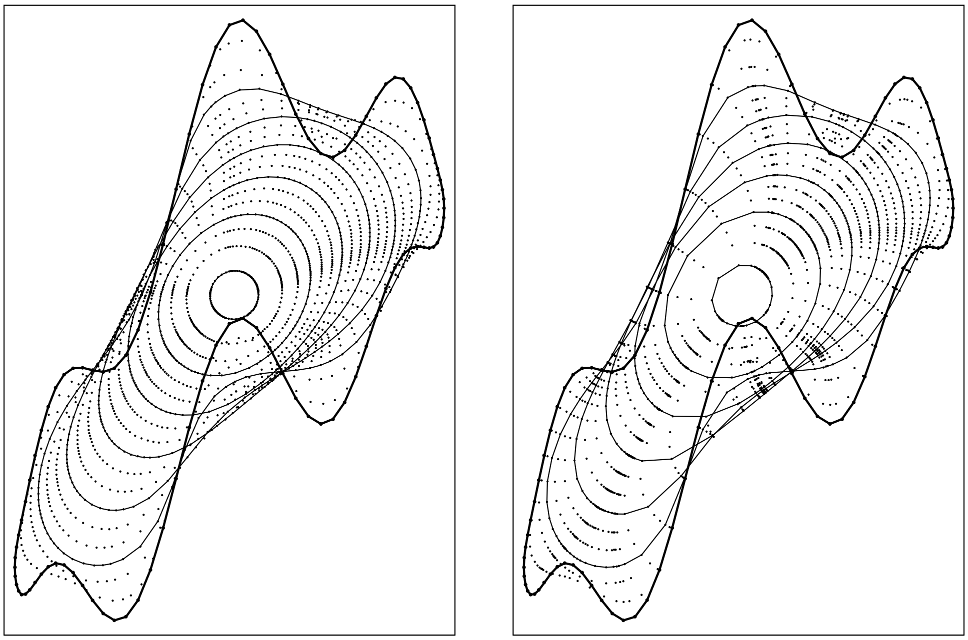
$$-\left[\frac{\beta}{k}\frac{\partial\vartheta}{\partial s}\right]_{x_{i-1}}^{x_i} = |r_i|\left(k_i\beta_i - \left(\sum_{l=1}^n |r_l|k_l\beta_l\right)\left(\sum_{l=1}^n |r_l|\right)^{-1}\right). \tag{7.8}$$

Approximating $\frac{\partial\vartheta}{\partial s}(x_i)$ by $2\frac{\vartheta_{i+1}-\vartheta_i}{(|r_i|+|r_{i+1}|)}$ and $\frac{\beta}{k}(x_i)$ by $\frac{1}{2}(\frac{\beta_i}{k_i} + \frac{\beta_{i+1}}{k_{i+1}})$ we end up with the system (7.3). Now, approximating $\frac{\partial x}{\partial t}$ by $\dot{x}_i$ inside V_i we obtain from (7.7) the system of ordinary differential equations

$$\frac{1}{2}(|r_i|\theta_{1,i} + |r_{i+1}|\theta_{1,i+1})\dot{x}_i = \frac{1}{\theta_{2,i+1}}\frac{x_{i+1}^j - x_i^j}{|r_{i+1}|} - \frac{1}{\theta_{2,i}}\frac{x_i^j - x_{i-1}^j}{|r_i|}. \tag{7.9}$$

There is a range of possibilities of how to solve this system. In order to obtain the scheme (7.1) we approximate the time derivative by the time difference of the new and previous discrete curve position where all nonlinear terms are taken from the previous time step and linear terms are considered at a new time level. The numerical simulations of section 8 show that such an approximation is sufficient in very general cases regarding accuracy and efficiency of computations. Moreover, using Proposition 7.1 we have guaranteed a kind of stability for numerical computations.

8. Discussion on numerical experiments. In this section we describe numerical results obtained by the algorithm (7.1)–(7.3) for solving the geometric equation (1.1). We test properties of the model and the numerical scheme in evolution of convex as well as nonconvex (and nonrectifiable) initial curves in the presence of nonlinearity and anisotropy in the shape of function β. The effect of redistribution of discrete



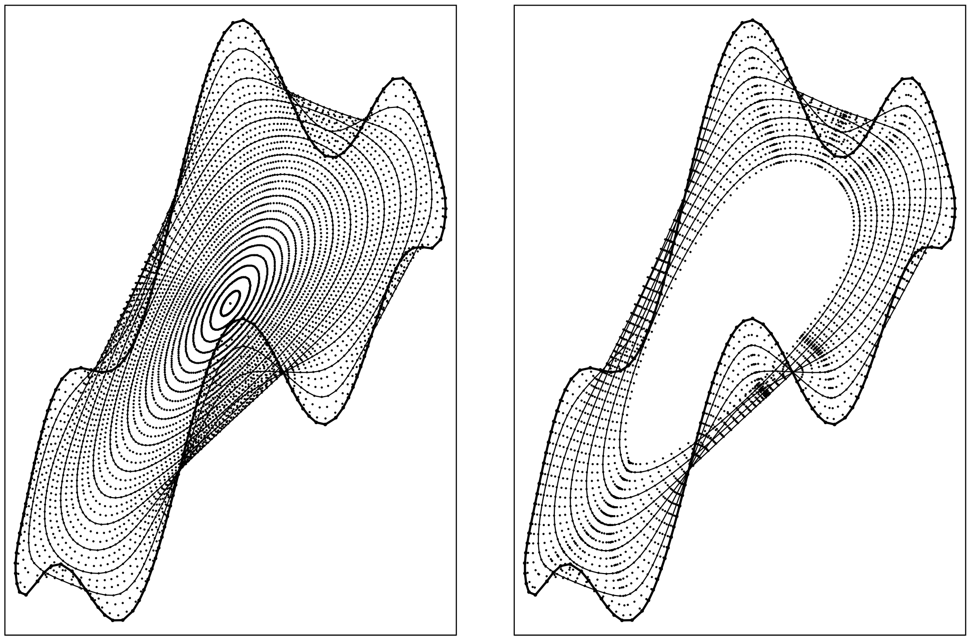
(a) (b)

FIG. 1. $\beta(k) = k$. (a) *Discrete evolution using tangential redistribution of points.* (b) *Without redistribution, only normal component of velocity is used.*

points representing an evolving curve is documented in the same time. We have found several examples where the uniform grid redistribution based on relative local length conservation of flowing curve segments is an important tool in correct handling of the curve evolution without other artificial operations like points removing or artificial cutting of the so-called swallow tails. The redistribution of grid points based on (6.2) preserves the initial discretization of a curve and thus makes its discrete representation smooth enough during evolution. First such examples are given in Figures 1(a) and 1(b). In those experiments $\beta(k) = k$; i.e., we have classical curve shortening, and we start with initial curve with large variations in the curvature, namely,

$$x_1(u) = \cos(2\pi u),$$
$$x_2(u) = \frac{1}{2}\sin(2\pi u) + \sin(x_1(u)) + \sin(2\pi u)(0.2 + \sin(2\pi u)\sin(6\pi u)\sin(6\pi u)),$$

$u \in [0, 1]$, and initial discretization is given by uniform division of the range of parameter u. The curve is represented by 100 discrete points. Addition of a nontrivial tangential velocity obeying (6.2) leads to the evolution plotted in Figure 1(a). In Figure 1(b) the points move only in the normal direction and one can see their fast merging in several regions and very poor discrete representation in other pieces of the curve. In all experiments we have used the uniform time step $\tau = 0.001$. The blowup time for the curvature was $T_{\max} = 0.363$. Isoperimetric ratio starting with 3.02 tends to 1.0, which is consistent with Grayson's theorem [24]. In both figures, we plot each 20th discrete time step using discrete points representing the evolving curve, and in each 60th time step we plot also by piecewise linear curve connecting those points.



(a) (b)

FIG. 2. $\beta(k) = k^{1/3}$. (a) *Discrete evolution using tangential redistribution of grids preserving the relative local length.* (b) *Without redistribution, computation collapses due to vanishing of the local length element* $|\partial_u x|$.

In Figures 2(a) and 2(b) we computed affine evolution of the same initial curve for the affine scale $\beta(k) = k^{1/3}$. The initial curve has been discretized almost uniformly. In Figure 2(a) we show how this discretization is then preserved in evolution when using the scheme (6.2). The blowup time $T_{\max} = 0.694$, a solution converges to an ellipse with the isoperimetric ratio stabilized on 1.33. This is in good agreement with analytical results of Sapiro and Tannenbaum [38]. On the other hand, without any grid redistribution we can see rapid merging of several points leading to degeneracy in the distance $|r_i|$ corresponding to discretization of the term $|\partial_u x|$ and subsequent collapse of computation. In Figure 2(b) one can see evolution until $t = 0.38$ just before numerical collapse of a solution.

In the figures below we have shown evolutions of the initial "∞-like" curve. In Figures 3(a) and 3(c) the tangential velocity preserving relative local length has been used, whereas in Figure 3(b) one sees that the computation without tangential redistribution cannot prevent vanishing of the term $|\partial_u x|$. In Figures 4(a) and 4(b) evolutions of general nonconvex curves are plotted.

In Figures 5(a) and 5(b) the affine invariant evolution of initial ellipse with half-axes ratio 3:1 is shown. In Figure 5(a) the exact blowup time $T_{\max} = 1.560$, while the numerically computed one is equal to 1.570 using time step $\tau = 0.001$ and 100 grid points for curve representation. The half-axes ratio as well as isoperimetric ratio were perfectly conserved during numerical evolution. Without any tangential velocity (i.e., $\alpha = 0$ and $\theta_2 = 1$), the numerical solution collapses, as should be obvious from Figure 5(b).

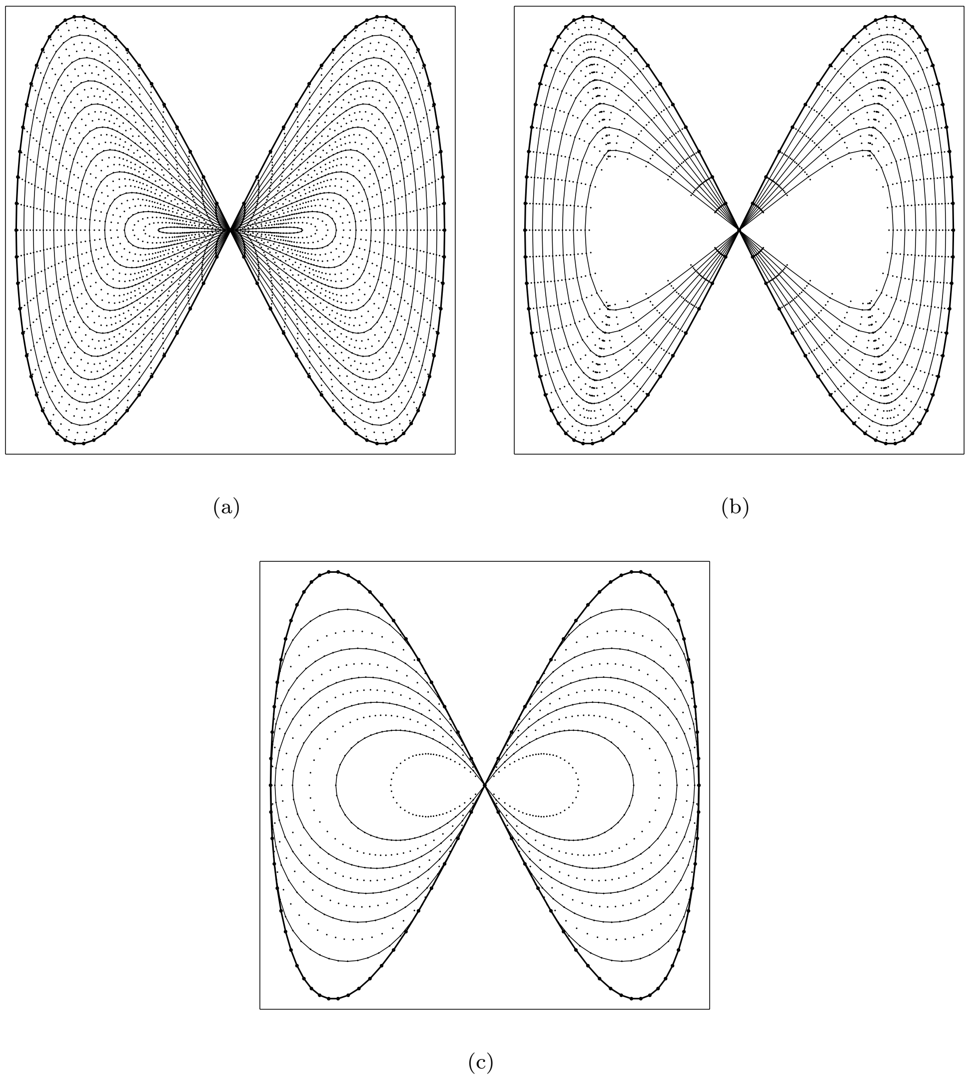
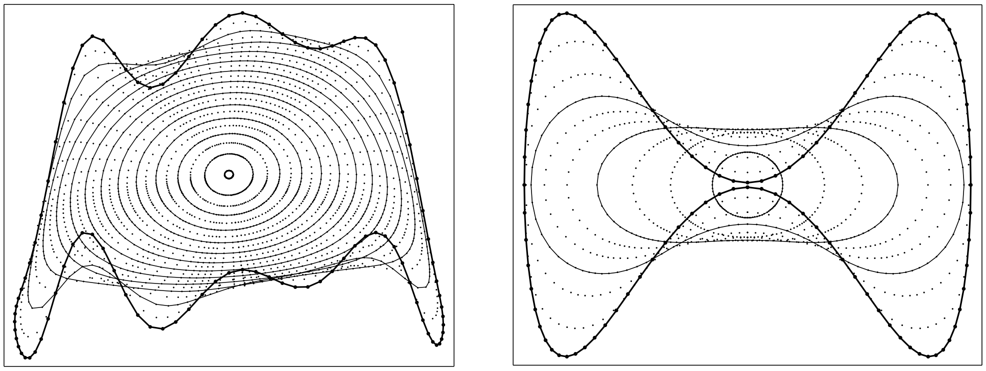


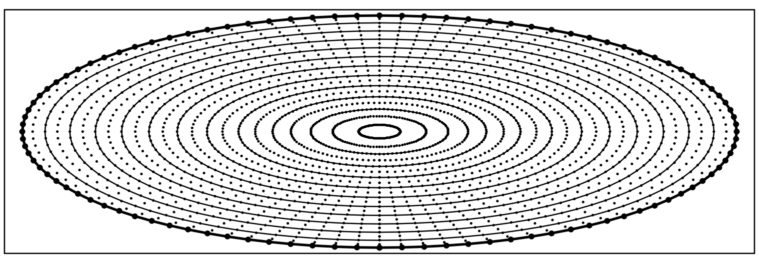
FIG. 3. $\beta(k) = k^{1/4}$. (a) *Evolution of "∞-like" curve using redistribution.* (b) *Evolution of "∞-like" curve without redistribution leading to merging of points.* (c) *Evolution of "∞-like" curve using tangential redistribution of points.*

In Figures 6 and 7 we present various computations including anisotropy in the model. For Figures 6(a)–6(d) we have chosen threefold anisotropy, while for Figures 7(a) and 7(b), a fourfold one. In Figure 6(a) we have computed linear anisotropic evolution of a unit circle by means of (7.1)–(7.3). In Figures 6(b) and 6(c) we have combined anisotropy with a nonlinear function of the curvature. In Figure 6(d) we have chosen the same initial curve and the velocity function as in Figure 6(c), but curves were computed without uniform grid redistribution. Curves are represented by 100 grid points and $\tau = 0.001$. In the first numerical experiment shown in Figure 6(a) the numerical blowup time $T_{\max} = 0.509$ (the exact one is 0.5). In this case the isoperimetric ratio tends to 1.048 and the curve approaches the Wulf shape for such

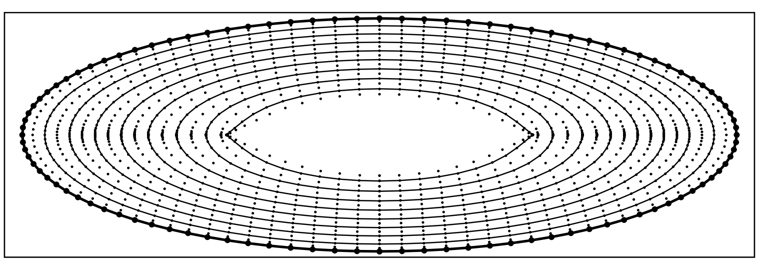


(a) (b)

FIG. 4. $\beta(k) = k^{1/2}$. *Evolution of general nonconvex curve using tangential redistribution of points.*



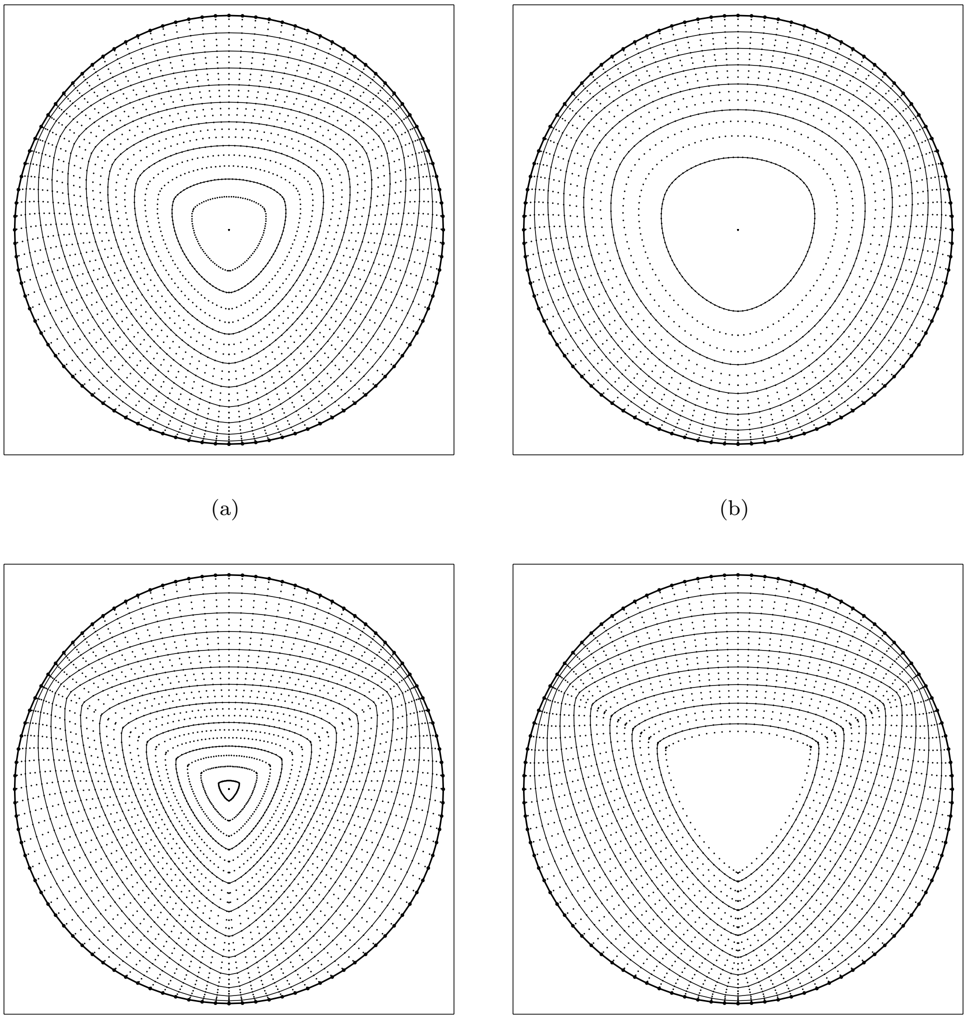
(a)



(b)

FIG. 5. (a) *Affine invariant motion of ellipse using tangential redistribution of points.* (b) *Computation using only normal component of velocity.*

an anisotropy function. In Figure 6(b) we chose $\beta(k) = k^m$, $m > 1$. The evolution is faster, numerical $T_{\max} = 0.373$ ($m = 2$), and the asymptotic isoperimetric ratio is 1.014. Taking $\beta(k) = k^m$, $m < 1$, the anisotropic evolution is slowed down, numerical $T_{\max} = 0.601$, $m = \frac{1}{2}$, the isoperimetric ratio tends to 1.13, and the asymptotical shape is more "sharp." In this example one sees that the initial uniform redistribu-



(a) (b) (c) (d)

FIG. 6. $\beta(k, \nu) = (1 - 7/9 \cos(3\nu))k$. (a)–(c) *Using redistribution.* (d) *Without redistribution.*

tion of grid points is not kept perfectly (in spite of results in Figure 6(b)). It very likely is caused by lack of well-conditioning of linear systems. This phenomenon is an objective of our future study. Further anisotropic experiments are presented in Figures 7(a)–7(c), where convergence to "oval square" is observed in both convex and nonconvex cases. The evolution of a nonconvex curve from Figure 7(c) is computed also for the case of the threefold anisotropy. Results are plotted in Figure 7(d). The last numerical experiment represents affine invariant evolution of a spiral. In Figure 8 we present several time moments of the motion until it is shrinking to a point.

9. Concluding remarks. In this paper we have studied the generalized mean curvature flow of planar curves. The normal velocity v of the flow is assumed to be a power-like function of the curvature k, and it may also depend on a spatial anisotropy

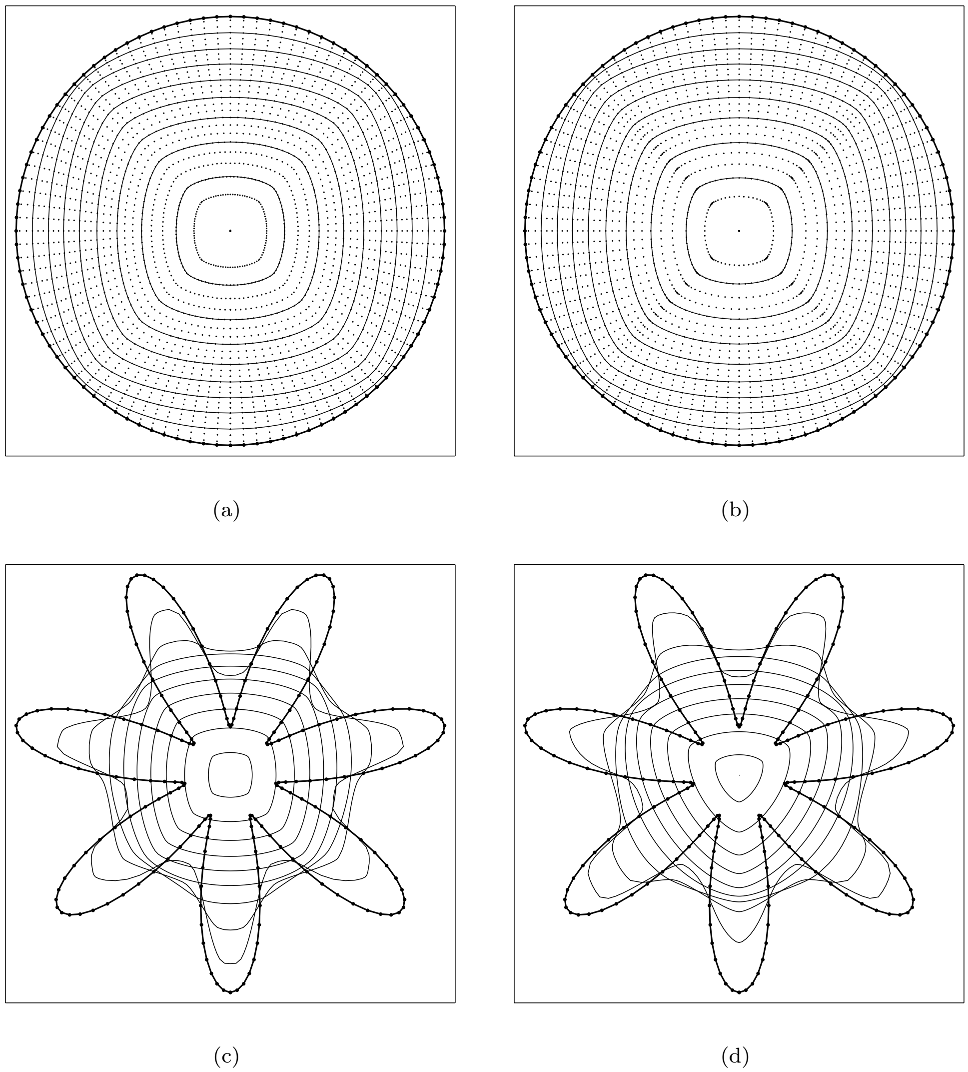


FIG. 7. $\beta = (1 - 0.8\cos(4(\nu - \pi/4)))k$. (a), (c), (d) *Using redistribution.* (b) *Without redistribution.*

γ, i.e., $v = \gamma k^m$, where $m > 0$. Our analysis covers both singular ($0 < m < 1$) and degenerate ($1 < m \leq 2$) cases. We followed the so-called direct approach. We have proposed and analyzed a governing intrinsic heat equation which is a parabolic equation for the position vector. This model is capable of describing both normal and tangential velocities of an evolving family of plane curves. We have also found that respect to choices of the tangential velocity numerical simulations may exhibit various instabilities. We overcome this difficulty by constructing a suitable tangential velocity functional yielding uniform redistribution of numerically computed grid points.

Acknowledgments. The authors are thankful to the anonymous referees for their valuable comments and for bringing to our attention the recent paper by Andrews.

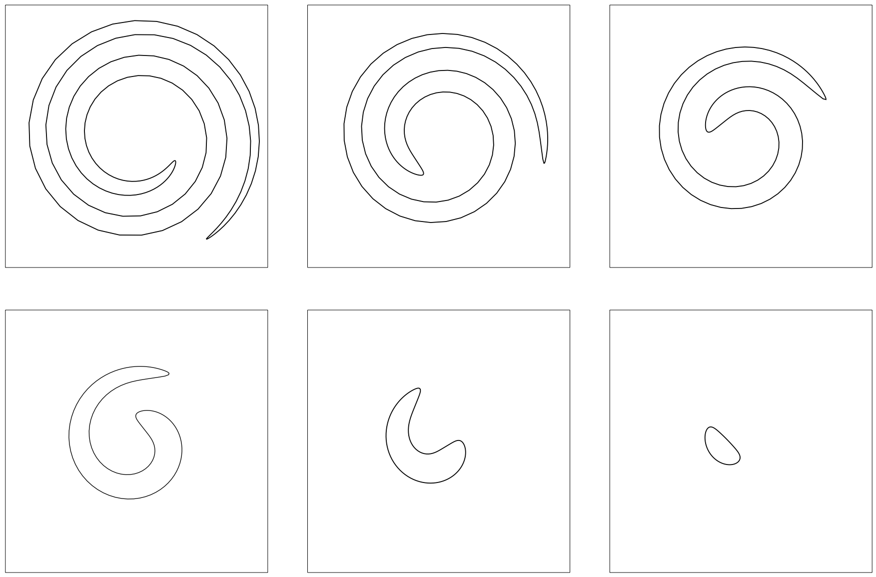


Fig. 8. *The sequence of evolving spirals for* $\beta(k,\nu) = k^{1/3}$ *using redistribution. The limiting curve is an ellipse rounded point.*

REFERENCES

[1] U. Abresch and J. Langer, *The normalized curve shortening flow and homothetic solutions*, J. Differential Geom., 23 (1986), pp. 175–196.

[2] L. Alvarez, F. Guichard, P. L. Lions, and J. M. Morel, *Axioms and fundamental equations of image processing*, Arch. Rational Mech. Anal., 123 (1993), pp. 200–257.

[3] L. Alvarez and J. M. Morel, *Formalization and computational aspects of image analysis*, Acta Numer., Cambridge University Press, Cambridge, UK, 1994, pp. 1–59.

[4] B. Andrews, *Evolving convex curves*, Calc. Var. Partial Differential Equations, 7 (1998), pp. 315–371.

[5] S. B. Angenent, *On the formation of singularities in the curve shortening flow*, J. Differential Geom., 33 (1991), pp. 601–633.

[6] S. B. Angenent, *Parabolic equations for curves on surfaces* I: *Curves with p-integrable curvature*, Ann. Math., 132 (1990), pp. 451–483.

[7] S. B. Angenent, *Parabolic equations for curves on surfaces* II: *Intersections, blow-up and generalized solutions*, Ann. Math., 133 (1991), pp. 171–215.

[8] S. B. Angenent, *Nonlinear analytic semiflows*, Proc. Roy. Soc. Edinburgh Sect. A, 115 (1990), pp. 91–107,

[9] S. B. Angenent and M. E. Gurtin, *Multiphase thermomechanics with an interfacial structure* 2. *Evolution of an isothermal interface*, Arch. Rational Mech. Anal., 108 (1989), pp. 323–391.

[10] S. B. Angenent and M. E. Gurtin, *Anisotropic motion of a phase interface. Well-posedness of the initial value problem and qualitative properties of the interface*, J. Reine Angew. Math., 446 (1994), pp. 1–47.

[11] S. B. Angenent and M. E. Gurtin, *General contact angle conditions with and without kinetics*, Quart. Appl. Math., 54 (1996), pp. 557–569.

[12] S. B. Angenent, G. Sapiro, and A. Tannenbaum, *On affine heat equation for non-convex curves*, J. Amer. Math. Soc., 11 (1998), pp. 601–634.

[13] M. Beneš and K. Mikula, *Simulations of anisotropic motion by mean curvature–comparison of phase field and sharp interface approaches*, Acta Math. Univ. Comenian. (N.S.), 67 (1998), pp. 17–42.

[14] G. Caginalp, *The dynamics of a conserved phase field system: Stefan-like, Hele-Shaw, and Cahn-Hilliard models as asymptotic limits*, IMA J. Appl. Math., 44 (1990), pp. 77–94.

[15] V. Caselles, R. Kimmel, G. Sapiro, and C. Sbert, *Minimal surfaces based object segmentation*, IEEE Trans. Pattern Anal. Mach. Intell., 19 (1997), pp. 394–398.

[16] K. Deckelnick, *Weak solutions of the curve shortening flow*, Calc. Var. Partial Differential Equations, 5 (1997), pp. 489–510.
[17] G. Dziuk, *Algorithm for evolutionary surfaces*, Numer. Math., 58 (1991), pp. 603–611.
[18] G. Dziuk, *Convergence of a semi discrete scheme for the curve shortening flow*, Math. Models Methods Appl. Sci., 4 (1994), pp. 589–606.
[19] G. Dziuk, *Discrete anisotropic curve shortening flow*, SIAM J. Numer. Anal., 36 (1999), pp. 1808–1830.
[20] C. M. Elliott, M. Paolini, and R. Schätzle, *Interface estimates for the fully anisotropic Allen–Cahn equation and anisotropic mean curvature flow*, Math. Models Methods Appl. Sci., 6 (1996), pp. 1103–1118.
[21] M. Gage and R. S. Hamilton, *The heat equation shrinking convex plane curves*, J. Differential Geom., 23 (1986), pp. 285–314.
[22] P. M. Girão, *Convergence of a crystalline algorithm for the motion of a simple closed convex curve by weighted curvature*, SIAM J. Numer. Anal., 32 (1995), pp. 886–899.
[23] P. M. Girão and R. V. Kohn, *Convergence of a crystalline algorithm for the heat equation in one dimension and for the motion of a graph by weighted curvature*, Numer. Math., 67 (1994), pp. 41–70.
[24] M. Grayson, *The heat equation shrinks embedded plane curves to round points*, J. Differential Geom., 26 (1987), pp. 285–314.
[25] M. Gurtin, *Thermomechanics of Evolving Phase Boundaries in the Plane*, Clarendon Press, Oxford, UK, 1993.
[26] A. Handlovičová, K. Mikula, and A. Sarti, *Numerical solution of parabolic equations related to level set formulation of mean curvature flow*, Comput. Visualization Sci., 1 (1998), pp. 179–182.
[27] M. Kass, A. Witkin, and D. Terzopoulos, *Constraints on deformable models: Recovering 3D shape and nongrid motion*, Artificial Intelligence, 36 (1988), pp. 91–123.
[28] M. Kimura, *Accurate numerical scheme for the flow by curvature* Appl. Math. Letters, 7 (1994), pp. 69–73.
[29] M. Kimura, *Numerical analysis for moving boundary problems using the boundary tracking method*, Japan J. Indust. Appl. Math., 14 (1997), pp. 373–398.
[30] R. Malladi, J. Sethian, and B. Vemuri, *Shape modeling with front propagation: a level set approach*, IEEE Trans. Pattern Anal. Mach. Intell., 17 (1995), pp. 158–174.
[31] K. Mikula and J. Kačur, *Evolution of convex plane curves describing anisotropic motions of phase interfaces*, SIAM J. Sci. Comput., 17 (1996), pp. 1302–1327.
[32] K. Mikula, *Solution of nonlinear curvature driven evolution of plane convex curves*, Appl. Numer. Math., 21 (1997) pp. 1–14.
[33] K. Mikula and D. Ševčovič, *Solution of nonlinearly curvature driven evolution of plane curves*, Appl. Numer. Math., 31 (1999), pp. 191–207.
[34] L. Moissan, *Affine plane curve evolution: A fully consistent scheme*, IEEE Trans. Image Process., 7 (1998), pp. 411–420.
[35] R. Nochetto, M. Paolini, and C. Verdi, *Sharp error analysis for curvature dependent evolving fronts*, Math. Models Methods Appl. Sci., 3 (1993), pp. 711–723.
[36] S. Osher and J. Sethian, *Fronts propagating with curvature dependent speed: Algorithm based on Hamilton-Jacobi formulation*, J. Comput. Phys., 79 (1988), pp. 12–49.
[37] S. Patankar, *Numerical Heat Transfer and Fluid Flow*, Hemisphere, New York, 1980.
[38] G. Sapiro and A. Tannenbaum, *On affine plane curve evolution*, J. Funct. Anal., 119 (1994), pp. 79–120.
[39] A. Schmidt, *Computation of three dimensional dendrites with finite elements*, J. Comput. Phys., 125 (1996), pp. 293–312.
[40] A. Schmidt, *Approximation of crystalline dendrite growth in two space dimensions*, Acta Math. Univ. Comenian. (N.S.), 67 (1998), pp. 57–68.
[41] J. Sethian, *Numerical algorithm for propagating interfaces: Hamilton-Jacobi equations and conservation laws*, J. Differential. Geom., 31 (1990), pp. 131–161.
[42] J. Sethian, *Level Set Methods. Evolving Interfaces in Geometry, Fluid Mechanics, Computer Vision, and Material Science*, Cambridge University Press, Cambridge, UK, 1996.
[43] J. Sethian, *Adaptive fast marching and level set methods for propagating interfaces*, Acta Math. Univ. Comenian. (N.S.), 67 (1998), pp. 3–16.
[44] T. Ushijima and S. Yazaki, *Convergence of a crystalline algorithm for the motion of a closed convex curve by a power of curvature* $V = K^{\alpha}$, SIAM J. Numer. Anal., 37 (2000), pp. 500–522.

Príloha 6.2.2

Reprint práce:

K. Mikula, D. Ševčovič: *Solution of nonlinearly curvature driven evolution of plane curves.* Applied Numerical Mathematics, 31 (1999), 191-207.

Applied Numerical Mathematics 31 (1999) 191–207
APPLIED NUMERICAL MATHEMATICS
www.elsevier.nl/locate/apnum

Solution of nonlinearly curvature driven evolution of plane curves

Karol Mikula [a,*], Daniel Ševčovič [b,1]

[a] *Department of Mathematics, Slovak Technical University, Radlinského 11, 813 68 Bratislava, Slovak Republic*
[b] *Institute of Applied Mathematics, Faculty of Mathematics and Physics, Comenius University, 842 15 Bratislava, Slovak Republic*

Abstract

The evolution of plane curves obeying the equation $v = \beta(k)$, where v is normal velocity and k curvature of the curve is studied. Morphological image and shape multiscale analysis of Alvarez, Guichard, Lions and Morel and affine invariant scale space of curves introduced by Sapiro and Tannenbaum as well as isotropic motions of plane phase interfaces studied by Angenent and Gurtin are included in the model. We introduce and analyze a numerical scheme for solving the governing equation and present numerical experiments. © 1999 Elsevier Science B.V. and IMACS. All rights reserved.

Keywords: Curve evolution; Image and shape multiscale analysis; Phase interface; Nonlinear degenerate parabolic equations; Numerical solution

0. Introduction

The goal of this paper is to investigate the evolution of closed smooth plane curves $\Gamma : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^2$. By contrast to the curve shortening flow studied in [1,7,14,17,18], we assume that the normal velocity of the curve Γ at its point x is a nonlinear function of the curvature k of Γ at x. More precisely, we study the evolution of plane curves obeying the geometrical equation

$$v = \beta(k), \tag{0.1}$$

where v is the normal velocity of evolving curves and $\beta : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a smooth function. As a typical example one can consider a function $\beta(k) = k^m$, where $m > 0$. Throughout this paper we adopt a convention according to which the curvature k of a curve Γ is always nonnegative whereas the normal vector N may change its orientation with respect to the tangent vector T.

* Corresponding author. E-mail: mikula@vox.svf.stuba.sk.
1 E-mail: sevcovic@fmph.uniba.sk.

PII: S0168-9274(98)00130-5

The geometrical equations like (0.1) are capable of describing various phenomena in physics, material sciences, computer vision, robotics and artificial intelligence. There are two main fields, in which the evolution of a plane curve plays an important role: (a) the multiscale analysis of images and shapes closely related to signal smoothing, edge detection and image representation (see, e.g., [3,19,27]); (b) the Stefan problem with surface tension and related interface motion models (see, e.g., [8,24,29]).

In the context of image processing, so-called *morphological image multiscale analysis* is widely used. This analysis is represented by a viscosity solution [13,16,12] of the following nonlinear degenerate parabolic equation in a two-dimensional rectangular domain

$$v_t = |\nabla v| g\big(\mathrm{div}\big(\nabla v/|\nabla v|\big)\big), \tag{0.2}$$

where g is a nondecreasing function [2,3,21]. It is a generalization of the so-called *level set equation* [25, 30] used for the classical mean curvature flow. The initial condition for (0.2) corresponds to the processed image and the solution v to its *scaling* version. In many situations, silhouettes (boundaries of distinguished shapes) in the image correspond to level lines of v. The morphological image multiscale analysis then leads to the silhouettes motion obeying the equation of the form (0.1). In the vision theory, *affine invariant scale space* has special conceptual and practical importance [2,28]. It is natural generalization of the linear curve shortening flow, and is given by (0.1) with $\beta(k) = k^{1/3}$. The *active contours* models (*snakes*) and *curvature-based multiscale shape representation*, related to edge detection, image segmentation and recognition, are other important fields in which geometrical equations are widely used [20,22].

In the context of multiphase thermomechanics with interfacial structure the plane curve evolution is a natural model for the *motion of phase interfaces*. The isotropic version of the theory of Angenent and Gurtin [8,9] has the form of equation (0.1). In this case, the nonlinearity expresses the dependence of the kinetic coefficient on the normal velocity (see [8, (4.11)]). For example, if the dependence is linear then we have $\beta(k) = k^{1/2}$. Under additional assumptions, a model corresponding to classical (i.e., anisotropic) curve shortening flow is derived and studied in [8, (4.13)]; for numerical approximation in this case we refer to [15,24]. If β is a strictly increasing function equation (0.1) has been studied in [5,6] as a model of curve evolution on arbitrary surfaces.

In the present paper, we suggest a *new computational method* for solving geometrical equation (0.1). The aim is to represent equation (0.1) by a so-called intrinsic heat equation governing the evolution of plane curves with the normal velocity obeying equation (0.1). Such a representation of the curve evolution is found for a general function β using an appropriate curve parameterization. In Section 1 we make use of the “Eulerian transformation” of the intrinsic heat equation (1.3) into a degenerate evolution partial differential equation (1.10) with spatial variable being independent on time and varying on a fixed interval. This equation is a generalization of the corresponding equation studied by Dziuk in [14]. The “intrinsic property” of the governing equation (1.10) causes that the spatial parameterization step is not involved in the approximation scheme and therefore only the spatial position of points of a curve Γ and the curvature of Γ play the role in the discretization scheme suggested in Section 4. In other words, given a discrete polygonal curve one can compute its evolution without knowing the normalized parameterization of the initial curve. The behavior of homothetic solutions is studied in Section 2. In Section 3 we prove some a-priori estimates of a smooth solution, which, in particular, imply the *curve shortening* property of the governing equation (1.10) which ensures, in some way, the stability of the method. The same property is proven for the time discretization scheme (4.1) in Section 4. In Section 5 the proposed numerical scheme is carefully tested by various examples of the nonlinear curvature driven evolution (0.1). We present a comparison of the numerical results with the exact homothetic solutions. In this section we also perform a comparison with previous results obtained by a conceptually different method introduced in [23]. It is worthwhile noting that the method of [23] can be applied only for the evolution of convex curves whereas the new method suggested in this paper is capable of capturing the nonlinear evolution of both convex as well as nonconvex curves.

Notice that coefficients in Eq. (1.10) may develop singularities either due to vanishing of x_u, or, by contrast to the case $\beta(k) = k$ studied in [14], also due to the presence of the extremal values of the curvature $k = 0$ or $k = \infty$. Moreover, Eq. (1.10) is written in a non-divergence form. This feature make the analysis particularly difficult. Therefore the careful analysis of the convergence as well as error estimates of the suggested approximation scheme are still open problems.

1. Governing equations

1.1. Parameterization of a plane curve

Let Γ be a smooth curve in the plane $\mathbb{R}^2$. By this we mean that Γ can be parameterized by a C^2 smooth function $x : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^2$ such that

$$\Gamma = \{x(u),\ u \in [0,1]\}. \tag{1.1}$$

We will henceforth write $\Gamma = \mathrm{Image}(x)$. To describe the time evolution $\{\Gamma^t\}$, $t \in [0, T_{\max})$, of a curve Γ^0 we adopt the notation

$$\Gamma^t = \{x(u,t),\ u \in [0,1]\}, \quad t \in [0, T_{\max}),$$

where $x \in C^2(\mathbb{R}/\mathbb{Z} \times [0, T_{\max}), \mathbb{R}^2)$. Obviously, any plane curve Γ admits various other parameterizations. Henceforth, the parameter s will always refer to the arc-length parameter of a plane curve Γ.

Example 1.1. Consider another parameterization s_* of a curve $\Gamma = \mathrm{Image}(x)$. Then it is easy to verify that $\det[\partial x/\partial s_*, \partial^2 x/\partial s_*^2] = \phi'(s)^{-3} \det[\partial x/\partial s, \partial^2 x/\partial s^2]$, where $s_* = \phi(s)$. As $k = |\det[\partial x/\partial s, \partial^2 x/\partial s^2]|$ we have

$$\left|\det[\partial x/\partial s_*, \partial^2 x/\partial s_*^2]\right| = 1 \tag{1.2}$$

provided that the new parameterization $s_* = \phi(s)$ has the property $\mathrm{d}s_* = \vartheta(s)\,\mathrm{d}s$ where $\vartheta = k^{1/3}$. A parameterization of a plane curve satisfying Eq. (1.2) is referred to as *the affine arc-length* (see [28]).

Throughout the paper we will use both notations x_ξ as well as $\partial x/\partial \xi$ in order to denote the partial derivative of x with respect to a variable ξ.

1.2. Intrinsic heat equation

The aim of this paper is to investigate the evolution of plane curves $\{\Gamma^t\}$ undergoing the intrinsic heat equation

$$\frac{\partial x}{\partial t} = \frac{\partial^2 x}{\partial s_*^2}, \tag{1.3}$$

where s_* is a new parameterization of a curve Γ^t obeying the law

$$\mathrm{d}s_* = \vartheta(s)\,\mathrm{d}s.$$

We will seek for a function ϑ such that the normal component of the curve-flow velocity v satisfies the equation $v = \beta(k)$. To this end, let us transform Eq. (1.3) using the arc-length parameterization. We obtain

$$\frac{\partial x}{\partial t} = \frac{1}{\vartheta(s)}\frac{\partial}{\partial s}\left(\frac{1}{\vartheta(s)}\frac{\partial x}{\partial s}\right) = \frac{1}{\vartheta^2(s)}kN - \frac{\vartheta'(s)}{\vartheta^3(s)}T, \tag{1.4}$$

where T is the unit tangent vector, $T = x_s$ and N is the unit normal vector satisfying Frenet's formula $T_s = kN$. Hence the normal velocity $v = (x_t, N)$ fulfills Eq. (0.1) iff

$$\vartheta = k^{1/2}\beta(k)^{-1/2}. \tag{1.5}$$

If the function β has the form $\beta(k) = k^{1/\alpha}$, $\alpha > 0$, we obtain $\vartheta = k^{(\alpha-1)/(2\alpha)}$. If $\alpha = 1$ then $x_t = kN$. On the other hand, if $\alpha = 3$ we have $\vartheta = k^{1/3}$ and $x_t = k^{1/3}N - \frac{1}{3}(k_s/k^{5/3})T$. Taking into account Example 1.1 we may conclude that for the affine arc-length parameterization satisfying (1.2) the normal velocity v of a curve Γ with the curvature k at a point x satisfies $v = k^{1/3}$ (see also [28]).

1.3. *Eulerian form of the governing equation*

It is worthwhile noting that the parameterization s_* occurring in (1.3) may depend on time t and its initial position u at $t = 0$. This is because of the requirement that the normal velocity should depend on the curvature only as it was prescribed by Eq. (0.1). Thus the evolution of the new parameterization $s_* = s_*(u,t)$ as well as the arc-length parameterization $s = s(u,t)$ depend on the solution x itself. This feature is similar, in spirit, to the transformation between Lagrangian (material) and Eulerian (spatial) coordinates in the classical mechanics. This is why the intrinsic heat equation (1.3) is not convenient when treating evolution of plane curves numerically. To overcome this difficulty, we rewrite (1.3) into a form involving a parameterization u independent of the time variable t and varying on the fixed interval $[0,1]$.

Let $u \in [0,1]$ be a time independent parameterization of a curve Γ. Then the arc-length parameterization s of Γ is related to u by $\mathrm{d}s = |x_u|\,\mathrm{d}u$. Furthermore, as $k = |x_{ss}|$ and

$$\frac{\partial^2 x}{\partial s^2} = \frac{1}{|x_u|}\frac{\partial}{\partial u}\left(\frac{1}{|x_u|}\frac{\partial x}{\partial u}\right) = \frac{1}{|x_u|^2}\left(x_{uu} - \frac{1}{|x_u|^2}(x_u, x_{uu})x_u\right)$$

we have $k = k(x_u, x_{uu})$, where

$$k(p,q) = |p|^{-3}\left(|p|^2|q|^2 - (p,q)^2\right)^{1/2}, \quad p, q \in \mathbb{R}^2, \tag{1.6}$$

where $(\cdot,\cdot)$ denotes the Euclidean scalar product in $\mathbb{R}^2$ and the corresponding norm is denoted by $|\cdot|$. Here and after we will assume that a function

(B) $\beta : [0,\infty) \to [0,\infty)$ is C^1-smooth on $(0,\infty)$ and is continuous on $[0,\infty)$, $\beta(k) > 0$ for $k > 0$.

Let us consider a new parameterization s_* satisfying $\mathrm{d}s_* = \vartheta(s)\,\mathrm{d}s$, where $\vartheta(s)$ is defined as in (1.5), i.e., $\vartheta = k^{1/2}\beta(k)^{-1/2}$. To facilitate the notation, let us define scalar valued functions $\theta_\beta, G_\beta : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}_0^+$,

$$\theta_\beta(p,q) = k(p,q)^{1/2}\beta\big(k(p,q)\big)^{-1/2}, \qquad G_\beta(p,q) = |p|\theta_\beta(p,q). \tag{1.7}$$

It is easy to verify that $k(p,q) = |p|^{-3}|\det[p,q]|$ and this is why the functions G_β and k have the following scaling and affine properties:

$$G_\beta(p, ap+q) = G_\beta(p,q), \qquad k(p, ap+q) = k(p,q), \qquad k(ap, bq) = a^{-2} b k(p,q) \tag{1.8}$$

for any $p, q \in \mathbb{R}^2$, $a, b \in \mathbb{R}$.

If $\beta(k) = k^{1/\alpha}$, $\alpha > 0$, then

$$G_\beta(p,q) = |p|^{(3-\alpha)/(2\alpha)}\big(|p|^2|q|^2 - (p,q)^2\big)^{(\alpha-1)/(4\alpha)},$$

and, in addition to (1.8), one has

$$G_\beta(ap, bq) = |a|^{1/\alpha}|b|^{(\alpha-1)/(2\alpha)} G_\beta(p,q) \quad \text{for any } p, q \in \mathbb{R}^2,\ a, b \in \mathbb{R}. \tag{1.9}$$

Now we are in a position to rewrite the intrinsic heat equation (1.3) into so-called “Eulerian form” with a parameterization u varying on fixed interval [0, 1] as follows:

$$\frac{\partial x}{\partial t} = \frac{1}{G_\beta(x_u, x_{uu})}\frac{\partial}{\partial u}\left(\frac{1}{G_\beta(x_u, x_{uu})}\frac{\partial x}{\partial u}\right), \quad (u,t) \in [0,1] \times [0, T_{\max}). \tag{1.10}$$

The fully nonlinear system of PDEs (1.10) is subject to the initial condition $x(u,0) = x^0(u)$, $u \in [0,1]$, and periodic boundary conditions at $u = 0, 1$, i.e., $x \in C^{2,1}(\mathbb{R}/\mathbb{Z} \times [0, T_{\max}), \mathbb{R}^2)$.

2. Special solutions

Throughout this section we will restrict ourselves to the case when $\beta(k) = k^{1/\alpha}$, $\alpha > 0$. We will seek for a solution $x(u,t)$ of (1.10) having the form

$$x(u,t) = \phi(t)\tilde{x}(u). \tag{2.1}$$

Suppose that $\tilde{x} \in C^2(\mathbb{R}/\mathbb{Z}; \mathbb{R}^2)$, $\tilde{x} \neq 0$, is a solution of the nonlinear eigenvalue problem

$$-\frac{1}{G_\beta(\tilde{x}_u, \tilde{x}_{uu})}\frac{\partial}{\partial u}\left(\frac{1}{G_\beta(\tilde{x}_u, \tilde{x}_{uu})}\frac{\partial \tilde{x}}{\partial u}\right) = \lambda \tilde{x}, \quad u \in [0,1]. \tag{2.2}$$

Clearly, if $\tilde{x} \neq 0$ is a solution of (2.2) then by taking the scalar product in $(L^2(0,1))^2$ of (2.2) with $G_\beta \tilde{x}$ we obtain

$$\lambda = \frac{\int_0^1 (1/G_\beta)|\tilde{x}_u|^2}{\int_0^1 G_\beta |\tilde{x}|^2} > 0.$$

Let ϕ be a solution of the initial value problem

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = -\lambda \phi^{-1/\alpha}, \qquad \phi(0) = \phi_0 > 0. \tag{2.3}$$

Then it should be obvious from the scaling property (1.9) that the function $x(u,t) = \phi(t)\tilde{x}(u)$ is a solution of (1.10) satisfying the initial condition $x(u,0) = \phi_0 \tilde{x}(u)$, $u \in [0,1]$. The explicit form of a solution of (2.3) is given by

$$\phi(t) = \left[\phi_0^{(1+\alpha)/\alpha} - \frac{1+\alpha}{\alpha}\lambda t\right]^{\alpha/(\alpha+1)}. \tag{2.4}$$

The life-span of a homothetic solution of the form (2.1) is the interval $[0, T_{\max})$, where

$$T_{\max} = \frac{\alpha\phi_0^{(1+\alpha)/\alpha}}{(1+\alpha)\lambda}. \tag{2.5}$$

Example 2.1. An ellipse $\widetilde{\Gamma} = \{(a\cos(2\pi u), b\sin(2\pi u))^{\mathrm{T}},\ u \in [0,1]\}$ is a solution of (2.2). Then

$$G_\beta(\tilde{x}_u, \tilde{x}_{uu}) = 2\pi(ab)^{(\alpha-1)/(2\alpha)}\big[a^2\sin^2(2\pi u) + b^2\cos^2(2\pi u)\big]^{(3-\alpha)/(4\alpha)}.$$

In the case $a = b$ (i.e., $\widetilde{\Gamma}$ is a circle) and $\alpha > 0$ we have $G_\beta = 2\pi a^{(\alpha+1)/(2\alpha)}$ and so $\tilde{x}$ is a solution of (2.2) iff $\lambda = a^{-(\alpha+1)/\alpha}$. If we choose the initial condition $\phi_0 = 1$ (i.e., $\Gamma^0 = \widetilde{\Gamma}$) then the life-span of a solution is $T_{\max} = (\alpha/(\alpha+1))a^{(\alpha+1)/\alpha}$. On the other hand, if $a \neq b$ and $\alpha = 3$ we obtain $G_\beta = 2\pi(ab)^{1/3}$ and so $\lambda = (ab)^{-2/3}$. Then $T_{\max} = \frac{3}{4}(ab)^{2/3}$.

By using the phase-space analysis argument, one can show that the only solution of Eq. (2.1) with normalized $\lambda = 1$ is either a circle for $0 < \alpha \neq 3$, or an ellipse for $\alpha = 3$. Thus a function $x \in C^{2,1}(\mathbb{R}/\mathbb{Z} \times [0,T), \mathbb{R}^2)$ of the form $x(u,t) = \phi(t)\tilde{x}(u)$ is a solution of (1.10) iff the family of its images $\Gamma^t = \mathrm{Image}(x(\cdot,t))$, $t \in [0,T)$, are either homothetically shrinking circles for $0 < \alpha \neq 3$ or homothetically shrinking ellipses for the case $\alpha = 3$. This is consistent with the result obtained by Sapiro and Tannenbaum [28] for the case $\beta(k) = k^{1/3}$.

3. A-priori estimates of solutions

The goal of this section is to derive a-priori estimates of solutions of the intrinsic heat equation (1.10). We will provide these estimates for the original equation (1.10) as well as for the modified equation

$$\frac{\partial x}{\partial t} = \frac{1}{G_{\beta,\varepsilon}(x_u, x_{uu})}\frac{\partial}{\partial u}\left(\frac{1}{G_{\beta,\varepsilon}(x_u, x_{uu})}\frac{\partial x}{\partial u}\right), \tag{3.1}$$

where $G_{\beta,\varepsilon}$, $\varepsilon \geqslant 0$, is a modification of G_β such that

(E)
$G_{\beta,\varepsilon} : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ is C^1-smooth, $0 < G_{\beta,\varepsilon}(p,q) < \infty$,
$G_{\beta,\varepsilon}(p, ap+q) = G_{\beta,\varepsilon}(p,q)$, for any $p, q \in \mathbb{R}^2$, $a, b \in \mathbb{R}$ and $\varepsilon > 0$, and
$G_{\beta,\varepsilon}(p,q) \to G_{\beta,0}(p,q) = G_\beta(p,q)$ as $\varepsilon \to 0^+$ for any $p, q \in \mathbb{R}^2$.

Definition 3.1. By a *nondegenerate evolving curve* we mean a function $x \in C^{2,1}(\mathbb{R}/\mathbb{Z} \times [0,T), \mathbb{R}^2)$ such that $0 < G_{\beta,\varepsilon}(x_u(u,t), x_{uu}(u,t)) < \infty$ for any $u \in \mathbb{R}$, $t \in [0,T)$. By a *nondegenerate classical solution* of Eq. (3.1) we mean a nondegenerate evolving curve $x \in C^{2,1}(\mathbb{R}/\mathbb{Z} \times [0,T), \mathbb{R}^2)$ satisfying (3.1).

Proposition 3.1. *Let x be a nondegenerate classical solution of Eq.* (3.1), $\varepsilon \geqslant 0$. *Then, for each $t \in (0, T_{\max})$,*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 |x_u(\cdot,t)| + \int_0^1 \omega_\varepsilon^2 k\beta(k)|x_u(\cdot,t)| = 0, \tag{3.2}$$

where $k = k(x_u, x_{uu})$ and $\omega_\varepsilon = G_\beta(x_u, x_{uu})/G_{\beta,\varepsilon}(x_u, x_{uu})$ for $\varepsilon > 0$, $\omega_0 = 1$.

Since $\mathrm{d}s = |x_u|\,\mathrm{d}u$ we have

$$\int_{\Gamma^t} \omega_\varepsilon^2\, k\beta(k)\,\mathrm{d}s = \int_0^1 \omega_\varepsilon^2\, k\beta(k)|x_u|\,\mathrm{d}u.$$

Here k stands for the curvature of Γ^t at a point $x \in \Gamma^t$. Therefore, Eq. (3.2) can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Length}\,(\Gamma^t) + \int_{\Gamma^t} \omega_\varepsilon^2 k\beta(k)\,\mathrm{d}s = 0.$$

Corollary 3.2. *Let $\Gamma^t,\ t \in [0, T_{\max})$, where $\Gamma^t = \mathrm{Image}(x(\cdot, t))$ be a flow of plane curves where x is a nondegenerate classical solution of (3.1). Then $(\mathrm{d}/\mathrm{d}t)|\Gamma^t| \leqslant 0$. In other words, the length $|\Gamma^t|$ of the curve Γ^t decreases along the time, i.e., $\{\Gamma^t\},\ t \in [0, T_{\max})$, is a curve shortening flow.*

Proof of Proposition 3.1. Denote $k = k(x_u, x_{uu})$, $G = G_{\beta,\varepsilon}(x_u, x_{uu})$ and $\theta = |x_u|^{-1}G$. Then

$$\begin{aligned}
|x_u|_t &= \frac{1}{|x_u|}(x_{ut}, x_u) = \theta\left(x_{ut}, \frac{x_u}{G}\right) = \theta\left\{\frac{\mathrm{d}}{\mathrm{d}u}\left(x_t, \frac{x_u}{G}\right) - \left(x_t, \frac{\mathrm{d}}{\mathrm{d}u}\frac{x_u}{G}\right)\right\} \\
&= \theta\left\{\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{1}{G}\frac{\mathrm{d}}{\mathrm{d}u}\frac{x_u}{G}, \frac{x_u}{G}\right) - (x_t, Gx_t)\right\} = -\theta^2|x_u||x_t|^2 + \theta\frac{\mathrm{d}}{\mathrm{d}u}\left\{\left(\frac{1}{\theta^2|x_u|}\frac{\mathrm{d}}{\mathrm{d}u}\frac{x_u}{\theta|x_u|}, \frac{x_u}{|x_u|}\right)\right\} \\
&= -\theta^2|x_u||x_t|^2 + \theta\frac{\mathrm{d}}{\mathrm{d}u}\left\{\frac{-\theta_u}{\theta^4|x_u|}\left(\frac{x_u}{|x_u|}, \frac{x_u}{|x_u|}\right) + \frac{1}{\theta^3|x_u|}\left(\frac{\mathrm{d}}{\mathrm{d}u}\frac{x_u}{|x_u|}, \frac{x_u}{|x_u|}\right)\right\} \\
&= -\theta^2|x_u||x_t|^2 - \theta\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{\theta_u}{\theta^4|x_u|}\right)
\end{aligned} \tag{3.3}$$

because

$$\left(\frac{\mathrm{d}}{\mathrm{d}u}\frac{x_u}{|x_u|}, \frac{x_u}{|x_u|}\right) = 0.$$

With regard to (1.6) we have $|x_u|^2|x_{uu}|^2 - (x_u, x_{uu})^2 = k^2|x_u|^6$. Therefore,

$$\begin{aligned}
|x_t|^2 &= \frac{1}{\theta^2|x_u|^2}\left|\frac{\mathrm{d}}{\mathrm{d}u}\frac{x_u}{\theta|x_u|}\right|^2 = \frac{1}{\theta^6|x_u|^6}\left|\theta|x_u|x_{uu} - \theta_u|x_u|x_u - \theta\frac{(x_u, x_{uu})}{|x_u|}x_u\right|^2 \\
&= \frac{1}{\theta^6|x_u|^6}\left\{\theta^2\left[|x_{uu}|^2|x_u|^2 - (x_u, x_{uu})^2\right] + \theta_u^2|x_u|^4\right\} = k^2\theta^{-4} + \frac{\theta_u^2}{\theta^6|x_u|^2}.
\end{aligned} \tag{3.4}$$

Taking into account (1.7), (3.3) and (3.4) we obtain

$$|x_u|_t = -|x_u|k^2\theta^{-2} - \frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{\theta_u}{\theta^3|x_u|}\right).$$

Finally, as

$$k^2\theta^{-2} = k^2|x_u|^2G_{\beta,\varepsilon}^{-2} = \frac{k|x_u|^2}{\beta(k)}G_{\beta,\varepsilon}^{-2}k\beta(k) = G_\beta^2 G_{\beta,\varepsilon}^{-2}k\beta(k) = \omega_\varepsilon^2 k\beta(k)$$

we conclude that

$$|x_u|_t = -\omega_\varepsilon^2\,|x_u|\,k\,\beta(k) - \frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{\theta_u}{\theta^3|x_u|}\right). \tag{3.5}$$

Since both $x(u,t)$ as well as $\theta(u,t)$ are 1-periodic functions in u Eq. (3.2) follows from (3.5) by integrating over the interval [0, 1]. □

4. Numerical scheme

In this section we present a time semi-discretization scheme for solving Eq. (3.1). Let $[0,T]$ be an interval and let $\tau = T/n,\ n \in \mathbb{N}$, denote the time discretization step. By $x^i,\ i = 0,1,\dots,n$, we denote the approximation of a true solution of (3.1) at time $t = i\tau$, i.e., $x^i(\cdot) = x(\cdot, i\tau)$. Let $\varepsilon > 0$ be fixed. The idea of the construction of a time discretization scheme is based on approximation of the intrinsic heat equation (1.3) by the backward Euler method

$$\frac{x^i - x^{i-1}}{\tau} = \frac{\partial^2 x^i}{ds_*^2}, \quad i = 1,2,\dots,n,$$

where the parameterization s_* is computed from the previous time step x^{i-1}. The "Eulerian form" of the above scheme reads as follows:

$$x^i - \frac{\tau}{g^{i-1}}\frac{\partial}{\partial u}\left(\frac{1}{g^{i-1}}\frac{\partial x^i}{\partial u}\right) = x^{i-1}, \quad i = 1,2,\dots,n, \tag{4.1}$$

where $g^{i-1} = G_{\beta,\varepsilon}(x_u^{i-1}, x_{uu}^{i-1})$ and x^0 is the initial condition.

In what follows we will investigate the discretization scheme (4.1). We will prove the existence of a sequence $x^i,\ i = 1,2,\dots,n$, as well as we will show that such a discretization of the governing equation (3.1) generates the curve shortening discrete semiflow.

Lemma 4.1. *Suppose that $g \in C^1(\mathbb{R}/\mathbb{Z};\mathbb{R}),\ g > 0,\ \tau > 0$, and $\bar{x} \in C(\mathbb{R}/\mathbb{Z};\mathbb{R}^2)$. Then there exists a unique solution $x \in C^2(\mathbb{R}/\mathbb{Z};\mathbb{R}^2)$ of the equation*

$$x - \frac{\tau}{g}\frac{\partial}{\partial u}\left(\frac{1}{g}\frac{\partial x}{\partial u}\right) = \bar{x}. \tag{4.2}$$

Moreover,

$$\int_0^1 g|x|^2 + 2\tau\int_0^1 \frac{1}{g}|x_u|^2 \leqslant \int_0^1 g|\bar{x}|^2 \tag{4.3}$$

and, in particular, $x = 0$ whenever $\bar{x} = 0$. Finally, if $\bar{x} \in C^1(\mathbb{R}/\mathbb{Z};\mathbb{R}^2)$ then

$$\int_0^1 |x_u| + \tau\int_0^1 \left(\frac{k|x_u|}{g}\right)^2 |x_u| \leqslant \int_0^1 |\bar{x}_u|, \tag{4.4}$$

where $k = k(x_u, x_{uu})$.

Proof. We first prove the uniqueness of a solution of (4.2). Since (4.2) is a linear nonhomogeneous equation for x the proof follows from (4.3) with $\bar{x}=0$. To prove (4.3) one can take the L^2 inner product of (4.2) with gx to obtain the estimate

$$2\int_0^1 g|x|^2 + 2\tau\int_0^1 \frac{1}{g}|x_u|^2 = 2\int_0^1 g(x,\bar{x}) \leqslant \int_0^1 g|x|^2 + \int_0^1 g|\bar{x}|^2$$

from which inequality (4.3) easily follows.

To prove the existence of a solution of (4.2) one can argue by Fredholm's alternative. Indeed, let $A: C(\mathbb{R}/\mathbb{Z};\mathbb{R}^2) \to C^2(\mathbb{R}/\mathbb{Z};\mathbb{R}^2)$ be a solution operator for the equation $x_{uu}+\pi^2 x = f$, i.e., $x = Af$. Then

$$Af(u) = -\frac{1}{2\pi}\int_0^1 \sin(\pi u - \pi s) f(s)\,\mathrm{d}s + \frac{1}{\pi}\int_0^u \sin(\pi u - \pi s) f(s)\,\mathrm{d}s$$

and this is why the linear operator A is bounded when operating from $C\to C^2$ and is compact as an operator from $C\to C^1$. Let L be a linear operator on $C^1(\mathbb{R}/\mathbb{Z};\mathbb{R}^2)$ defined as $Lx = A(g_u g^{-1}x_u + g^2\tau^{-1}x + \pi^2 x)$. Then $L: C^1\to C^1$ is compact and therefore $I-L$ is a Fredholm mapping of zero index. As a consequence of (4.3) we have that the kernel of $I-L$ is trivial. Therefore, the equation $x - Lx = -A(g^2\tau^{-1}\bar{x})$ has a solution $x\in C^1$. In fact, $x\in C^2$ and x solves (4.2).

Finally we prove (4.4). The proof is similar, in technique, to that of Proposition 3.1. Let us denote $\theta = |x_u|^{-1}g$, $\delta_\tau x = (x-\bar{x})/\tau$. Then, following the lines of the proof of Proposition 3.1 one obtains

$$\begin{aligned}\left(\delta_\tau x_u, \frac{x_u}{|x_u|}\right) &= \theta\Big(\delta_\tau x_u, \frac{x_u}{g}\Big) = \theta\left\{\frac{\mathrm{d}}{\mathrm{d}u}\left(\delta_\tau x, \frac{x_u}{g}\right) - \left(\delta_\tau x, \frac{\mathrm{d}}{\mathrm{d}u}\frac{x_u}{g}\right)\right\}\\ &= -\theta^2|x_u||\delta_\tau x|^2 - \theta\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{\theta_u}{\theta^4|x_u|}\right).\end{aligned}$$

Using the same argument as in (3.4) yields

$$|\delta_\tau x|^2 = \frac{1}{\theta^2|x_u|^2}\left|\frac{\mathrm{d}}{\mathrm{d}u}\frac{x_u}{\theta|x_u|}\right|^2 = k^2\theta^{-4} + \frac{\theta_u^2}{\theta^6|x_u|^2}.$$

Hence,

$$\left(\delta_\tau x_u, \frac{x_u}{|x_u|}\right) = -k^2\theta^{-2}|x_u| - \frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{\theta_u}{\theta^3|x_u|}\right).$$

On the other hand,

$$\left(\delta_\tau x_u, \frac{x_u}{|x_u|}\right) = \frac{1}{\tau}\left(x_u - \bar{x}_u, \frac{x_u}{|x_u|}\right) = \frac{1}{\tau}\left(|x_u| - \left(\bar{x}_u, \frac{x_u}{|x_u|}\right)\right).$$

Therefore,

$$\int_0^1 |x_u| + \tau\int_0^1 \left(\frac{k}{\theta}\right)^2 |x_u| = \int_0^1 \left(\bar{x}_u, \frac{x_u}{|x_u|}\right) \leqslant \int_0^1 |\bar{x}_u|$$

and the proof of the lemma follows. □

We claim that we have assured the existence of a sequence $x^i \in C^2(\mathbb{R}/\mathbb{Z};\mathbb{R}^2)$, $i = 0,1,\ldots,n$, generated by the iteration scheme (4.1) provided that $x^0 \in C^3(\mathbb{R}/\mathbb{Z};\mathbb{R}^2)$. Indeed, for $\varepsilon > 0$ we have $g^0 > 0$ and $g^0 \in C^1$. Now, if $g^{i-1} \in C^1$ then according to Lemma 4.1 there exists the unique solution $x^i \in C^2$ of (4.1). Then

$$x^i_{uu} = \frac{g^{i-1}_u}{g^{i-1}} x^i_u + \frac{(g^{i-1})^2}{\tau}\left(x^i - x^{i-1}\right).$$

By the hypothesis (E) and the property (1.8) we may conclude that the function $g^i = G_{\beta,\varepsilon}(x^i_u, x^i_{uu}) = G_{\beta,\varepsilon}(x^i_u, (g^{i-1})^2\tau^{-1}(x^i - x^{i-1}))$ is, in effect, C^1-smooth and $g^i > 0$. Then an induction argument enables to conclude that the sequence x^i, $i = 0,1,\ldots,n$, is well defined and all x^i, $i = 0,1,\ldots,n$, are C^2-smooth.

Summarizing the above considerations we obtain the following result.

Proposition 4.2. *Let $x^0 \in C^3(\mathbb{R}/\mathbb{Z};\mathbb{R}^2)$ and $\varepsilon > 0$. Then there exists a unique sequence x^i, $i = 0,1,\ldots,n$, generated according to the iteration scheme* (4.1). *Moreover,*

$$\int_0^1 |x^i_u| + \tau \int_0^1 \left(\frac{k^i |x^i_u|}{g^{i-1}}\right)^2 |x^i_u| \leqslant \int_0^1 |x^{i-1}_u| \quad \textit{for } i = 1,2,\ldots,n, \tag{4.5}$$

where $k^i = k(x^i_u, x^i_{uu})$ and $g^{i-1} = G_{\beta,\varepsilon}(x^{i-1}_u, x^{i-1}_{uu})$. In particular, the length of the curve $\Gamma^i = \mathrm{Image}(x^i)$ decreases along the discrete evolution generated by (4.1).

We end this section by discussing the full space-time discretization scheme to be used in all numerical simulations below. To derive the fully discrete analogue of (4.1) we use the uniform spatial grid $u_j = jh$ ($j = 0,\ldots,m$) with $h = 1/m$. The smooth solution x is then approximated by the discrete values x^i_j corresponding to $x(jh, i\tau)$. Using quite natural finite difference approximations of spatial differential terms in (4.1) we end up with the following semi-implicit difference scheme

$$\frac{1}{2}\left(g^{i-1}_j + g^{i-1}_{j+1}\right)\frac{x^i_j - x^{i-1}_j}{\tau} = \frac{x^i_{j+1} - x^i_j}{g^{i-1}_{j+1}} - \frac{x^i_j - x^i_{j-1}}{g^{i-1}_j}, \quad i = 1,\ldots,n,\ j = 1,\ldots,m, \tag{4.6}$$

where

$$g^{i-1}_j = h^{i-1}_j \sqrt{\frac{k^{i-1}_j + \varepsilon}{\beta(k^{i-1}_j + \varepsilon)}}$$

and

$$h^{i-1}_j = |x^{i-1}_j - x^{i-1}_{j-1}|,$$

$$k^{i-1}_j = \frac{\left|\arccos\left((x^{i-1}_{j+1} - x^{i-1}_{j-1}, x^{i-1}_j - x^{i-1}_{j-2})/(|x^{i-1}_{j+1} - x^{i-1}_{j-1}||x^{i-1}_j - x^{i-1}_{j-2}|)\right)\right|}{h^{i-1}_j}.$$

The scheme is subject to the periodic boundary conditions $x^i_{j+m} = x^i_j$ ($j = -1,0,1$). In each discrete computational time step $i\tau$ the scheme (4.6) leads to solving of two tridiagonal systems for the new curve position, which are computed in a very fast way. Let us mention that (4.6) does not involve the spatial grid parameter h and in the linear case $\beta(k) = k$ it coincides with Dziuk's scheme [14].

5. Discussion on numerical experiments

Now, we present numerical results obtained by the approximation scheme (4.1) in the fully discrete version (4.6).

It follows from (2.4) that a special solution of (0.1) with $\beta(k) = k^m,\ m > 0$, is a circle homothetically shrinking to the center; its radius $R(t)$ being given by $R(t) = (R(0)^{m+1} - (m+1)t)^{1/(m+1)}$. Using this formula we obtain exact blow up time for curvature. Table 1 shows relationship between exact and numerically computed blow up times for the power like function $\beta(k) = k^m$ for various $m > 0$. It shows the exact blow up times $T_{\max}$ (see (2.5)), and numerically computed ones for time steps $\tau = 0.01$, $\tau = 0.001$, $\tau = 0.0001$, respectively. The equidistant time step is used until the curvature begins to growth beyond a threshold value. After this moment we adaptively refine the time step to obtain numerical blow up (curvature of order 10^5). We use the mesh containing 100 space grid points in order to represent the position of the curve.

In the case $\beta(k) = k^{1/3}$, arbitrary ellipse is a homothetic solution (see [2,28] and Section 2 of this paper). This property is also confirmed by our numerical simulations. During the time evolution the ratio a/b of halfaxes stays constant up to the moment very close to the exact time of shrinking. The shape selfsimilarity during the evolution is justified by computing the isoperimetric ratio $\mathrm{Iso} = L^2/(4\pi S)$,

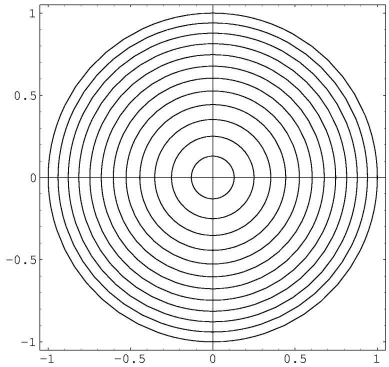


Fig. 1. Shrinking of the unit circle by nonlinear curve shortening with $\beta(k) = k^{1/2}$; numerical blow up time for the curvature is 0.671, plotting time step is 0.1.

Table 1
Relationship between exact and numerically computed curvature blow up times for initial unit circle, $\beta(k) = k^m$

m	$\frac{1}{10}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$	1	0
$T_{\max}$-exact	$0.9\overline{09}$	0.8	0.75	$0.6\overline{6}$	0.5	1.0
$\tau = 0.01$	0.942	0.835	0.785	0.701	0.536	1.02
$\tau = 0.001$	0.913	0.8047	0.754	0.671	0.5048	1.003
$\tau = 0.0001$	0.9098	0.8007	0.7506	0.6673	0.5005	1.0007

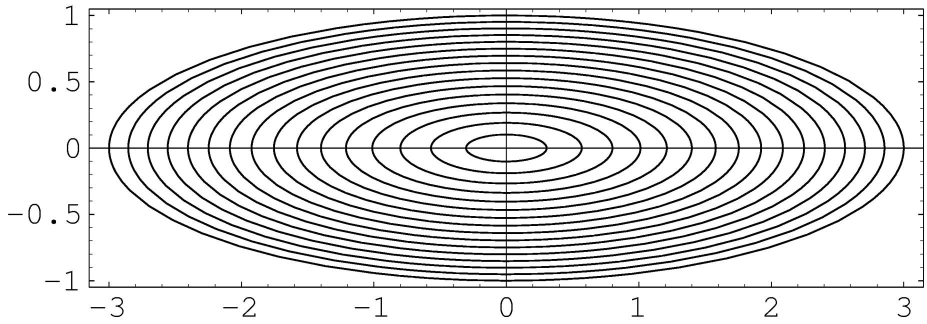


Fig. 2. Affine invariant evolution of the ellipse, initial halfaxes $a = 3,\ b = 1$; halfaxes ratio and isoperimetric ratio are conserved during the computations.

Table 2
Relationship between exact and numerically computed curvature blow up times for initial ellipses; $\beta(k) = k^{1/3}$

Halfaxes ratio	2 : 1	3 : 1	4 : 1
T_{max}-exact	1.191	1.560	1.890
$\tau = 0.001$	1.195	1.564	1.893
Iso	1.188	1.508	1.864

Table 3
Evolution of the isoperimetric ratio

Time/Iso	0	0.6	1	1.4	1.5	1.519
$\beta(k) = k^{1/2}$	1.508	1.36	1.25	1.11	1.07	1.04
$\beta(k) = k^{1/4}$	1.508	1.63	1.79	2.39	3.36	4.12

where L is the length of the curve and S is the enclosed area. It turns out that also this quantity is practically constant for the numerical solution.

For $m = \frac{1}{3}$, the exact blow up time for a shrinking ellipse can be computed (see Example 2.1) and is equal to $\frac{3}{4}(ab)^{2/3}$. In the next table we compare the numerical and exact blow up times for several ellipses. The ratio of halfaxes is printed in the headline of Table 2. The mesh containing 200 space grid points has been used for discretization of the curve. We also print the isoperimetric ratio which is conserved up to 4 digits during numerical evolution.

In spite of conservation of the isoperimetric ratio for $m = \frac{1}{3}$, it tends to 1 in numerical computations with $m = \frac{1}{2}$ and to ∞ for $m = \frac{1}{4}$, respectively. We print values of $\mathrm{Iso}_t,\ t \in [0, T_{max})$, in these two cases for initial ellipse with halfaxes ratio 3 : 1.

In Figs. 3 and 4 we present the comparison of the numerical results obtained by two rather different methods. Namely, the tested method (4.1), based on the computing of the curve's position vector, and the method introduced in [23], based on the computing of the curvature of evolving curve. In the second case, the real motion is reconstructed from the computed curvature in discrete time steps. This method is based

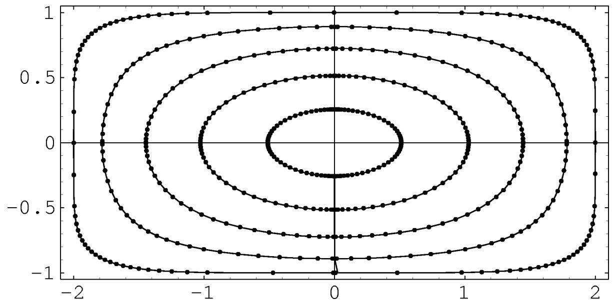


Fig. 3. Comparison of two different methods for evolution of convex curve: tick marks—method (4.1); solid lines—method from [23].

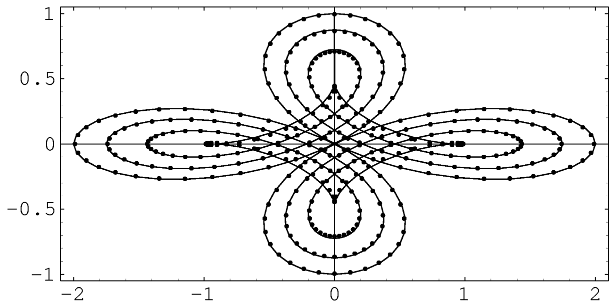


Fig. 4. Comparison of two different methods for evolution of selfintersecting "convex" curve: tick marks—method (4.1); solid lines—method from [23]. The evolving curve is plotted at the same discrete time moments until the "hair" singularity is formed. The method from [23] cannot continue beyond singularity.

on solving the nonlinear parabolic equation of porous-medium type and its convergence for (0.1) is the consequence of the results of [24]. However, it is restricted to convex cases (including selfintersections).

In Fig. 3 one sees suitable redistribution (from initial to the first plotted step) of computational grid points due to the presence of the tangential component of the velocity (see (1.4)). Due to the shape of β the points with high curvatures are moving along the curves ($m < 1$) and it works against the degeneracy of equation. In spite of this, the effect is opposite for $m > 1$ and leads to serious computational difficulties in that case. This phenomenon can be explained, in a satisfactory manner, by Eq. (1.4). It follows from (1.4) and (1.5) that the tangential velocity is proportional to k_s times the sign of $m - 1$. Therefore in the case $m \leqslant 1$ the tangential component of the velocity drives the grids away from the pieces of the curve with increasing curvature whereas its action is opposite in the case $m > 1$.

The results discussed above are very accurate already for reasonable large computational time steps. It indicates the usefulness and effectiveness of the method even in cases when no exact solutions are known. In Figs. 5–8 we show evolutions of several initially nonconvex curves with the different choices of β. We also present the passage through singularities in some examples of immersed curves. A simple *point removing algorithm* has been built into the scheme preventing the "$|x_u| = 0$" kind of singularity and, moreover, it is very useful tool in order to pass through singularities and other situations when the grid points representing the curve move very close to each other.

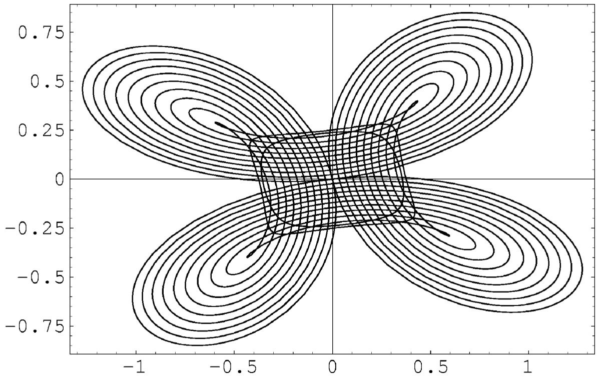


Fig. 5. Evolution of the affine transformed 4-petal through singularities to an ellipse-rounded point, $\beta(k)=k^{1/3}$.

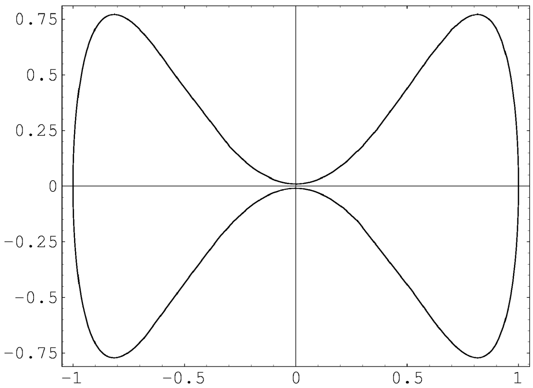


Fig. 6. The initial nonconvex curve.

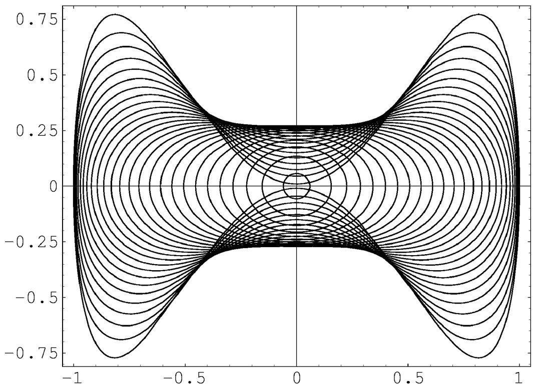


Fig. 6a. The case $\beta(k)=k$. Time interval is $[0, 0.26]$.

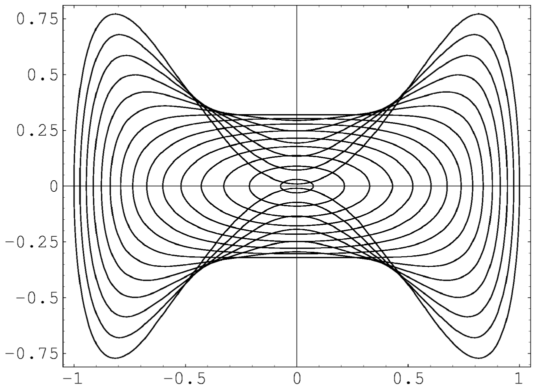


Fig. 6b. The case $\beta(k)=k^{1/3}$. Time interval is $[0, 0.56]$.

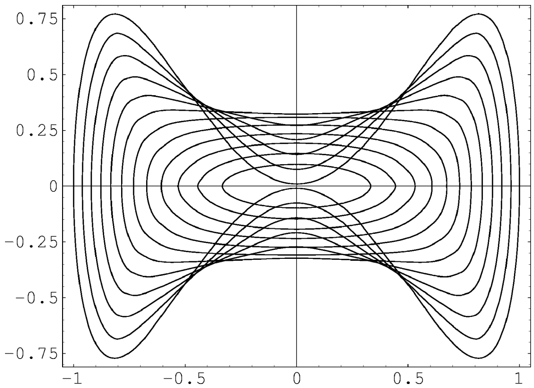


Fig. 6c. The case $\beta(k)=k^{1/5}$. Time interval is $[0, 0.56]$.

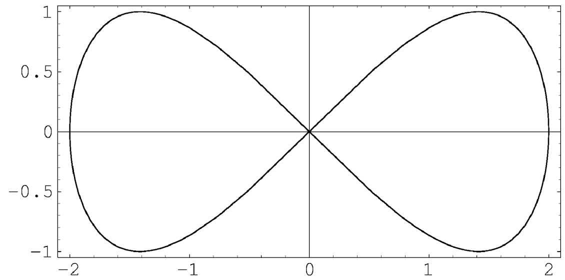


Fig. 7. The initial ∞-like selfintersecting curve.

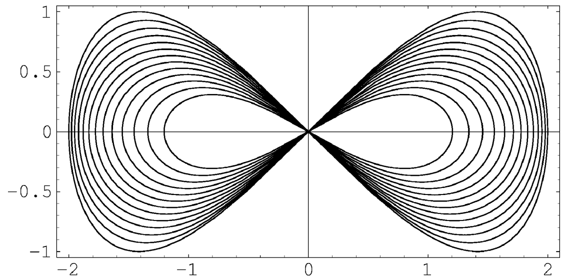


Fig. 7a. The case $\beta(k) = k$. Time interval is $[0, 0.3]$.

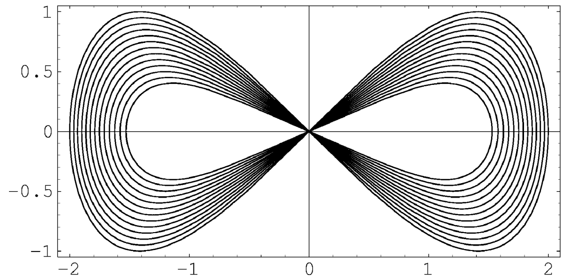


Fig. 7b. The case $\beta(k) = k^{1/3}$. Time interval is $[0, 0.3]$.

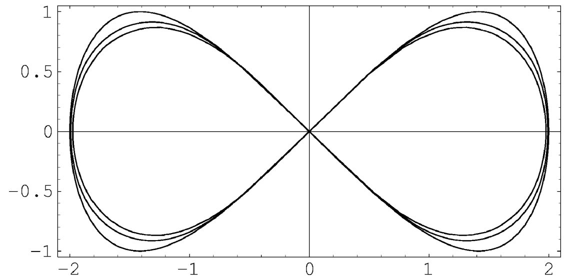


Fig. 7c. The case $\beta(k) = k^2$. Time interval is $[0, 0.08]$.

Figs. 6a–6c show the time evolution of the initial curve depicted in Fig. 6. The plotting step is 50. The initial curve has the parameterization $\{(\cos(2\pi u), 2\sin(2\pi u) - 1.99\sin(2\pi u)^3),\ u \in [0,1]\}$ and the spatial mesh contained 100 equally distributed points. In Fig. 6a, respectively 6b, with $\beta(k) = k$, respectively $\beta(k) = k^{1/3}$, one can see the evolution of a nonconvex curve to a circle-rounded point, respectively to an ellipse-rounded point, in Fig. 6c we have plotted evolution of the nonconvex curve with the blow up of isoperimetric ratio for the case $\beta(k) = k^{1/5}$.

Figs. 7a–7c show the time evolution of the initial curve depicted in Fig. 7. The plotting step is 40 (Figs. 7a, 7b) and 30 (Fig. 7c). The initial curve has the parametrization $\{(2\cos(2\pi u), \sin(4\pi u)),\ u \in [0,1]\}$ and the spatial mesh contained 100 equally distributed points. In Fig. 7c with $\beta(k) = k^2$ one sees very different behavior for the parts of curve with curvature close or equal to 0 in comparison with Fig. 7b, where $\beta(k) = k^{1/3}$. We did not trace the evolution of curves for $\beta(k) = k^2$ beyond the time interval $[0, 0.1]$ as it becomes unstable for large time intervals of simulation. The pieces of the curve with the curvature close to zero do not move for a long time in the case $\beta(k) = k^2$ (see Fig. 7c) whereas they move quickly apart from each other in the case $\beta(k) = k^{1/3}$ (see Fig. 7b). This phenomenon can be related to the effect of the the slow and fast diffusion diffusion effect known from the analysis of porous medium equations.

Another example of the time evolution of an irregular initial curve with the large variation in the curvature is demonstrated by Figs. 8a–8c with initial curve depicted in Fig. 8. We have used the time step $\tau = 0.001$, plotting step 50, and the mesh contained 200 grids.

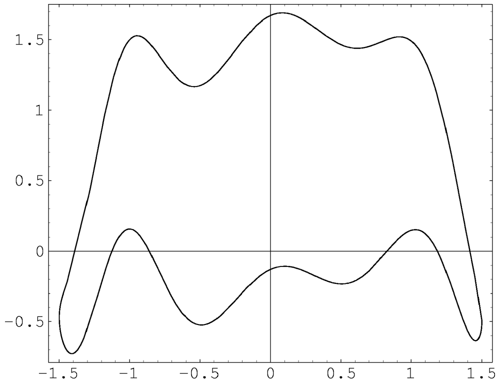


Fig. 8. The initial irregular plane curve.

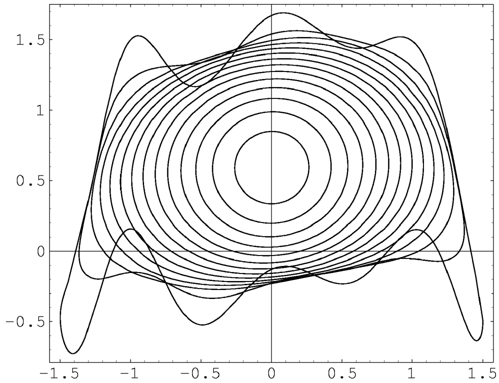


Fig. 8a. The case $\beta(k) = k$. Time interval is $[0, 0.65]$.

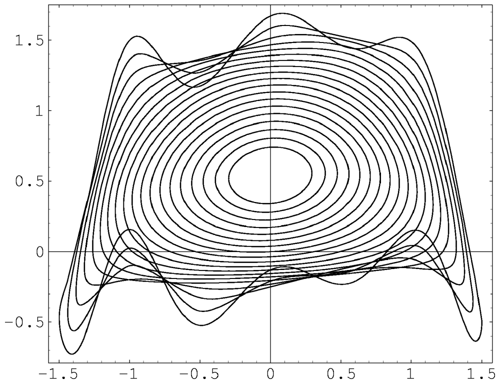


Fig. 8b. The case $\beta(k) = k^{1/3}$. Time interval is $[0, 0.9]$.

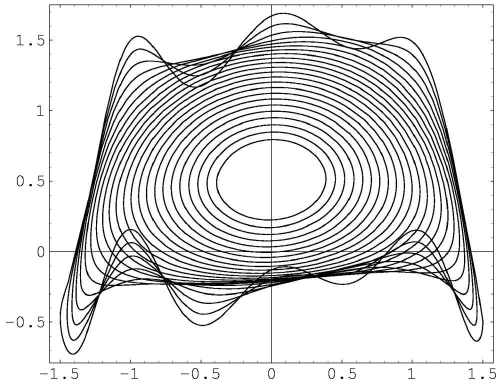


Fig. 8c. The case $\beta(k) = \arctan(k)$. Time interval is $[0, 1.1]$.

References

[1] U. Abresch and J. Langer, The normalized curve shortening flow and homothetic solutions, *J. Differential Geometry* 23 (1986) 175–196.

[2] L. Alvarez, F. Guichard, P.L. Lions and J.M. Morel, Axioms and fundamental equations of image processing, *Arch. Rat. Mech. Anal.* 123 (1993) 200–257.

[3] L. Alvarez and J.M. Morel, Formalization and computational aspects of image analysis, *Acta Numer.* (1994) 1–59.

[4] S.B. Angenent, Nonlinear analytic semiflows, *Proc. Roy. Soc. Edinburgh Sect. A* 115 (1990) 91–107.

[5] S.B. Angenent, Parabolic equations for curves on surfaces I: Curves with p-integrable curvature, *Ann. Math.* 132 (3) (1990) 451–483.

[6] S.B. Angenent, Parabolic equations for curves on surfaces II: Intersections, blow-up and generalized solutions, *Ann. Math.* 133 (1) (1991) 171–215.

[7] S.B. Angenent, On the formation of singularities in the curve shortening flow, *J. Differential Geometry* 33 (1991) 601–633.

[8] S.B. Angenent and M.E. Gurtin, Multiphase thermomechanics with an interfacial structure 2. Evolution of an isothermal interface, *Arch. Rat. Mech. Anal.* 108 (1989) 323–391.
[9] S.B. Angenent and M.E. Gurtin, Anisotropic motion of a phase interface. Well-posedness of the initial value problem and qualitative properties of the interface, *J. Reine Angew. Math.* 446 (1994) 1–47.
[10] S. Altschuler, Singularities of the curve shrinking flow for space curves, *J. Differential Geometry* 34 (1991) 491–514.
[11] S. Altschuler and M.A. Grayson, Shortening space curves and flow through singularities, *J. Differential Geometry* 35 (1992) 283–298.
[12] Y.-G.Chen, Y. Giga and S. Goto, Uniqueness and existence of viscosity solutions of generalized mean curvature flow equation, *J. Differential Geometry* 33 (1991) 749–786.
[13] M.G. Crandall, H. Ishii and P.L. Lions, User's guide to viscosity solutions of second order partial differential equations, *Bull. Amer. Math. Soc.* 27 (1992) 1–67.
[14] G. Dziuk, Convergence of a semi discrete scheme for the curve shortening flow, *Math. Models Methods Appl. Sci.* 4 (4) (1994) 589–606.
[15] G. Dziuk, Convergence of a scheme for anisotropic curvature flow, Abstract (from the Conference on Generalized Stefan Problem: Analysis and Numerical Methods, Pavia, 1995) in *FBP News* 9 (1995) 5.
[16] L.C. Evans and J. Spruck, Motion of level sets by mean curvature. I, *J. Differential Geometry* 33 (1991) 635–681.
[17] M. Gage and R.S. Hamilton, The heat equation shrinking convex plane curves, *J. Differential Geometry* 23 (1986) 285–314.
[18] M. Grayson, The heat equation shrinks embedded plane curves to round points, *J. Differential Geometry* 26 (1987) 285–314.
[19] J. Kačur and K. Mikula, Solution of nonlinear diffusion appearing in image smoothing and edge detection, *Appl. Numer. Math.* 17 (1995) 47–59.
[20] M. Kass, A. Witkin and D. Terzopoulos, Snakes: active contours models, in: *Proc. 1st International Computer Vision Conference* (1987) 259–269.
[21] P.L. Lions, Axiomatic derivation of image processing models, *Math. Models Methods Appl. Sci.* 4 (1994) 467–475.
[22] A. Mackworth and F. Mokhtarian, A theory of multiscale, curvature-based shape representation for planar curves, *IEEE Trans. Pattern Anal. Machine Intell.* 14 (1992) 789–805.
[23] K. Mikula, Solution of nonlinear curvature driven evolution of plane convex curves, *Appl. Numer. Math.* 21 (1997) 1–14.
[24] K. Mikula and J. Kačur, Evolution of convex plane curves describing anisotropic motions of phase interfaces, *SIAM J. Sci. Comput.* 17 (6) 1996.
[25] S. Osher and J. Sethian, Fronts propagating with curvature dependent speed: algorithm based on Hamilton–Jacobi formulation, *J. Comput. Phys.* 79 (1988) 12–49.
[26] S. Patankar, *Numerical Heat Transfer and Fluid Flow* (Hemisphere Publ. Corp., New York, 1980).
[27] B.M. ter Haar Romeny (Ed.), *Geometry Driven Diffusion in Computer Vision* (Kluwer, Dordrecht, 1994).
[28] G. Sapiro and A. Tannenbaum, On affine plane curve evolution, *J. Funct. Anal.* 119 (1) (1994) 79–120.
[29] A. Schmidt, Computation of three dimensional dendrites with finite elements, *J. Comput. Phys.* 125 (2) (1996) 293–312.
[30] J. Sethian, Numerical algorithm for propagating interfaces: Hamilton–Jacobi equations and conservation laws, *J. Differential Geometry* 31 (1990) 131–161.

Príloha 6.3.1

Reprint práce:

M. Revallo, D. Ševčovič, Brestenský, S. Ševčík: *Viscously controlled nonlinear magnetoconvection in a non-uniformly stratified horizontal fluid layer.* Physics of the earth and planetary interiors 111, 1-2 (1999), 83-92.

ELSEVIER

Physics of the Earth and Planetary Interiors 111 (1999) 83–92

PHYSICS OF THE EARTH AND PLANETARY INTERIORS

Viscously controlled nonlinear magnetoconvection in a non-uniformly stratified horizontal fluid layer

M. Revallo [a], D. Ševčovič [b], S. Ševčík [a], J. Brestenský [a,*]

[a] Dept of Geophysics, Faculty of Math and Physics, Comenius University, 842 15 Bratislava, Slovak Republic
[b] Inst. of Applied Mathematics, Faculty of Math and Physics, Comenius University, 842 15 Bratislava, Slovak Republic

Received 20 February 1998; accepted 20 July 1998

Abstract

Weakly nonlinear analysis is adopted in order to study a model of magnetoconvection in a rotating horizontal fluid layer. The layer is supposed to be non-uniformly stratified and is permeated by an azimuthal magnetic field. The only nonlinearity brought in this convecting system is due to presence of Ekman layers along the horizontal mechanical boundaries. The governing equations for this model together with the expression for geostrophic flow, i.e., modified Taylor's constraint are analysed with help of perturbation methods. As a result, the bifurcation structure in the vicinity of the critical Rayleigh number is revealed.

Keywords: Magnetoconvection; Modified Taylor's constraint; Non-uniform stratification; Perturbation techniques; Weakly nonlinear analysis

1. Introduction

In a large class of MHD models, the assumption is often made that the primary force balance in the Earth's core is entered by Lorentz, Coriolis, buoyancy and pressure forces in the equation of motion. Such a force balance, familiar also as magnetostrophic approximation, has a solution, if and only if so-called Taylor's constraint is satisfied (Taylor, 1963). In the case of small but non-zero viscosity, the magnetostrophic approximation still holds as a primary force balance, but Taylor's constraint has to be modified to include viscous effects. In this case, modified Taylor's constraint can be taken as a predictive formula for evaluation of geostrophic flow which is thus expressed explicitly (Fearn, 1994).

In this paper, we focus our attention on a finite amplitude rotating magneto-convection in a horizontal layer permeated by azimuthal magnetic field. The linearized version of this problem for the model of uniformly stratified rotating layer with infinite horizontal extension has been studied by Soward (1979) and Brestenský and Ševčík (1994). In the model of rotating annulus (Skinner and Soward, 1988, 1991), the effect of geostrophic flow was incorporated making the whole problem nonlinear. Here, conditions

* Corresponding author. E-mail: brestensky@fmph.uniba.sk

0031-9201/99/$ - see front matter
PII: S0031-9201(98)00148-4

for the onset of instability in the regime of so-called Taylor state [1] were found.

Unlike the above mentioned references, in our recent studies made in Brestenský et al. (1997) and Revallo et al. (1997), [2] we restricted our attention on the early evolution of instability in the vicinity of the basic static state, i.e., in the vicinity of critical Rayleigh number R_c. This leads to a specific weakly nonlinear problem where the ensuing magnetoconvection is affected by the presence of geostrophic flow. The interested reader is referred to BRS for some mathematical aspects as well as for the method of solution. In RSB we considered a simple model of radially bounded horizontal rotating layer with free infinitely electrically and thermally conducting boundaries. We found that the oscillatory convection in this system sets in via Hopf bifurcation which is typically supercritical for $q = 0.005$. Furthermore the convective instability has a form of travelling waves whose frequency has a decreasing tendency as the Rayleigh number is increased beyond its critical value.

In this paper, we pursue the weakly nonlinear analysis made in BRS and RSB for a more complicated model of magnetoconvection where non-uniform stratification of the layer is incorporated. Our modification of the model is based upon an idea originally proposed by Bod'a (1988) and later developed by Ševčík (1989). [3] In these references a linear problem of magnetoconvection in a horizontally unbounded geometry was set up in which the density gradient changes its sign across the layer (non-uniform stratification). The concept of a non-uniformly stratified layer appears to be reasonable as it incorporates more realistic conditions in the Earth's interior which is known to be non-uniformly stratified.

Note that several interesting features were isolated in the model introduced by Bod'a (1988), e.g., existence of the magnetic mode in the layer gravitationally stable in the top half and unstable in the bottom half. Moreover, under the assumption of non-uniform stratification, the excitation of thermal mode was observed in S89 even for the layer cooled from below and heated from above (the case of negative Rayleigh number).

The paper is organised as follows. In Section 2 we formulate the nonlinear problem and we state the expression for the geostrophic flow. In Section 3 we present a system of nonlinear PDE's governing motion. Such a motion is periodic in both time and in the azimuthal variable. Section 4 refers back to the original linear problem for a non-uniformly stratified layer solved in S89. In Section 5 we briefly outline the solution of PDE's by a perturbation technique and we quote resulting amplitude equation. The results of bifurcation analysis are presented in Section 6. The corresponding bifurcation diagrams are also shown in this section. Section 7 is devoted to conclusions.

2. Description of the nonlinear model

The model under consideration is a weakly bounded cylinder [4] of width d, $z \in \langle 0, d \rangle$ and radius s_n, $s \in \langle 0, s_n \rangle$, rotating rapidly with angular velocity $\Omega_0 \hat{\boldsymbol{z}}$ about the vertical rotation axis. The cylinder contains an electrically conducting Boussinesq fluid permeated by an azimuthal magnetic field B_0 linearly growing with the distance from the rotation axis. The instability of this system can be caused by the vertical temperature gradient. Therefore, we consider the temperature difference ΔT maintained between the lower, T_1, and the upper boundaries, $T_1 - \Delta T$. Non-uniform stratification can be modelled by negative heat sources, $H < 0$, distributed within the layer. This has a consequence of non-linear (quadratic) dependence of basic temperature profile, T_0.

Assuming small but non-zero viscosity leads towards formation of viscous boundary layers (the Ekman layers) along the horizontal boundaries. As a result, non-zero geostrophic flow $\Omega(s)$ is induced by Ekman suction, which couples the interaction between boundary layer and the rest of the fluid, making the whole problem nonlinear.

[1] In this asymptotic regime the magnitude of geostrophic flow is such that viscous forces no longer have major influence on the convection and the net torque on geostrophic cylinders vanishes.

[2] Henceforth referred to as BRS and RSB.

[3] Henceforth referred to as S89.

[4] The radial extension of the layer is much greater than its thickness.

The ensuing convective instability manifests itself by perturbations of the velocity $\boldsymbol{u}$, the magnetic field $\boldsymbol{b}$ and the temperature $\tilde{\vartheta}$ which relate to the basic state represented by

$$\begin{aligned} &\boldsymbol{U}_0 = \boldsymbol{0}, \\ &\boldsymbol{B}_0 = B_M \frac{s}{d} \hat{\boldsymbol{\varphi}}, \\ &T_0 = T_l - \Delta T \frac{z}{d}\left(1 - \frac{z-d}{2 z_M * - d}\right). \end{aligned} \tag{1}$$

The quantity $z_M * = -\rho_0 c_p \kappa \Delta T/(dH) + d/2$, referred to as stratification parameter, is the z-coordinate of the level dividing the layer into the stably and unstably stratified sublayers; $\rho_0 c_p \kappa$ is the thermal conductivity. The temperature $T_0(z)$ reaches minimum and its gradient changes direction at the level $z = z_M *$. Note that the cases of uniform stratification can be obtained as the limiting cases $z_M * \to \pm\infty$.

We non-dimensionalise the basic equations with the use of characteristic length d, magnetic diffusion time d^2/η, magnetic field B_M, and temperature difference across the layer ΔT. The equations in the cylindrical polar coordinates (s, φ, z) governing the evolution of perturbations $\boldsymbol{u}$, $\boldsymbol{b}$, $\tilde{\vartheta}$ are cast as follows

$$\hat{\boldsymbol{z}} \times \boldsymbol{u} = -\nabla p + \Lambda[(\nabla \times s\hat{\boldsymbol{\varphi}}) \times \boldsymbol{b} + (\nabla \times \boldsymbol{b}) \times s\hat{\boldsymbol{\varphi}}] + R\tilde{\vartheta}\hat{\boldsymbol{z}}, \tag{2}$$

$$\frac{\partial \boldsymbol{b}}{\partial t} - \nabla \times (s\Omega(s)\hat{\boldsymbol{\varphi}} \times \boldsymbol{b}) = \nabla \times (\boldsymbol{u} \times s\hat{\boldsymbol{\varphi}}) + \nabla^2 b, \tag{3}$$

$$q\left(\frac{\partial \tilde{\vartheta}}{\partial t} + (s\Omega(s)\hat{\boldsymbol{\varphi}} \cdot \nabla)\tilde{\vartheta}\right) = -\boldsymbol{u} \cdot \nabla T_0 + \nabla^2 \tilde{\vartheta}, \tag{4}$$

$$\nabla \cdot \boldsymbol{b} = 0, \quad \nabla \cdot \boldsymbol{u} = 0 \tag{5}$$

where $\hat{\boldsymbol{z}}$ and $\hat{\boldsymbol{\varphi}}$ are the axial and azimuthal unit vectors, respectively.

In Eqs. (2)–(5) the dimensionless parameters, the modified Rayleigh number R, the Elsasser number Λ, the Ekman number E and the Roberts number q, are defined by

$$R = \frac{g d \Delta T \alpha}{2\Omega_0 \kappa}, \quad \Lambda = \frac{B_M^2}{2\Omega_0 \rho_0 \eta \mu}, \quad E = \frac{\nu}{2 d^2 \Omega_0}, \quad q = \frac{\kappa}{\eta}$$

where κ and η are the thermal and magnetic diffusivities, ν is the kinematic viscosity, α is the coefficient of thermal expansion, g is the acceleration due to gravity, μ is the permeability and ρ_0 is the density.

In the case of non-uniform stratification, the temperature gradient entering Eq. (4) can be expressed in terms of the dimensionless parameters in the following form (S89)

$$\begin{aligned} &\frac{dT_0}{dz} = -(2az - a + 1) \text{ for } \Delta T > 0, \\ &\frac{dT_0}{dz} = +(2az - a + 1) \text{ for } \Delta T < 0 \end{aligned} \tag{6}$$

where the coefficient a relates to the dimensionless stratification parameter z_M via

$$a = \frac{1}{1 - 2z_M}. \tag{7}$$

Here $z_M = z_M */d$. Note that taking the coefficient $a = 0$ corresponds to uniform stratification.

The angular velocity $\Omega(s)$ of geostrophic flow entering the convective non-linearities in the above set of equations is determined by modified Taylor's constraint

$$\Omega(s) = \frac{\Lambda}{(2E)^{1/2} s} \int_{z_B}^{z_T} \langle F_{M\varphi} \rangle^{\varphi} dz \text{ with } \langle F_{M\varphi} \rangle^{\varphi} = \langle [(\nabla \times \boldsymbol{b}) \times \boldsymbol{b}]_{\varphi} \rangle^{\varphi} \tag{8}$$

where z_B and z_T delimit the horizontal boundaries of the layer, the $\langle \dots \rangle^{\varphi} \equiv 1/(2\pi) \int_0^{2\pi} \dots d\varphi$ stands for averaging over φ and $F_{M\varphi} \equiv [(\nabla \times \boldsymbol{B}) \times \boldsymbol{B}]_{\varphi}$ denotes an azimuthal component of Lorentz force.

Eqs. (2)–(5) have to be solved subject to boundary conditions corresponding to rigid [5] perfectly

[5] Only due to the effect of Ekman secondary flow.

electrically and thermally conducting horizontal boundaries, i.e.,

$$u_z = \tilde{\vartheta} = b_z = 0, \quad \hat{\boldsymbol{z}} \times \frac{\partial \boldsymbol{b}}{\partial z} = \boldsymbol{0} \quad \text{at} \quad z = 0, d. \quad (9)$$

In addition, we will assume that on the sidewall boundaries, delimited by $s = s_n$ the perturbations are vanishing (see following section).

3. Formulation of the nonlinear problem

It is convenient to rearrange the system of nonlinear equations (Eqs. (2)–(5)) with help of the poloidal–toroidal decomposition of vector fields (for details see BRS). For the velocity perturbation $\boldsymbol{u}$ and magnetic field perturbation $\boldsymbol{b}$ we assume

$$\boldsymbol{u} = k^{-2}\left[\nabla \times (\nabla \times \tilde{w}\hat{\boldsymbol{z}}) + \nabla \times \tilde{\omega}\hat{\boldsymbol{z}}\right],$$
$$\boldsymbol{b} = k^{-2}\left[\nabla \times (\nabla \times \tilde{b}\hat{\boldsymbol{z}}) + \nabla \times \tilde{j}\hat{\boldsymbol{z}}\right]. \quad (10)$$

Here, k is a radial wave number and the representing functions $\tilde{w}$, $\tilde{\omega}$, $\tilde{b}$, $\tilde{j}$ depend on coordinates z, s, φ and time t and will be symbolized by $\tilde{f}(z, s, \varphi, t)$, or shortly $\tilde{f}$, as in Brestenský and Ševčík (1994). The same notation applies for the temperature perturbation, i.e., $\tilde{\vartheta}$.

The representing functions of $\tilde{f}$ can be sought in the form

$$\tilde{f}(z,s,\varphi,t) = \Re e\{A(\varepsilon^p t) f_m(z,s)\exp(im\varphi + \lambda t)\}. \quad (11)$$

Hereafter the symbol $f_m(z,s)$ stands for one of the complex functions $w_m(z,s)$, $\omega_m(z,s)$, $b_m(z,s)$, $j_m(z,s)$ and $\vartheta_m(z,s)$ dependent on coordinates z and s. Unlike the assumption often made in the linear case, e.g, (Soward, 1979; Šimkanin et al., 1997), each of the functions $f_m(z,s)$ above is modulated by a complex amplitude $A(\varepsilon^p t)$ varying in the so-called slow time scale $\varepsilon^p t$ where ε is a small unfolding parameter and p is a natural number to be specified later. Furthermore, m is an azimuthal wave number (integer) and λ is a complex frequency (related to a real frequency via $\lambda = i\sigma$).

Upon the above assumption, the reduced system of nonlinear equations can be derived from Eqs. (2)–(5). Hereafter, the notation $\tau = \varepsilon^p t$ and $\dot{A}(\tau) = \mathrm{d}A(\tau)/\mathrm{d}\tau$ will be used. The partial differential equations for representing functions take the following forms

$$0 = -Dw_m(z,s) + 2\Lambda Db_m(z,s) - im\Lambda j_m(z,s),$$
$$0 = -D\omega_m(z,s) + 2\Lambda Dj_m(z,s) + im\Lambda\left(D^2 - k^2\mathscr{I}_m\right)b_m(z,s) - Rk^2\vartheta_m(z,s),$$
$$\lambda A(\tau) b_m(z,s) + \varepsilon^p \dot{A}(\tau) b_m(z,s) + A(\tau)|A(\tau)|^2 P_m(z,s) = imA(\tau) w_m(z,s) + A(\tau)\left(D^2 - k^2\mathscr{I}_m\right)b_m(z,s),$$
$$\lambda A(\tau) j_m(z,s) + \varepsilon^p \dot{A}(\tau) j_m(z,s) + A(\tau)|A(\tau)|^2 T_m(z,s) = imA(\tau)\,\omega_m(z,s) + A(\tau)\left(D^2 - k^2\mathscr{I}_m\right)j_m(z,s),$$
$$(1/q)\left(\lambda A(\tau)\vartheta_m(z,s) + \varepsilon^p \dot{A}(\tau)\vartheta_m(z,s) + A(\tau)|A(\tau)|^2 S_m(z,s)\right) = A(\tau)\zeta(z) \times\mathscr{I}_m w_m(z,s) + A(\tau)\left(D^2 - k^2\mathscr{I}_m\right)\vartheta_m(z,s) \quad (12)$$

where $\zeta(z) = -\mathrm{d}T_0/\mathrm{d}z$.

The representing functions for nonlinearities $P_m(z,s)$, $T_m(z,s)$, $S_m(z,s)$ are expressed in terms of

$$P_m(z,s) = im\Omega(s) b_m(z,s) - im\mathscr{I}_m^{-1}\{\mathscr{P}_\Omega\, b_m(z,s)\},$$
$$T_m(z,s) = im\Omega(s) j_m(z,s) + \mathscr{I}_m^{-1}\{\mathscr{T}_\Omega\, Db_m(z,s)\},$$
$$S_m(z,s) = im\Omega(s)\vartheta_m(z,s). \quad (13)$$

Here $D = \partial/\partial z$ and $\mathscr{I}_m^{-1}$ is the inverse operator to the linear Bessel differential operator $\mathscr{I}_m$

$$\mathscr{I}_m \equiv -\frac{1}{k^2}\left(\frac{\partial^2}{\partial s^2} + \frac{1}{s}\frac{\partial}{\partial s} - \frac{m^2}{s^2}\right)$$

and P_Ω, T_Ω are differential operators

$$\mathscr{P}_\Omega = -\frac{1}{k^2}\left\{\frac{\partial^2\Omega(s)}{\partial s^2} + \frac{\partial\Omega(s)}{\partial s}\left[2\frac{\partial}{\partial s} + \frac{1}{s}\right]\right\},$$

$$\mathscr{T}_\Omega = -\frac{1}{k^2}\left\{s\frac{\partial^2\Omega(s)}{\partial s^2}\frac{\partial}{\partial s} + s\frac{\partial\Omega(s)}{\partial s} \times\left[\frac{m^2}{s^2} + \frac{2}{s}\frac{\partial}{\partial s} + \frac{\partial^2}{\partial s^2}\right]\right\}. \quad (14)$$

The expression for the geostrophic flow $\Omega(s)$ in terms of $b_m(z,s)$ and $j_m(z,s)$ can be derived directly from Eq. (8) (see RSB). Assuming the separability $f_m(z,s) = f(z) J_m(ks)$, it simplifies to

$$\Omega(s) = \mathscr{Z} \frac{1}{s} \frac{d}{ds} J_m^2(ks) \text{ where } \mathscr{Z} = \frac{\Lambda}{(8E)^{1/2} k^2} \Re e \int_{z_B}^{z_T} j(z) \overline{Db(z)} dz \quad (15)$$

is a functional depending on the functions $b(z)$ and $j(z)$ known from the linear study.

For the representing functions $f_m(z,s)$, the corresponding boundary conditions can be obtained from Eq. (9)

$$w_m(z,s) = \vartheta_m(z,s) = b_m(z,s) = Dj_m(z,s) = 0,$$
$$\text{for all } z = 0, d \text{ and } s \in (0, s_n). \quad (16)$$

In a radial direction we impose the following boundary conditions

$$\vartheta_m(z,s) = b_m(z,s) = j_m(z,s) = 0,$$
$$\text{for all } s = 0, s_n, \text{ and } z \in (0,d). \quad (17)$$

In our model, the parameter s_n, which delimits the layer in a radial direction, has to be chosen to coincide with the n-th root of the Bessel function i.e., $J_m(ks_n) = 0$ for sufficiently large integer n. Since the values of radial wave number k are taken from the linear analysis, the above condition is met for certain values of s_n only.

4. Linearized problem and its solution

Considering small perturbations the whole problem can be linearized (see e.g., Bod'a, 1988; S89). Specifically, in our model the linearization can be fixed by the condition $\Omega(s) = 0$. The linear case allows for the separation of representing functions

$$f_m(z,s) = f(z) J_m(ks) \quad (18)$$

where $J_m(ks)$ is the Bessel function of the first kind, k is a radial wavenumber (real) and $f(z)$ is the complex function of z-coordinate. Respecting the boundary conditions, for each particular $f(z)$ we have

$$w(z) = \sum_{n=1}^{\infty} w_n \sin(\pi n z),$$
$$\omega(z) = \omega_0 + \sum_{n=1}^{\infty} \omega_n \cos(\pi n z),$$
$$b(z) = \sum_{n=1}^{\infty} b_n \sin(\pi n z),$$
$$j(z) = j_0 + \sum_{n=1}^{\infty} j_n \cos(\pi n z),$$
$$\vartheta(z) = \sum_{n=1}^{\infty} \vartheta_n \sin(\pi n z). \quad (19)$$

Inserting the above expansions into the linearized equations, after a series of standard operations we obtain a set of algebraic equations for complex coefficients w_n, ω_n, b_n, j_n, ϑ_n. In S89, the critical Rayleigh number R_c, the critical frequency λ_c the critical radial wave number k_c and the complex coefficients were computed for various sets of parameters (Λ, q, m, a).

5. Solution of the nonlinear problem

A standard way is to represent nonlinear equations (Eq. (12)) in a matrix form (in BRS referred to as an abstract nonlinear problem)

$$A(\tau) \mathscr{L} \boldsymbol{\psi} = N(A(\tau), \dot{A}(\tau), \boldsymbol{\psi}) \quad (20)$$

where $A(\tau)$ is the complex amplitude, $\mathscr{L}$ is the linear operator and $\boldsymbol{\psi}$ is the vector function

$$\boldsymbol{\psi}^T \equiv (w_m(z,s), \omega_m(z,s), b_m(z,s), j_m(z,s), \vartheta_m(z,s)) \quad (21)$$

and the right-hand side vector $N(A(\tau), \dot{A}(\tau), \boldsymbol{\psi})$ contains the nonlinearities.

Due to the structure of the geostrophic term, a cubic nonlinearity appears in the system. Taking the symmetry properties into account, the integer parameter p has to be set $p = 2$ and the increment R–R_c of the Rayleigh number will be fixed by the condition

$$R - R_c = \pm \varepsilon^2 \quad (22)$$

which ensures the supercritical or subcritical character of bifurcation. Here ε has the meaning of a small unfolding parameter. The vector of representing functions $\boldsymbol{\psi}$ as well as the complex amplitude $A(\tau)$ have to be expanded in terms of ε, $(\varepsilon \ll 1)$

$$\boldsymbol{\psi} = \boldsymbol{\psi}_1 + \varepsilon \boldsymbol{\psi}_2 + \varepsilon^2 \boldsymbol{\psi}_3 + \cdots,$$
$$A(\tau) = \varepsilon A_1(\tau) + \varepsilon^2 A_2(\tau) + \varepsilon^3 A_3(\tau) + \cdots, \tag{23}$$

where τ is the slow time associated with the physical time through the relation $\tau = \varepsilon^2 t$.

Inserting the above expansions into (20) and collecting terms of the same power of ε, yields a series of non-homogeneous matrix equations. In order to ensure their solvability, the complex amplitudes A_i, $(i = 1, 2, \ldots)$ must be adjusted at each stage of expansion, giving rise to amplitude equations. At the leading order the final form of amplitude equation reads

$$\frac{\mathrm{d}A(\varepsilon^2 t)}{\mathrm{d}t} = (R - R_c)\,\alpha A(\varepsilon^2 t) - \beta A(\varepsilon^2 t)|A(\tau)|^2, \tag{24}$$

which describes the evolution of the amplitude $A(\tau)$ in the physical time t instead of τ.

The equation quoted above is the normal form for the Hopf bifurcation in $R = R_c$. Stability analysis of this normal form enables us to identify the super- or subcriticality, stability and the frequency response of the convecting system in the vicinity of $R = R_c$. We are able to discuss these properties in terms of the complex coefficients α and β which depend on the parameters m, Λ, E, q as well as on the critical parameters R_c, k_c and σ_c as shown in Appendix A. Motivated by RSB we introduce the following notation

$$R_2 = \frac{\beta_r}{\alpha_r}, \quad \sigma_2 = \alpha_i \frac{\beta_r}{\alpha_r} - \beta_i. \tag{25}$$

The above expressions are directly associated with the Hopf bifurcation properties, namely if $R_2 > 0$ the Hopf bifurcation at R_c is supercritical, otherwise it is subcritical. The sign of α_r causes the change of stability of the solutions in the vicinity of R_c. In case $\sigma_2 > 0$ the frequency response of the nonlinear system is such that frequency of the oscillations grows; in case $\sigma_2 < 0$ frequency of the oscillations decreases (if $\sigma_c > 0$).

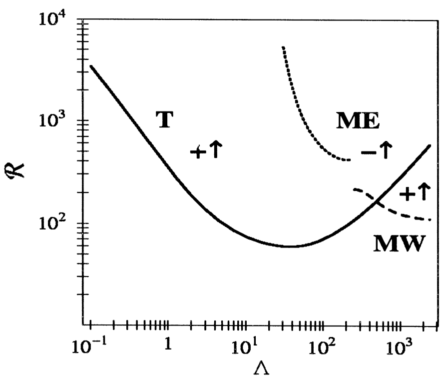


Fig. 1. T, MW and ME modes for $m = 1$, $z_M = 0.6$ and $q = 0.005$.

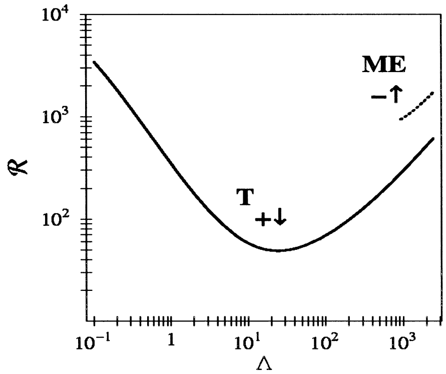


Fig. 2. T and ME modes for $m = 1$, $z_M = 0.6$ and $q = 0.5$.

6. Results

In the numerical experiments to follow the values of the critical Rayleigh number R_c, the critical radial wave number k_c and the critical complex frequency $\lambda_c = i\sigma_c$ were obtained from the linear stability analysis made in S89. We have evaluated the coefficients R_2 and σ_2 numerically for various sets of parameters m, q, Λ.

Namely, we studied two particular cases related to the azimuthal wave numbers $m = 1$ and $m = 2$ with the Elsasser number Λ ranging from 10^{-3} to 2500. The values of the Roberts number were chosen $q = 0.005$ and $q = 0.5$. Note that the choice of the Ekman number is irrelevant in this case. It can be scaled out of the problem when the nonlinearity is only due to geostrophic flow.

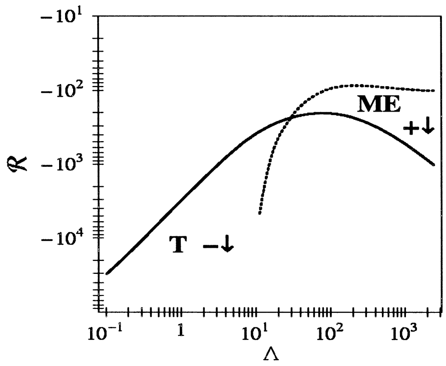


Fig. 3. T and ME modes for $m = 1$, $z_M = 0.4$ and $q = 0.005$.

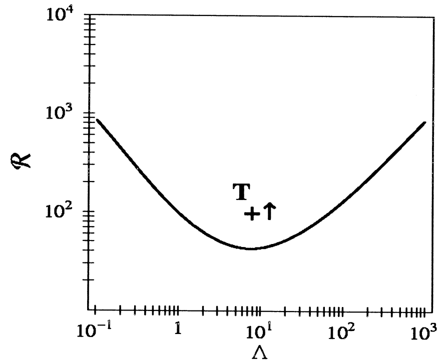


Fig. 5. T modes for $m = 2$, $z_M = 0.6$ and $q = 0.005$.

The figures below are bifurcation diagrams in the space of parameters R (Rayleigh number) and Λ (Elsasser number) where the marked curve $R_c = R_c(\Lambda)$ shows the dependence of the critical Rayleigh number on the Elsasser number for each particular convective mode. Knowing the coefficients α and β of the amplitude equation enables us to classify the domains separated by $R_c = R_c(\Lambda)$. In each of the diagrams, the domains below the curves correspond to trivial conductive solutions whilst the domains above correspond to oscillatory convective solutions.

Here T and MW symbolize thermally and magnetically driven waves propagating westwards, respectively and ME denotes magnetically driven waves propagating eastwards. Other notation adopted here differs from that used in RSB. Hereafter + and − in the diagrams stand for supercritical and subcritical Hopf bifurcation. In both cases the trivial solution loses stability when the parameter R passes its critical value R_c. Recalling properties of the Hopf bifurcation, the arising subcritical and supercritical oscillations are unstable and stable, respectively. It must be emphasized, however, that all of what was said of the stability holds in the case when $\alpha_r > 0$. Analyzing the normal form for $\alpha_r < 0$, we deduce that the stability of the trivial and nontrivial solutions is reversed. Specifically, the case of supercritically bifurcated oscillations which are unstable (only ME modes for $q = 0.5$) will be denoted by $+U$. The arrow symbols ↑ and ↓ in the graphs below denote increase or decrease in frequency of nonlinear convective oscillations.

At this stage, we must realize that also negative values of the Rayleigh number can be considered in the underlying model. This is actually the case when the lower horizontal boundary of the layer is cooled and the upper one is heated. From the physical point

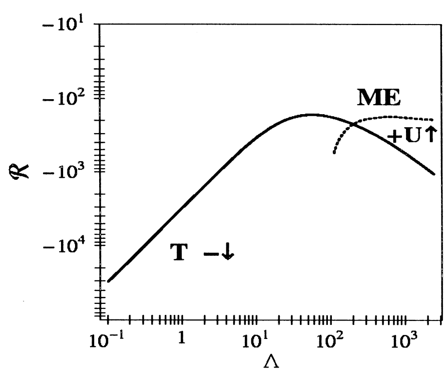


Fig. 4. T and ME modes for $m = 1$, $z_M = 0.4$ and $q = 0.5$.

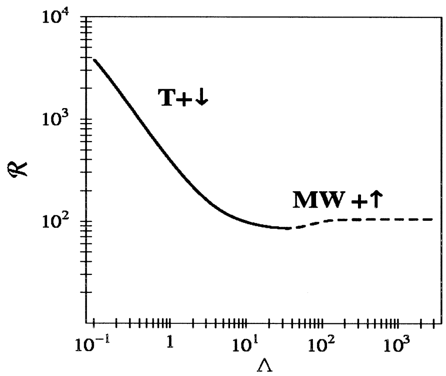


Fig. 6. T and MW modes for $m = 1$, $z_M = 0.9$ and $q = 0.005$.

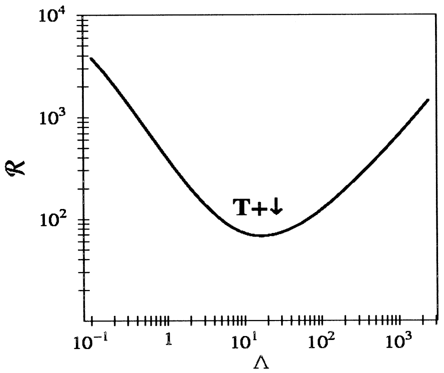


Fig. 7. T modes for $m=1$, $z_M=0.9$ and $q=0.5$.

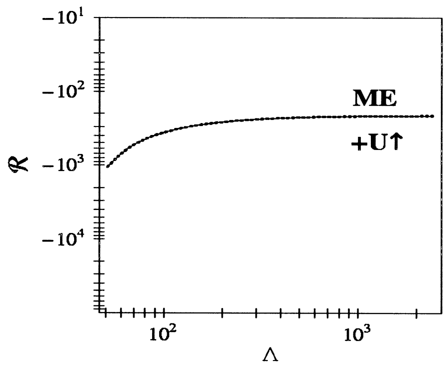


Fig. 9. ME modes for $m=1$, $z_M=0.1$ and $q=0.5$.

of view only the absolute value of the Rayleigh number is of relevance as it is directly related to the energy input into the system. Realizing this fact in what follows, we classify the type of bifurcation (supercritical or subcritical) with respect to the absolute value of the Rayleigh number.

Figs. 1–5 are bifurcation diagrams where the stratification parameter was set $z_M=0.6$ and $z_M=0.4$. This choice of z_M relates to the cases of positive and negative Rayleigh number, respectively. The weakly nonlinear behaviour of particular kinds of modes shows some characteristic features. In Figs. 1–5, it can be seen that the value of dimensionless stratification parameter z_M, measuring the thickness of unstably stratified sublayer, is directly related to the sub- or supercriticality of the T, MW and ME modes. Typically, the T modes bifurcate supercritically and the ME modes bifurcate subcritically for $z_M=0.6$, i.e., when thickness of the unstably stratified sublayer is larger than that of the stably stratified sublayer. On the other hand, for $z_M=0.4$ the T modes appear to be subcritical and the ME modes are supercritical.

The same applies for different configuration of stratification when z_M was chosen $z_M=0.9$ or $z_M=0.1$, as it is presented in Figs. 6–10. This choice of z_M means that the unstably and stably stratified sublayers become more distinct from each other as for their thickness. That is why only one kind of the convective oscillatory mode was isolated for each particular stratification. As for Figs. 8 and 9, only ME modes are depicted. Here, the T modes are off the scale due to the high Rayleigh number R. For $m=1$, $z_M=0.9$ and $q=0.005$ (see Fig. 1) an observation has been made in the linear study that at $\Lambda \sim 50$ the T mode is continuously transformed into MW mode.

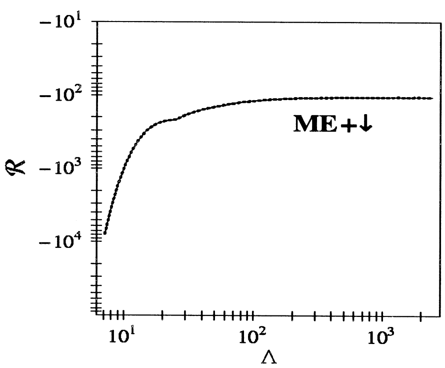


Fig. 8. ME modes for $m=1$, $z_M=0.1$ and $q=0.005$.

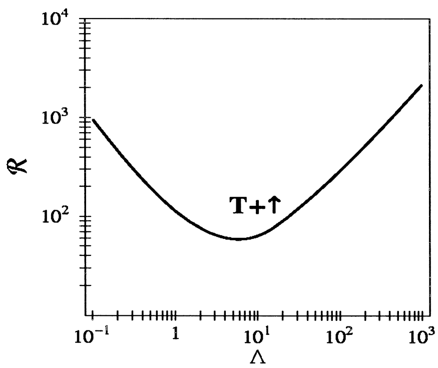


Fig. 10. T modes for $m=2$, $z_M=0.9$ and $q=0.005$.

Table 1
The values of the functional $\mathscr{Z}$ for T modes for $m=1$, $q=0.005$ and various Λ and z_M

	$z_M=0.1$	$z_M=0.4$	$z_M=0.6$	$z_M=0.9$
$\Lambda=100$	0.0245	0.13	0.019	0.0128
$\Lambda=500$	0.0056	0.023	0.0243	0.0016
$\Lambda=1000$	0.0031	0.029	0.0199	0.0032
$\Lambda=2500$	0.0024	0.043	0.032	0.0025

Realizing that frequency of the nonlinear convective oscillations changes at $\Lambda \sim 50$, the weakly nonlinear analysis is capable of identifying this interface as well. It is also remarkable that for $m=1$, $q=0.5$ and for $z_M=0.4$ or $z_M=0.1$ the supercritical oscillations corresponding to ME modes were found to be unstable (the domain below the dotted curve in Figs. 4 and 9). This was the only case when the unstable supercritical convection was observed in our magnetoconvection model.

In Table 1 we show the dependence of the functional $\mathscr{Z}$ on the Elssaser number Λ and the stratification parameter z_M. Recall that $\mathscr{Z}$ enters the expression (15) for the geostrophic flow $\Omega(s)$, i.e., it can be thought as the amplitude of $\Omega(s)$. It turned out that the dependence of $\mathscr{Z}$ on Λ is non-monotonous. More interestingly, for fixed Λ, the factor $\mathscr{Z}$ exhibits local maximum as a function of z_M located near $z_M=0.5$.

7. Concluding remarks

In this paper we studied the weakly nonlinear effect of geostrophic flow on the marginal convection in the non-uniformly stratified horizontal fluid layer. Under the assumption of weak boundedness of the layer, the analysis presented here is based on the data available for the unbounded linear version of the model. We found out that the convective instability in our model sets in via Hopf bifurcation and classified its properties.

For the azimuthal wave number $m=1$, it is apparent from the bifurcation diagrams that the weakly nonlinear behaviour of T and ME modes under the action of geostrophic flow is different while T and MW modes are not distinguished from each other. The global observation says that, for each considered value q, Λ, z_M and for $R>0$, the T modes bifurcate supercritically and the ME modes bifurcate subcritically. On the other hand, for $R<0$ the bifurcations corresponding to T and ME modes are subcritical and supercritical, respectively. Varying the values of the parameters q, Λ, z_M may only cause changes in stability of solutions or change in frequency response.

Two interesting features were isolated for particular modes in certain parametric regimes. Firstly, when $q=0.005$ and stably stratified sublayer is thin enough as it is expressed in terms of stratification parameter equal $z_M=0.9$, the continuous transition between T and MW modes occur. This phenomenon, known from linear study of Soward (1979) and S89, was observed also in our nonlinear problem as a change of frequency of instabilities. Secondly, for $q=0.5$ and the stratification parameter $z_M=0.4$, the ME modes though being preferred to T modes at the linear stage, were identified as unstable supercritically bifurcated ones. Irrespective of the choice of parameters, the convective oscillatory modes with the azimuthal wave number $m=2$, which are always thermally driven, set in via supercritical bifurcation and their frequency grows.

We also found that respect to the choice of the stratification parameter z_M, the maximal amplitude of the geostrophic flow can be expected for the stratification characterised by $z_M=0.5$, i.e., when the stably and unstably stratified sublayers have the same thickness.

In the following, we comment briefly on some mathematical aspects of our analysis. Inserting the perturbation expansions into the modified Taylor constraint, the resulting formula for geostrophic flow $\Omega(s)$ gains quite a simple form (Eq. (15)) which is usable for analytical calculations. Moreover, the structure of the expression for $\Omega(s)$ implies that in this nonlinear problem there is no interaction of oscillatory modes with different azimuthal wave numbers m. Note also that growing the radial extension of the layer, measured in terms of s_n, makes only the amplitude of perturbations vanish, having no impact on the bifurcation properties. This fact emerges from the assumption of weak boundedness of the layer.

Another notable feature is that the Hopf bifurcation is a direct consequence of symmetry of the governing equations which is due to the presence of cubic nonlinearities. Therefore, the same type of bifurcation would appear in spherical geometry where more realistic problem of this kind could be formulated.

Acknowledgements

Financial support from the Scientific Grant Agency VEGA under grant No. 1/4324/97 is acknowledged. The second author was partially supported by VEGA grant 1/4190/97 and by the Swiss National Science Foundation under Project No. 7IP 051638.

Appendix A

The normal form Eq. (24) coefficients α and β are

$$\alpha = -k_c^2 \frac{\langle \vartheta(z)\,\omega^+(z)\rangle^z}{M},$$

$$\beta = 4\mathscr{Z}\frac{I_2}{I_1}\frac{\langle Db(z)\,j^+(z)\rangle^z}{M}$$

where

$$M = \langle b(z)\,b^+(z)\rangle^z + \langle j(z)\,j^+(z)\rangle^z + (1/q)\langle \vartheta(z)\,\vartheta^+(z)\rangle^z$$

and $\mathscr{Z}$ is a functional defined by Eq. (15).

The bracketed terms denote the integrals over the z coordinate

$$\langle f(z)\overline{f^+(z)}\rangle^z = \int_{z_B}^{z_T} f(z)\overline{f^+(z)}\,\mathrm{d}z$$

where the functions $f^+(z)$ solve the corresponding adjoint problem.

The coefficient β involves the integrals over the radial coordinate

$$I_1 = \int_0^{s_n} J_m^2(k_c s)\,s\,\mathrm{d}s,$$

$$I_2 = \int_0^{s_n} J_m^2(k_c s)\left(\frac{d}{\mathrm{d}s}J_m(k_c s)\right)^2 s\,\mathrm{d}s.$$

Being positive for each choice of s_n and irrespective of k_c, these integrals do not affect the properties of the Hopf bifurcation and their ratio $I_2/I_1 \to 0$ as $s_n \to \infty$. The consequence of this asymptotics is that the amplitude of solution decreases as s_n becomes larger, as would be naturally expected from configuration of the model.

References

Bod'a, J., 1988. Thermal and magnetically driven instabilities in a non-constantly stratified fluid layer. Geophys. Astrophys. Fluid Dynamics 44, 77–90.

Brestenský, J., Ševčík, S., 1994. Mean electromotive force due to magnetoconvection in rotating horizontal layer with rigid boundaries. In: Proctor, M.R.E. (Ed.), Geodynamo Modelling. Geophys. Astrophys. Fluid Dynamics, Vol. 77, No. 1–4, pp. 191–208.

Brestenský, J., Revallo, M., Ševčovič, D., 1997. Finite amplitude magnetoconvection determined by modified Taylor's constraint. Acta Astron. et Geophys., Univ. Comenianae 18, 1–18.

Fearn, D.R., 1994. Nonlinear planetary dynamos. In: Proctor, M.R.E., Gilbert, A.D. (Eds.), Lectures on Solar and Planetary Dynamos. Cambridge University Press.

Revallo, M., Ševčovič, D., Brestenský, J., 1997. Analysis of the model of magnetoconvection with nonlinearity due to modified Taylor's constraint. In: Brestenský, J., Ševčík, S. (Eds.), Stellar and Planetary Magnetoconvection (Bratislava 1996). Acta Astron. et Geophys. Univ. Comenianae, Vol. 19, pp. 317–335.

Ševčík, S., 1989. Thermal and magnetically driven instabilities in a non-constantly stratified rapidly rotating fluid layer with azimuthal magnetic field. In: Fearn, D.R., Proctor, M.R.E. (Eds.), Geodynamo Modelling. Geophys. Astrophys. Fluid Dynamics, Vol. 49, pp. 195–211.

Šimkanin, J., Brestenský, J., Ševčík, S., 1997. Dependence of rotating magnetoconvection in horizontal layer on boundary conditions and stratification. In: Brestenský, J., Ševčík, S. (Eds.), Stellar and Planetary Magnetoconvection (Bratislava 1996). Acta Astron. et Geophys., Univ. Comenianae, Vol. 19, pp. 195–220.

Skinner, P.H., Soward, A.M., 1988. Convection in a rotating magnetic system and Taylor's constraint. Geophys. Astrophys. Fluid Dynamics 44, 91–116.

Skinner, P.H., Soward, A.M., 1991. Convection in a rotating magnetic system and Taylor's constraint II. Geophys. Astrophys. Fluid Dynamics 60, 335–356.

Soward, A.M., 1979. Thermal and magnetically driven convection in a rapidly rotating fluid layer. J. Fluid Mech. 90, 669–684.

Taylor, J.B., 1963. The magnetohydrodynamics of a rotating fluid and the Earth's dynamo problem. Proc. R. Soc. London A 274, 274–283.

Príloha 6.3.2

Reprint práce:

M. Revallo, D. Ševčovič, J. Brestenský: *Analysis of the model o magnetoconvection with nonlinearity due to modified Taylor's constraint.* Acta Astron. et Geophys. Univ. Comenianae 19 (1997) 317-336.

Analysis of the model of magnetoconvection with nonlinearity due to modified Taylor's constraint

M. Revallo[1], D. Ševčovič[2] and J. Brestenský[1]

[1] *Department of Geophysics, Faculty of Mathematics and Physics, Comenius University, 842 15 Bratislava, Slovak Republic*

[2] *Institute of Applied Mathematics, Faculty of Mathematics and Physics, Comenius University, 842 15 Bratislava, Slovak Republic*

Abstract. A problem of magnetoconvection is considered where the nonlinear effect of geostrophic flow determined by Ekman suction is included. Perturbation techniques are adopted in order to construct slowly varying periodic solution branching from the steady state conductive solution. The analysis is also used to determine the relevant bifurcation structure in the vicinity of the critical Rayleigh number.

Key words: Magnetoconvection, modified Taylor's constraint, perturbation techniques, weakly nonlinear analysis

1. Introduction

The fluid motion in Earth-like planet cores can be characterized by magnetostrophic approximation with dominating Lorentz, Coriolis, buoyancy and pressure forces in the equation of motion. The approximation with zero viscous forces has a solution, only if so-called Taylor's constraint is satisfied (see Section 2). A specific problem arises when magnetostrophic approximation holds but small viscous forces in the Ekman boundary layers are present. In this case a non-zero geostrophic flow is induced by the viscous flow in thin Ekman layers and nonlinear dynamics of the whole magnetoconvecting system is affected through the so-called Ekman suction mechanism.

The question is, if such a nonlinear viscous system, which reflects more realistically conditions in the Earth's core, could possibly evolve into the Taylor state. At this particular state, viscous forces have no longer major influence on the dynamics and Taylor's condition is met. The problem of possible achievement of the Taylor state has been studied in simpler planar or cylindrical and also in spherical geometry for both kinematic dynamos and magnetoconvection

Received: March 9, 1997; Accepted: June 10, 1997
Acta Astron. et Geophys. Univ. Comenianae **XIX**, (1997), 317 – 336.

models. It has been shown by Fearn, Proctor and Sellar (1994) that some specific simplifications can be made in the case of magnetoconvection models. Namely, non-axisymmetric instabilities of magnetic field only have to be considered for computation of geostrophic flow, whereas contributions from basic axisymmetric magnetic fields can be neglected (see Section 2).

In this paper we study a problem of finite amplitude rotating magnetoconvection affected by Ekman suction. The investigation has been motivated by the linear stability analysis developed by Soward (1979), (see also Brestenský and Ševčík 1994, 1995, and Šimkanin et al 1997 in this Issue) as well as the nonlinear problem studied in Skinner and Soward (1988, 1991). In contrast to the approach applied in the nonlinear study done by Skinner and Soward (1988, 1991), the purpose of the present paper is to study state of magnetoconvection near the critical Rayleigh number R_c.

The methods and techniques of this paper are based on the regular perturbation theory, linear and nonlinear functional analysis and bifurcation theory. The main idea is to expand a solution into power series in terms of a small unfolding parameter ε corresponding to the small increase in the Rayleigh number beyond its critical value R_c. Let us emphasize that this approach can describe local bifurcation structure near R_c only.

The underlying geometry is a weakly bounded cylinder, i.e. the cylinder with a radius strongly exceeding its height. It can sufficiently approximate the laterally unbounded geometry used in the linear study (Soward 1979). We must emphasize that the finite extension in the radial direction is a crucial assumption of the theory. The reason for dealing with the bounded geometry is twofold. Firstly, it enables us to set up suitable function spaces and operators we will work with. Secondly, as a consequence of the boundedness of the cylinder, the third order approximation of the power series expansion is capable of describing the Hopf bifurcation phenomenon in the amplitude equation (51) in Section 3.3. On the other hand, the main disadvantage of this approach is that we have to set up boundary conditions on vertical boundaries of the cylinder. In this paper we consider the simplest case of Dirichlet boundary conditions which seem to be less physically meaningful. The more realistic boundary conditions will be treated in the forthcoming paper.

The outline of this paper is as follows. In Section 2 we derive a system of nonlinear PDE's governing the motion periodic in both time and the azimuthal variable. Section 3.1 is devoted to the study of the constructed system of nonlinear equations. We present a method on how to obtain a power series expansion of a solution in terms of a small unfolding parameter. Using the so-called solvability condition known from Fredholm's alternative in the functional analysis, we determine leading coefficients of the expansions in Section 3.2. In Section 3.3 we sketch a procedure how to derive an ordinary differential equation for the time dependent amplitude. Numerical results are reported in Section 4. In the Appendix we present formulae for the leading terms in the power series expansions.

2. Formulation of the nonlinear problem

2.1. Basic leading equations

The aim of this paper is a local stability analysis of a nonlinear system of PDE's governing a specified model of magnetoconvection.

The model considered is an infinite horizontal layer of width d rotating rapidly with angular velocity $\Omega_0\hat{\mathbf{z}}$. The layer contains an electrically conducting Boussinesq fluid permeated by an azimuthal magnetic field linearly growing with the distance from the vertical rotation axis. An unstable temperature gradient is maintained by heating the fluid from below and cooling from above. The fluid layer is supposed to have free perfectly electrically and thermally conductive horizontal boundaries.

The convective instability in this rotating system is caused by the vertical temperature gradient and manifests itself by perturbations of the velocity $\mathbf{u}$, the magnetic field $\mathbf{b}$ and the temperature ϑ which refer to the basic state represented by $\mathbf{U}_0$, $\mathbf{B}_0$, T_0 .

In this paper, we investigate the existence of periodic solution for these perturbations in the vicinity of the basic state determined by

$$\mathbf{U}_0 = \mathbf{0}\,, \quad \mathbf{B}_0 = B_M \frac{s}{d}\,\hat{\boldsymbol{\varphi}}\,, \quad T_0 = T_1 - \frac{\Delta T}{d}\,(z + \frac{d}{2})\,. \tag{1}$$

We non-dimensionalise the problem with the use of characteristic length d , magnetic diffusion time d^2/η , magnetic field B_M , and temperature difference across the layer ΔT . In the cylindrical polar coordinates $(s,\ \varphi,\ z)$ the equations governing the evolution of perturbations $\mathbf{u}$, $\mathbf{b}$, $\widetilde{\vartheta}$ of the basic state gain the following form

$$\hat{\mathbf{z}} \times \mathbf{u} = -\nabla p + \Lambda\,[\,(\,\nabla \times s\,\hat{\boldsymbol{\varphi}}) \times \mathbf{b} + (\,\nabla \times \mathbf{b}\,) \times s\,\hat{\boldsymbol{\varphi}}] + R\,\vartheta\,\hat{\mathbf{z}}\,, \tag{2}$$

$$\frac{\partial \mathbf{b}}{\partial t} - \nabla \times (\,s\,\Omega(s)\,\hat{\boldsymbol{\varphi}} \times \mathbf{b}\,) = \nabla \times (\,\mathbf{u} \times s\,\hat{\boldsymbol{\varphi}}\,) + \nabla^2 \mathbf{b}\,, \tag{3}$$

$$\frac{1}{q_R}\left(\frac{\partial \widetilde{\vartheta}}{\partial t} + (\,s\,\Omega(s)\,\hat{\boldsymbol{\varphi}} \cdot \nabla\,)\,\widetilde{\vartheta}\right) = -\,\mathbf{u} \cdot \nabla T_0 + \nabla^2 \widetilde{\vartheta}\,, \tag{4}$$

$$\nabla \cdot \mathbf{b} = 0\,, \tag{5}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{6}$$

where $\hat{\boldsymbol{\varphi}}$ and $\hat{\mathbf{z}}$ are the unit azimuthal and axial vectors, respectively. The dimensionless parameters, the modified Rayleigh number R , the Elsasser number Λ , the Ekman number E and the Roberts number q_R , are defined by

$$R = \frac{gd\Delta T\alpha}{2\Omega_0\kappa}\,, \quad \Lambda = \frac{B_M^2}{2\Omega_0\rho_0\eta\mu}\,, \quad E = \frac{\nu}{2d^2\Omega_0}\,, \quad q_R = \frac{\kappa}{\eta}$$

where κ and η are the thermal and magnetic diffusivities, ν is the kinematic viscosity, α is the coefficient of thermal expansion, g is the acceleration due to gravity, μ is the permeability and ρ_0 is the density.

The model of magnetoconvection includes the effect of Ekman suction which is associated with a nontrivial geostrophic flow. This gives rise to the presence of nonlinear terms encountered in the above differential equations, namely in (3) and (4). It is known that geostrophic flow can be expressed via so-called modified Taylor's constraint (see Fearn 1994).

Let $\langle \dots \rangle^\varphi \equiv 1/(2\pi) \int_0^{2\pi} \dots d\varphi$ be averaging over the azimuthal component φ. Denote by $F_{M\varphi} \equiv [(\nabla \times \mathbf{B}) \times \mathbf{B}]_\varphi$ the azimuthal component of Lorentz force. Then splitting magnetic field $\mathbf{B}$ on basic field $\mathbf{B}_0$ and perturbation $\mathbf{b}$, $\mathbf{B} \equiv \mathbf{B}_0 + \mathbf{b}$ ($\langle \mathbf{B} \rangle^\varphi = \mathbf{B}_0,\ \langle \mathbf{b} \rangle^\varphi = \mathbf{0}$), the angular velocity $\Omega(s)$ of geostrophic flow in our magnetoconvection model can be expressed in terms of the magnetic field perturbation $\mathbf{b}$, i.e. (see e.g. Skinner and Soward 1988)

$$\Omega(s) = \frac{\Lambda}{(2E)^{1/2} s} \int_{z_B}^{z_T} \langle F_{M\varphi} \rangle^\varphi \, dz \quad \text{with} \quad \langle F_{M\varphi} \rangle^\varphi = \langle [(\nabla \times \mathbf{b}) \times \mathbf{b}]_\varphi \rangle^\varphi . \tag{7}$$

It is significant for the model under consideration that the possible contribution $\langle [(\nabla \times \mathbf{B}_0) \times \mathbf{B}_0]_\varphi \rangle^\varphi$ from basic field to azimuthally averaged Lorentz force $\langle F_{M\varphi} \rangle^\varphi$ vanishes (see also Fearn, Proctor and Sellar 1994). We note that the expression (7) is well-known as modified Taylor's constraint.

The vector nonlinear equations (2 - 6) together with the expression for geostrophic flow (7) seem rather complicated to be solved analytically. We therefore restrict solutions to a smaller phase space of functions having special structure. Roughly speaking, the main idea is to express all the vector fields in terms of their scalar representing functions which are supposed to have a form of travelling waves, as it is described below.

We split the velocity perturbation $\mathbf{u}$ as well as the magnetic field perturbation $\mathbf{b}$ into their poloidal and toroidal parts

$$\mathbf{u} \;=\; k^{-2}(\nabla \times (\nabla \times \widetilde{w}\,\hat{\mathbf{z}}) + \nabla \times \widetilde{\omega}\,\hat{\mathbf{z}}) , \tag{8}$$

$$\mathbf{b} \;=\; k^{-2}(\nabla \times (\nabla \times \widetilde{b}\,\hat{\mathbf{z}}) + \nabla \times \widetilde{j}\,\hat{\mathbf{z}}) . \tag{9}$$

Similarly as in the papers Brestenský and Ševčík (1994) and Brestenský, Revallo and Ševčovič (1997)[1] we have adopted the tilde notation for representing poloidal and toroidal functions as well as for thermal function. Each of the representing functions $\widetilde{w}$, $\widetilde{\omega}$, $\widetilde{b}$, $\widetilde{j}$, $\widetilde{\vartheta}$ (all symbolized as $\widetilde{f}$) depends on coordinates z, s, φ and time t.

Suppose that the representing functions $\widetilde{f}$ can be decomposed as

$$\tilde{f}(z, s, \varphi, t) = \Re e\{ f_m(z, s) \exp(im\varphi + \lambda t)\} \tag{10}$$

[1]Henceforth the abbreviations (BS) and (BRS) will be used.

where the functions of $f_m(z,s)$, i.e. $b_m(z,s)$, $j_m(z,s)$, $w_m(z,s)$, $\omega_m(z,s)$ and $\vartheta_m(z,s)$ depend on vertical and radial coordinates z and s . Here m is an integer azimuthal wave number, k is a real radial wave number and λ is a complex frequency related to a real frequency via $\lambda = i\sigma$.

Inserting the above ansatz into the governing equations for perturbations (2-6) and into the expression for modified Taylor's constraint enables us to set up a system of nonlinear equations for representing functions $f_m(z,s)$. The resulting nonlinear system is well posed on a suitable function space as it has been yet shown in (BRS). Hereafter, this system of equations will be referred to as an abstract nonlinear problem.

2.2. Abstract nonlinear problem

The procedure leading towards the abstract nonlinear problem presented below is straightforward but rather technically tedious. It is discussed in a more detail in (BRS).

The equations for the representing functions $f_m(z,s)$ can be finally written as follows

$$\begin{aligned}
0 \;&=\; -\,Dw_m(z,s) + 2\Lambda\, Db_m(z,s) - im\Lambda\, j_m(z,s)\,,\\
0 \;&=\; -\,D\omega_m(z,s) + 2\Lambda\, Dj_m(z,s) + im\Lambda\,(D^2 - k^2\,\mathcal{J}_m)\, b_m(z,s) - Rk^2\,\vartheta_m(z,s)\,,\\
\lambda\, b_m(z,s) + P_m(z,s) \;&=\; im\, w_m(z,s) + (D^2 - k^2\,\mathcal{J}_m)\, b_m(z,s)\,,\\
\lambda\, j_m(z,s) + T_m(z,s) \;&=\; im\,\omega_m(z,s) + (D^2 - k^2\,\mathcal{J}_m)\, j_m(z,s)\,,\\
(1/q_R)\;(\,\lambda\,\vartheta_m(z,s) + S_m(z,s)\,) \;&=\; \mathcal{J}_m\, w_m(z,s) + (D^2 - k^2\,\mathcal{J}_m)\,\vartheta_m(z,s)
\end{aligned} \tag{11}$$

where the nonlinearities $P_m(z,s)$, $T_m(z,s)$ and $S_m(z,s)$ are expressed in terms of $f_m(z,s)$ and the angular velocity $\Omega(s)$ of geostrophic flow as follows

$$\begin{aligned}
P_m(z,s) &= im\,\Omega(s)\, b_m(z,s) \;-\; im\,\mathcal{J}_m^{-1}\,\{\mathcal{P}_\Omega\; b_m(z,s)\,\}\,,\\
T_m(z,s) &= im\,\Omega(s)\, j_m(z,s) \;+\; \mathcal{J}_m^{-1}\,\{\mathcal{T}_\Omega\; D\, b_m(z,s)\,\}\,,\\
S_m(z,s) &= im\,\Omega(s)\,\vartheta_m(z,s)\,.
\end{aligned} \tag{12}$$

Here $D = \partial/\partial z$ and $\mathcal{J}_m^{-1}$ is the inverse operator to the linear Bessel differential operator $\mathcal{J}_m$. The operator $\mathcal{J}_m$ is defined as

$$\mathcal{J}_m \equiv\; -\frac{1}{k^2}\,\left(\frac{\partial^2}{\partial s^2} + \frac{1}{s}\,\frac{\partial}{\partial s} - \frac{m^2}{s^2}\right) \tag{13}$$

and for the Bessel function $J_m(ks)$ it has a useful property $\mathcal{J}_m\,\{J_m(ks)\} = J_m(ks)$. Furthermore, $\mathcal{P}_\Omega$, $\mathcal{T}_\Omega$ are differential operators

$$\mathcal{P}_\Omega = -\frac{1}{k^2}\,\left\{\frac{\partial^2\Omega(s)}{\partial s^2} + \frac{\partial\Omega(s)}{\partial s}\,\left[2\,\frac{\partial}{\partial s} + \frac{1}{s}\right]\right\}\,, \tag{14}$$

$$\mathcal{T}_\Omega = -\frac{1}{k^2}\,\left\{s\,\frac{\partial^2\Omega(s)}{\partial s^2}\,\frac{\partial}{\partial s} + s\,\frac{\partial\Omega(s)}{\partial s}\,\left[\frac{m^2}{s^2} + \frac{2}{s}\,\frac{\partial}{\partial s} + \frac{\partial^2}{\partial s^2}\right]\right\} \tag{15}$$

where the partial derivative $\partial/\partial s$ reflects the fact of $\Omega(s)$ being a functional (see below). The interested reader is referred to (BRS) for the complete derivation of the above system of nonlinear PDE's. Furthermore, it has been shown in Appendix of (BRS) that $\mathcal{J}_m^{-1}$ is a well defined bounded linear operator on a suitable function space. We notice that the above expressions for P_m and T_m in (12) emerge after decomposition of the vector nonlinearity in induction equation (3) into poloidal and toroidal fields. The expression for S_m represents the scalar nonlinearity in the heat equation (4).

The geostrophic flow $\Omega(s)$ entering the set of equations (12) is given by formula which can be directly obtained by inserting (9) together with the ansatz (10) into (7). A straightforward series of calculations yields

$$\Omega(s) = \frac{\Lambda}{2\,(2E)^{1/2}}\,\frac{1}{s}\cdot\Re e\left\{\frac{1}{s^2}\frac{\partial}{\partial s}\left[\,s^2\,I(s)\,\right] - B(s)\right\} \tag{16}$$

where

$$I(s) = \frac{1}{k^4}\int_{z_B}^{z_T}\left(\frac{m^2}{s^2}\,j_m(z,s)D\overline{b_m(z,s)} \;-\; \frac{\partial}{\partial s}j_m(z,s)\,\frac{\partial}{\partial s}D\overline{b_m(z,s)}\right)dz$$

is the integral part and

$$B(s) = \frac{1}{k^2}\,\frac{\partial}{\partial s}j_m(z,s)\;\mathcal{J}_m\overline{b_m(z,s)}\,\Big|_{z_B}^{z_T}$$

is the boundary term. Here $D = \partial/\partial z$ and an overbar denotes the complex conjugation of $b_m(z,s)$.

It is remarkable that the complex conjugation in the expression for geostrophic flow $\Omega(s)$ eliminates exponentials of the tilded representing functions $\tilde{b}(z,s,\varphi,t)$ and $\tilde{j}(z,s,\varphi,t)$. Therefore upon assumption (10), the expression for $\Omega(s)$ does not involve the variables φ, t and is entered by simpler functions $b_m(z,s)$ and $j_m(z,s)$ only. This is the important fact which approves the choice of $f_m(z,s)$ as representing functions for our nonlinear problem. At this stage it is yet easy to see that $P_m(z,s)$, $T_m(z,s)$ and $S_m(z,s)$ are cubic nonlinearities in $f_m(z,s)$.

For the special case of infinitely electrically and thermally conducting horizontal boundaries and vanishing viscosity[2] the following boundary conditions have to be satisfied

$$w_m(z,s) = \vartheta_m(z,s) = b_m(z,s) = Dj_m(z,s) = 0\,, \qquad \text{for all } z = z_B,\ z_T,\ \text{and } s\in(0,s_n)\,. \tag{17}$$

Notice that the above choice of boundary conditions makes the boundary term in expression (16) vanish.

[2]Recall that viscosity in our model is to be taken non-zero only within the Ekman layers along the horizontal boundaries. It is actually the viscous flow in the Ekman layers which is responsible for Ekman suction and geostrophic flow given by (16).

In a radial direction we impose the following boundary conditions

$$w_m(z,s) = \vartheta_m(z,s) = b_m(z,s) = j_m(z,s) = 0\,,\\ \text{for all } s = 0,\ s_n,\ \text{and } z \in (z_B, z_T)\,. \tag{18}$$

Here and after s_n, which delimites the layer in a radial direction, will always stand for the n-th root of the scaled Bessel function $J_m(ks)$, i.e.

$$J_m(ks_n) = 0 \quad \text{for all } n = 1, 2, \dots\,. \tag{19}$$

Notice that the Dirichlet-like boundary conditions (18) for the representing functions have been set up especially due to mathematical purposes. It should be emphasized again that in our approach the bounded geometry is needed in order to apply some functional analytical results. Roughly speaking, the choice of boundary conditions (18) enables us to guarantee the existence of the inverse operator $\mathcal{J}_m^{-1}$ and, as a consequence, to justify the definitions of the cubic nonlinearities $P_m(z,s)$, $T_m(z,s)$ introduced in (12).

Given a parameter $k > 0$, in our case from the linear stability study for the unbounded geometry, we are forced to restrict ourselves to a certain set of possible radii of the underlying cylinder. Namely, these radii must meet the condition (19).

The relation (19) represents itself a kind of a duality for the choice of the pair $(k,\ s)$; 1) either we firstly fix k and subsequently restrict the radial extension to s_n, or 2) we prescribe the radius, say S, first and then we find a set of possible values of k's satisfying the relation $J_m(k_n S) = 0$. Although both approaches are beneficial, in this paper we discuss the first approach only.

We also notice that in the approach 1) the minimisation of $R(k)$ leading to the critical R_c and k_c is performed over a continuum of values of k whereas in the approach 2) minimisation is performed over a discrete set of k's. Finally, we remark that the discrete set of k's is asymptotically dense in $(0, \infty)$ as $S \to \infty$. Therefore, for large values of the radius S, both approaches appear to be the same from numerical point of view.

3. Solution of abstract nonlinear problem by perturbation methods

3.1. Properties of the adjoint operator

In this section we recall derivation of the so called solvability condition made in (BRS). The computations to follow are based on methods of the functional analysis, namely on the Fredholm alternative argument which is applicable to linear operators on Hilbert spaces. In this paper we will not report all the relevant mathematics except of some remarks on the choice of function spaces setting.

Following the idea of a matrix representation (see e.g. Proctor and Weiss 1982) we rewrite the linear part of equations (11) in the matrix form

$$\mathcal{L} \equiv \begin{pmatrix} -D & 0 & 2\Lambda\, D & -im\Lambda & 0 \\ 0 & -D & im\Lambda\mathcal{D}^2 & 2\Lambda D & -R_c k^2 \\ im & 0 & (\mathcal{D}^2-\lambda_c) & 0 & 0 \\ 0 & im & 0 & (\mathcal{D}^2-\lambda_c) & 0 \\ \mathcal{J}_m & 0 & 0 & 0 & (\mathcal{D}^2-\lambda_c/q_R) \end{pmatrix} \tag{20}$$

where $\mathcal{D}^2 = D^2 - k^2\mathcal{J}_m$. Thus the linear part of (11) has the form $\mathcal{L}\psi$ where ψ is a vector function

$$\psi(z,s) \equiv \, (w_m(z,s),\, \omega_m(z,s),\, b_m(z,s),\, j_m(z,s),\, \vartheta_m(z,s))^T \, .$$

The linear kernel problem, i.e. the homogeneous matrix equation

$$\mathcal{L}\,\psi = 0 \tag{21}$$

has been studied in Soward (1979) where the critical values of Rayleigh number R_c , the complex frequency $\lambda_c = i\sigma_c$ as well as the solution ψ have been found.

The full nonlinear problem (11) can be rewritten as

$$\mathcal{L}\,\psi = N(\psi) \tag{22}$$

where the term $N(\psi)$ contains all the nonlinearities $P_m(z,s)$, $T_m(z,s)$, $S_m(z,s)$ involved in (11).

At this stage it is worthwhile noting that the nonlinear problem (11) has an important symmetry, i.e. the vector function $\psi = (w_m, \omega_m, b_m, j_m, \vartheta_m)^T$ solves (11) if and only if $-\psi$ does. This is based upon the useful property of the nonlinearities $P_m(z,s)$, $T_m(z,s)$ and $S_m(z,s)$ being cubic in representing functions $f_m(z,s)$.

To solve the above semilinear problem by means of the functional analysis we have to find the kernel of the corresponding adjoint operator $\mathcal{L}^+$, i.e. a solution ψ^+ of the adjoint linear equation

$$\mathcal{L}^+\psi^+ = 0\,. \tag{23}$$

A solution of the above problem will be taken for as so-called test function in order to determine higher order terms in power series expansion for a solution ψ of (22).

We define a bilinear form $\langle\, .\, |\, .\, \rangle$ as follows

$$\langle\, \psi \,|\, \chi\, \rangle = \langle\, \psi\, \overline{\chi}\, \rangle^{zs} \equiv \sum \int_G f(z,s)\, \overline{g(z,s)}\; s\, ds\, dz \tag{24}$$

where $\sum$ denotes the summation over all components f and g of vectors ψ and χ , respectively. Here G_n is a bounded domain of the vertical and radial variable, $G_n = (z_B, z_T) \times (0, s_n)$.

Now we are in a position to define an adjoint operator to $\mathcal{L}$ with respect to the inner product $\langle\,.\,|\,.\,\rangle$. The adjoint linear operator $\mathcal{L}^+$ is completely determined by the relation

$$\langle\,\mathcal{L}\,\psi\,|\,\psi^+\,\rangle = \langle\,\psi\,|\,\mathcal{L}^+\,\psi^+\,\rangle \qquad \text{for all } \psi \in X\,,\ \psi^+ \in X^+ \tag{25}$$

where X and X^+ are domains of definitions of the linear operators $\mathcal{L}$ and $\mathcal{L}^+$, respectively. Applying Green's formula on $\langle\,\mathcal{L}\,\psi\,|\,\psi^+\,\rangle$ yields

$$\langle\,\mathcal{L}\,\psi\,|\,\psi^+\,\rangle = \langle\,\psi\,|\,\mathcal{L}^+\,\psi^+\,\rangle + \mathcal{B} \tag{26}$$

where $\mathcal{B}$ is a boundary term. With the use of (26) it can be shown that the matrix linear operator

$$\mathcal{L}^+ = \begin{pmatrix} D & 0 & -im & 0 & \mathcal{J}_m \\ 0 & D & 0 & -im & 0 \\ -2\Lambda\,D & -im\Lambda\mathcal{D}^2 & (\mathcal{D}^2+\lambda_c) & 0 & 0 \\ im\Lambda & -2\Lambda\,D & 0 & (\mathcal{D}^2+\lambda_c) & 0 \\ 0 & -R_c k^2 & 0 & 0 & (\mathcal{D}^2+\lambda_c/q_R) \end{pmatrix} \tag{27}$$

obeys the definition (25) (i.e. the boundary term $\mathcal{B}$ vanishes), provided that $\psi(z,s)$ satisfies the boundary conditions (17, 18) and $\psi^+(z,s) = (w_m^+(z,s), \omega_m^+(z,s), b_m^+(z,s), j_m^+(z,s), \vartheta_m^+(z,s))^T$ satisfies dual boundary conditions at $z = z_B,\ z_T$

$$\omega_m^+(z,s) = \vartheta_m^+(z,s) = b_m^+(z,s) = Dj_m^+(z,s) = 0\,, \qquad \text{for all } z = z_B, z_T \text{ and } s \in (0, s_n) \tag{28}$$

and radial boundary conditions at $s = 0,\ s_n$

$$\psi^+(z,0) = \psi^+(z,s_n) = 0\,, \qquad \text{for all } s = 0, s_n \text{ and } z \in (z_B, z_T)\,. \tag{29}$$

We proceed by construction of a kernel function ψ^+ satisfying the adjoint equation $\mathcal{L}^+\psi^+ = 0$. The components of a vector $\psi^+ = (w_m^+, \omega_m^+, b_m^+, j_m^+, \vartheta_m^+)^T$ are assumed to be separated as follows

$$f_m^+(z,s) = f^+(z)\,J_m(ks) \tag{30}$$

where the adjoint functions $f^+(z)$ depend only on a vertical coordinate while the radial dependence is expressed here by the Bessel function $J_m(ks)$. Plugging the above ansatz into the matrix equation $\mathcal{L}^+\psi^+ = 0$, we obtain a system of linear differential equations in z variable (see BRS). The existence of a nontrivial solution of this adjoint system satisfying the dual boundary conditions (28) in the z variable is a consequence of the spectral theorem for the adjoint operator and the fact that the equation $\mathcal{L}\psi = 0$ has a solution decomposable in each vector component to the form $f_m(z,s) = f(z)J_m(ks)$ (see BS).

The linear operators $\mathcal{L}$ and $\mathcal{L}^+$ are defined on suitable Hilbert spaces X and X^+, respectively, with values in a Hilbert space Z. These function spaces can be constructed with respect to boundary conditions for vector functions ψ and ψ^+, respectively. It turns out that these spaces are subclasses of Sobolev spaces $W^{2,2}(G_n)$. The space Z is the weighted Lebesgue space $L^2_\varrho(G_n)$ with the weight $\varrho(s) = s$. The reader is referred to the analysis made in (BRS) for further details of construction and properties of the underlying function spaces.

Let us emphasize that the crucial assumption of the theory is that we operate with function spaces defined on a bounded domain G_n. Then the operator $\mathcal{J}_m$ defined on a subclass of a Sobolev space has a discrete spectrum bounded away from zero. This justifies the usage of the inverse operator $\mathcal{J}_m^{-1}$ in (12). Furthermore, the boundedness of the domain implies that the coefficients β defined in Appendix and consequently R_2 determined in (43) are generically non-zero. Thus the amplitude equation (51) in Section 3.3 is indeed a prototype for the Hopf bifurcation phenomenon.

3.2. Derivation of the solvability condition

At this stage, we are yet able to make use of perturbation techniques and adjointness properties in order to solve the abstract nonlinear problem (11) in its matrix representation (22).

Suppose that the unknown function ψ and the Rayleigh number R (the system parameter) can be expanded into a power series in terms of a small unfolding parameter ε, ($\varepsilon \ll 1$)

$$\psi = \varepsilon\,\psi_1 + \varepsilon^2\,\psi_2 + \varepsilon^3\,\psi_3 + \dots\,, \tag{31}$$

$$R = R_c + \varepsilon\,R_1 + \varepsilon^2\,R_2 + \dots \tag{32}$$

where the first order term ψ_1 is identical to the solution of the linearized problem (21) and R_c is a critical value of Rayleigh number known from linear stability analysis made in (BS). Higher order coefficients in the expansion are assumed to satisfy $\psi_k \notin \mathrm{Ker}(\mathcal{L})$ for $k \geq 2$.

The nonlinear system (11), however, when being driven through the critical value R_c within its parameter regime, gives rise to the oscillatory instability. Therefore a complex frequency λ has to be expanded into a power series as well

$$\lambda = \lambda_c + \varepsilon\,\lambda_1 + \varepsilon^2\,\lambda_2 + \dots \tag{33}$$

where λ_c is a critical frequency corresponding to R_c. Now we can insert the above expansions (31-33) into the system (11). Collecting the terms of the same power of ε and using the well-known matrix representation one obtains a series of linear problems.

In the first order of ε^1, we obtain a homogeneous linear problem

$$\mathcal{L}\,\psi_1 = 0 \tag{34}$$

where the components of the vector ψ_1 can be sought in the form $f_{m1}(z,s) = f(z)J_m(ks)$. The exact expression for each vector component $f(z)$ can be found e.g. in (BS) or in (Šimkanin et al 1997) in this Issue.

In the second order of ε^2, we have

$$\mathcal{L}\,\psi_2 = \begin{pmatrix} 0 \\ R_1 k^2\,\vartheta_{m1}(z,s) \\ \lambda_1\,b_{m1}(z,s) \\ \lambda_1\,j_{m1}(z,s) \\ (\lambda_1/q_R)\,\vartheta_{m1}(z,s) \end{pmatrix} \tag{35}$$

where the components $f_{m2}(z,s)$ of a vector ψ_2 are yet unknown. At this order of perturbation expansion the influence of the cubic nonlinearities $P_m(z,s)$, $T_m(z,s)$ and $S_m(z,s)$ is still not present. Taking the inner product $\langle\,.\,|\,.\,\rangle$ of (35) with the dual kernel function ψ^+ yields a simple complex equation

$$-\,\alpha_1\,R_1 + \lambda_1 = 0\,. \tag{36}$$

With regard to the requirement $\lambda_1 = i\sigma_1$, σ_1 is real, the unique solution of this equation is $R_1 = 0$, $\lambda_1 = 0$ and so $\mathcal{L}\psi_2 = 0$. As ψ_2 does not belong to the kernel of $\mathcal{L}$ we finally obtain $\psi_2 = 0$. This property can be also seen from the symmetry of the abstract nonlinear problem.

In the third order of ε^3, the solvability condition yields a nonhomogeneous problem

$$\mathcal{L}\,\psi_3 = \begin{pmatrix} 0 \\ R_2 k^2 \vartheta_{m1}(z,s) \\ P_{m1}(z,s) \,+\, \lambda_2 b_{m1}(z,s) \\ T_{m1}(z,s) \,+\, \lambda_2 j_{m1}(z,s) \\ (1/q_R) S_{m1}(z,s) \,+\, (\lambda_2/q_R)\vartheta_{m1}(z,s) \end{pmatrix}\,. \tag{37}$$

It is obvious that the nonlinear terms in first order representing functions $f_{m1}(z,s)$, namely $P_{m1}(z,s)$, $T_{m1}(z,s)$ and $S_{m1}(z,s)$, arise at this order of expansion. Now the angular velocity $\Omega(s)$ of geostrophic flow (in its leading term) is a function of $b_{m1}(z,s)$ and $j_{m1}(z,s)$. We therefore adopt the notation $\Omega_1(s)$ for convenience.

We briefly sum up the notation used for this stage of perturbation method. All the nonlinearities are functions of $f_{m1}(z,s)$ which are separable in z and s coordinate. They can be therefore expressed in terms of the simple representing functions $f(z)$, known from the linear stability study, as follows

$$\begin{aligned} P_{m1}(z,s) &= im\,\Omega_1(s)\,J_m(ks)\,b(z) - im\,\mathcal{J}_m^{-1}\{\mathcal{P}_{\Omega_1}\,J_m(ks)\}\,b(z)\,, \\ T_{m1}(z,s) &= im\,\Omega_1(s)\,J_m(ks)\,j(z) + \mathcal{J}_m^{-1}\{\mathcal{T}_{\Omega_1}\,J_m(ks)\}\,Db(z)\,, \\ S_{m1}(z,s) &= im\,\Omega_1(s)\,J_m(ks)\,\vartheta(z) \end{aligned} \tag{38}$$

with $\mathcal{P}_{\Omega_1}$, $\mathcal{T}_{\Omega_1}$ corresponding to $\mathcal{P}_{\Omega}$, $\mathcal{T}_{\Omega}$ in (14, 15) where $\Omega(s)$ has been substituted by $\Omega_1(s)$.

Following (16) and the boundary conditions (17), for geostrophic flow $\Omega_1(s)$ in terms of the simple representing functions $f(z)$ we have

$$\Omega_1(s) = \mathcal{Z} \cdot \Omega_s(s)\,. \tag{39}$$

Here

$$\mathcal{Z} = \frac{\Lambda}{2\,(2E)^{1/2}\,k^2} \cdot \Re e \left\{ \int_{z_B}^{z_T} j(z)\overline{Db(z)}\,dz \right\} \tag{40}$$

is the functional involving the functions $b(z)$ and $j(z)$ and

$$\Omega_s(s) = \frac{1}{k^2\,s^3}\,\frac{\partial}{\partial s}\left[m^2 J_m^2(ks) - s^2 \left(\frac{\partial}{\partial s} J_m(ks) \right)^2 \right]$$

describes the radial dependence of geostrophic flow. Using the property of the Bessel differential operator $\mathcal{J}_m$ defined by (13), the above expression can be simplified and written as

$$\Omega_s(s) = \frac{1}{s}\,\frac{d}{ds} J_m^2(ks)\,. \tag{41}$$

The solvability condition for the 3-rd order of the expansion yields an inner product equation

$$\langle\, F_3 \,|\, \psi^+ \,\rangle = 0 \tag{42}$$

where F_3 is a vector of right-hand side terms in (37) and ψ^+ is the previously constructed solution of $\mathcal{L}^+\psi^+ = 0$. By straightforward integrations one finds the solvability condition schematically written as

$$-\,\alpha\, R_2 + \lambda_2 - \beta = 0\,. \tag{43}$$

This condition can be thought of as a complex equation for determining the parameters R_2 and $\lambda_2 = i\sigma_2$ where σ_2 is real, giving us information about bifurcation and frequency response of the dynamical system in the vicinity of the critical Rayleigh number R_c.

The complex coefficients α and β entering (43) depend on the parameters m, Λ, E, q_R as well as on the critical parameters R_c, k_c and λ_c. Their full form is given in terms of analytical expressions (see Appendix).

Now the solution ψ of the nonlinear problem $\mathcal{L}\psi = N(\psi)$ has the power series expansion

$$\psi = \varepsilon\psi_1 + \varepsilon^3\psi_3 + o(\varepsilon^3)\,. \tag{44}$$

Similarly, up to the second order terms, we have

$$R \sim R_c + \varepsilon^2 R_2 \,, \tag{45}$$

$$\lambda \sim \lambda_c + \varepsilon^2 \lambda_2 \,. \tag{46}$$

Finally, if we put

$$\varepsilon = \sqrt{(R - R_c)/R_2} \tag{47}$$

then, in the first order approximation, the representing functions $\tilde{f}(z, s, \varphi, t)$ associated to a solution of the evolution problem (2-6) through (8, 9) can be written as

$$\tilde{f}(z, s, \varphi, t) \sim \sqrt{\frac{R - R_c}{R_2}}\, \Re e\{f(z) J_m(ks) \exp(im\varphi + \lambda t)\} \,. \tag{48}$$

The expression $\sqrt{(R - R_c)/R_2}$ therefore relates to the amplitude of representing functions $\tilde{f}(z, s, \varphi, t)$. It can be seen that if $R_2 > 0$, the Hopf bifurcation arising in R_c is supercritical. On the other hand, if $R_2 < 0$, the bifurcation is subcritical. The complex frequency in the neighbourhood of R_c varies according to

$$\lambda \sim \lambda_c + \varepsilon^2 \lambda_2 = \lambda_c + \frac{R - R_c}{R_2}\, \lambda_2 \,. \tag{49}$$

Some useful properties of the constructed solution, i.e. its dependence on the system parameters and its asymptotics, are presented on Figures 1 - 4 below.

3.3. The amplitude modulation and stability properties of the solution

In the previous paragraph it has been shown that the nonlinear problem (2 - 6) has a nontrivial periodic solution when Rayleigh number R is increased beyond its critical value R_c. This periodic solution, branching at R_c from trivial one, can be either supercritical or subcritical, depending on the sign of parameter R_2. Such a behaviour should indicate the Hopf bifurcation arising at the critical Rayleigh number R_c.

The above analysis, however, does not cover stability properties of the periodic solution constructed above. To analyze stability of the basic state and the bifurcating periodic orbit we have to study a larger phase space than the space of all functions periodic in t and φ variable as it has been proposed by ansatz (10). To this end, one may enlarge this class of functions by assuming that the representing functions $\widetilde{b}$, $\widetilde{j}$, $\widetilde{w}$, $\widetilde{\omega}$ and $\widetilde{\vartheta}$ have the form

$$\tilde{f}(z, s, \varphi, t) = \Re e\{A(\varepsilon^2 t)\, f_m(z, s)\, \exp(im\varphi + \lambda_c t)\} \,. \tag{50}$$

Notice that in (50) each of the functions $f_m(z, s)$ is modulated by complex amplitude $A(\varepsilon^2 t)$ varying in the so-called slow time scale $\varepsilon^2 t$ where ε is a small unfolding parameter as in (31 - 33). As it is indicated by expansions (45, 46) we are forced to choose the scale $\varepsilon^2 t$ in order to capture slowly varying periodic solutions with the complex frequency $\lambda \sim \lambda_c + \varepsilon^2 \lambda_2$. The meaning of all other variables and parameters involved in (50) is left unchanged.

Under the above assumption, straightforward computations based on the same Fredholm alternative argument and on the same function spaces setting can be carried out to derive solvability condition. It can be shown that in this case solvability condition gains a form of an ordinary differential equation for the time dependent complex amplitude $A(\varepsilon^2 t)$. For the modulus $|A(\varepsilon^2 t)|$ the third order approximation of the corresponding ordinary differential equation reads as follows

$$\frac{1}{\alpha_r} \frac{d|A(\varepsilon^2 t)|}{dt} = (R - R_c)\, |A(\varepsilon^2 t)| - R_2\, |A(\varepsilon^2 t)|^3 \tag{51}$$

where the coefficients α_r (the real part of α) and R_2 are the same as in solvability condition (43).

Notice that the amplitude equation (51) is a prototype for the Hopf bifurcation phenomenon and therefore can be conceived as normal form for the Hopf bifurcation. Both the trivial solution and the bifurcating periodic (nontrivial) solution can be sought as stationary solutions (fixed points) of amplitude equation (51). The only nontrivial steady state solution of the ODE (51) is the constant function

$$|A| = \sqrt{\frac{R - R_c}{R_2}} \tag{52}$$

which in fact coincides with the unfolding parameter ε . Therefore inserting the steady state amplitude (52) into (50) yields the same periodic solution as the one previously constructed in Section 3.2.

As a result, depending on the sign of R_2 one observes either supercritical or subcritical type of the Hopf bifurcation. The stability of both steady state and periodic solutions depends on the sign of coefficient α_r . More details concerning the amplitude modulation as well as derivation and analysis of the normal form equation (51) will be presented in the forthcoming paper.

4. Bifurcation diagrams and asymptotic properties of the solution

In our numerical experiments the values of the critical Rayleigh number R_c , the critical radial wave number k_c and the critical complex frequency $\lambda_c = i\sigma_c$ were obtained from the linear stability analysis made in (BS). We studied four particular cases related to the azimuthal wave numbers $m = 1,\ 2,\ 3$ and 5 , with the Elsasser number Λ ranging from 10^{-3} to 2500 . The Ekman number and the Roberts number were chosen $E = 3 \times 10^{-7}$ and $q_R = 0.005$, respectively. More details concerning the typical values of the critical parameters can be found e.g. in Šimkanin et al (1997) in this Issue.

The Figures 1, 2 are bifurcation diagrams in the space of system parameters R (Rayleigh number) and Λ (Elsasser number). The dependence $R_c = R_c(\Lambda)$ is known from linear stability studies made in Soward (1979) and (BS). The weakly nonlinear analysis from previous sections is capable of describing behaviour of solutions (trivial and nontrivial one) and their stability properties in the underlying space of parameters. This enables us to classify qualitatively the bifurcation diagrams to follow.

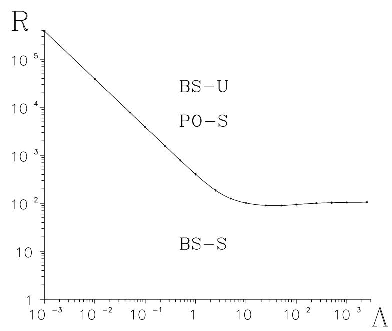


Fig.1. T and MW modes for the azimuthal wave number $m = 1$.

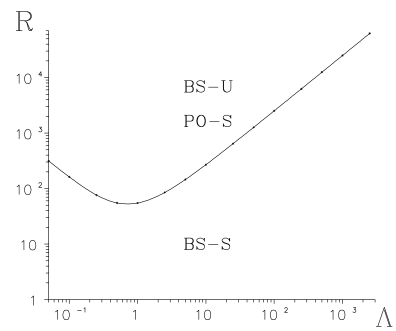


Fig.2. T modes for the azimuthal wave number $m = 5$.

The marked curves in Figures 1, 2 show the dependence of the critical Rayleigh number R_c on the Elsasser number Λ for azimuthal wave numbers $m = 1, 5$. Here T and MW are to symbolize thermally and magnetically driven waves propagating westwards, respectively, as they have been classified in (BS); in Figure 1 the T wave changes into MW wave by increasing Λ at $\Lambda \sim 100$. The parameter space (Λ, R) divided by the curve $R_c = R_c(\Lambda)$, splits into two regions. In the region labeled by $BS-S$ there is no periodic orbit near the locally stable basic state whereas in the region $BS-U$, $PO-S$ the basic state is unstable and there is a stable periodic solution. Here the abbreviation BS stands for "Basic State" and PO for "Periodic Orbit".

The other studied cases of the azimuthal wave number $m = 2, 3$ result into qualitatively same plots and therefore are omitted. We only mention that for large values of the Elsasser number, there is an indication for the Hopf bifurcation to be subcritical for the case $m = 2$. This is due to the change in sign of the coefficient α_r. This special case however needs to be investigated in a more detail. Note that in Skinner and Soward (1990) the subcritical behaviour has been observed for q_R of order unity and for smaller Λ only.

The Figures 3, 4 show asymptotic properties of the finite amplitude solution when the radius of the layer becomes larger. We remind ourselves that the radial extension of the layer measured by s_n has to be finite as it has been proclaimed in previous sections.

Recall that in general the critical Rayleigh number R_c and the critical complex frequency $\lambda_c = i\sigma_c$ are functions of the critical radial wave number k_c. In the linear stability study in (BS) related to the unbounded geometry, for any value of Elsasser number Λ, the wave number $k = k_c$ has been chosen such that the corresponding R_c was minimal. For the particular case of $m = 5$ and for the choice of $\Lambda = 1.0$, it follows from (BS) that $k_c = 5.16$.

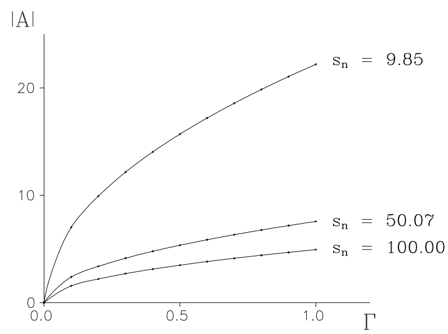


Fig.3. The modulus $|A|$ of the amplitude versus Γ for the azimuthal wave number $m = 5$, the Elsasser number $\Lambda = 1$ and various radii s_n.

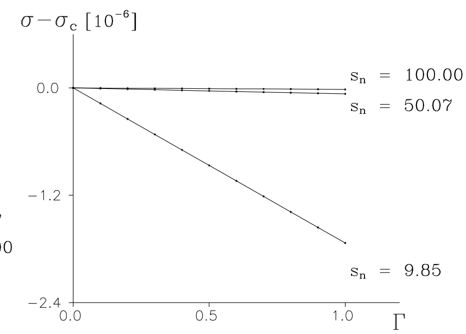


Fig.4. The difference $\sigma - \sigma_c$ of frequencies versus Γ for the azimuthal wave number $m = 5$, the Elsasser number $\Lambda = 1$ and various radii s_n.

Figure 3 above depicts the dependence of the modulus of the amplitude $|A|$, given by (52), on the so-called surplus thermal energy $\Gamma = (R - R_c)s_n^2$. More precisely, the quantity Γ is qualitatively proportional to the thermal energy needed to heat the bottom circular domain of the radius s_n which is, in effect, associated with increase of the Rayleigh number R beyond R_c. This picture can be also viewed as a supercritical bifurcation diagram. Indeed, if $R < R_c$ (i.e. $\Gamma < 0$) there is no periodic solution in the vicinity of the stable basic state. On the other hand, when $R > R_c$ (i.e. $\Gamma > 0$) there is a stable periodic orbit with the modulus of amplitude equal to $|A|$ and the basic state is unstable. The bifurcation curves are plotted for various radial extensions s_n of the cylinder. The reason for introducing the quantity Γ is to compare bifurcation curves for various radii s_n. In terms of the new system parameter Γ, for the amplitude we have $|A| = (\Gamma/R_2)^{1/2}s_n^{-1}$ instead of (52).

It follows from (43) and the expressions for α and β in Appendix that $R_2 = O(s_n^{-1})$ as $s_n \to +\infty$. Therefore for fixed values of the parameter Γ we have

$|A| = O(s_n^{-1/2})$ as $s_n \to +\infty$. This is in agreement with an observation that if the input energy proportional to Γ is constant, the amplitude of motion becomes smaller with growth of the radial extension of the layer.

One has to be careful, however, about the asymptotics like this. The proof of existence of finite amplitude periodic solution based on the weakly nonlinear theory is limited to the parameter range $R_c \leq R < \hat{R}(s_n)$ only. Gathering from the expression $\varepsilon = \sqrt{(R - R_c)/R_2}$, where ε has to be chosen small (i.e. $\varepsilon \ll 1$), and from the asymptotics $R_2 = O(s_n^{-1})$ as $s_n \to +\infty$, we can see that $\hat{R}(s_n) \to R_c$ as $s_n \to +\infty$, i.e. the region of parameter space evaporates.

Figure 4 shows the dependence of the complex frequency $\lambda = i\sigma$ on Γ . For $m = 5$ and $\Lambda = 1$ the critical frequency is $\lambda_c = i\sigma_c$ with $\sigma_c = 0.0376392$. Actually, the difference $\sigma - \sigma_c$ has been plotted versus Γ . In terms of Γ we have $\sigma = \sigma_c + (\Gamma\sigma_2)/(R_2 s_n^2)$. Therefore the dependence of σ on Γ is linear.

Notice that the Γ scale in Figures 3, 4 is magnified in order to show the qualitative features of behaviour of amplitude modulus and frequency response of the nonlinear system. The maximal value of the parameter Γ , however, must be chosen small enough as it is interrelated with the small unfolding parameter ε through the relation $\Gamma = \varepsilon^2 R_2 s_n^2$.

5. Conclusions

It has been shown in this paper that the weakly nonlinear analysis is capable of proving the existence of a nontrivial periodic solution in the vicinity of the critical Rayleigh number R_c for a nonlinear model of rotating magnetoconvection affected by Ekman suction. Although the basic governing equations together with modified Taylor's constraint yield a rather complicated structure, they can be solved analytically in the vicinity of R_c . It has been shown that besides the trivial (zero) solution, there is a periodic solution of the nonlinear problem representing wave propagation in the azimuthal direction.

The existence of a non-trivial periodic solution is neither an obvious matter emerging from the corresponding linearized theory nor a direct consequence of the form of nonlinear governing equation. Among the assumptions guaranteeing the existence of such a solution a crucial role is played by boundedness of the underlying geometry. In case of a rotating horizontal layer it naturally means a restriction to the radially bounded cylinder.

The symmetry of governing equations which is due to cubic nonlinearities implies that the transition from a trivial (conductive) solution towards a non-trivial (convective) periodic solution is via Hopf bifurcation. Applying methods and techniques of the functional analysis, namely solvability conditions from Fredholm's alternative, leads towards derivation of the normal form for the Hopf bifurcation and analytical expressions of its coefficients.

The obtained analytical formulae for the normal form coefficients were evaluated numerically. The bifurcation diagrams showing domains of existence and stability of the solutions have been depicted for the parameter space (Λ, R) . Also the asymptotic properties of the amplitude and frequency of periodic solution for different radial extensions of the layer have been portrayed.

Acknowledgements. This work was supported by the Scientific Grant Agency VEGA (grants No. 1/1817/94 and No. 1/1492/94). We are indebted to M. R. E. Proctor for encouraging us to solve the presented problem. We have also benefited from valuable comments on presentation of the results by two referees, A. M. Soward and unknown one. We are also grateful to F.H. Busse, the guest editor of this Issue, for his helpful remarks.

References

Adams, R. A. : 1975, *Sobolev spaces*, Academic press, London

Brestenský, J. and Ševčík, S.: 1994, *Geophys. Astrophys. Fluid Dyn.* **77**, 191. Referred to as (BS) in text.

Brestenský, J. and Ševčík, S. : 1995, *Acta Astronomica et Geophysica Univ. Comenianae* **17**, 1

Brestenský, J. , Revallo, M. and Ševčovič, D.: 1997, *Acta Astronomica et Geophysica Univ. Comenianae* **18**, 1. Referred to as (BRS) in text.

Fearn, D. R. : 1994, In : *Lectures on Solar and Planetary Dynamos*, eds. Proctor, M. R. E. and Gilbert, A. D. , Cambridge University Press

Fearn, D. R. , Proctor, M. R. E. and Sellar, C. C. : 1994, *Geophys. Astrophys. Fluid Dyn.* **77**, 111

Manneville, P. : 1990, *Dissipative structures and weak turbulence*, Academic press, London

Proctor, M. R. E. and Weiss N. O. : 1982, *Rep. Prog. Phys.* **45**, 1317

Roberts P. H. and Stewartson K. : 1974, *Phil. Trans. R. Soc. London* **227**, 93

Šimkanin, J. , Brestenský, J. and Ševčík, S.: 1997, In *this Issue*, pp. 195 – 220

Skinner, P. H. and Soward, A. M. : 1988, *Geophys. Astrophys. Fluid Dyn.* **44**, 91

Skinner, P. H. and Soward, A. M. : 1990, *Geophys. Astrophys. Fluid Dyn.* **60**, 335

Soward, A. M. : 1979, *J. Fluid Mech.* **90**, 669

Soward, A. M. : 1986, *Geophys. Astrophys. Fluid Dyn.* **35**, 329

Appendix

The coefficients α and β in the solvability condition (55) are

$$\alpha = -\,k_c^2\,\frac{\langle\,\vartheta(z)\,\omega^+(z)\,\rangle^z}{M}\,, \qquad \beta = 4\mathcal{Z}\,\frac{I_2}{I_1}\,\frac{\langle\,Db(z)\,j^+(z)\,\rangle^z}{M}$$

where $M = \langle\,b(z)\,b^+(z)\,\rangle^z + \langle\,j(z)\,j^+(z)\,\rangle^z + (1/q_R)\,\langle\,\vartheta(z)\,\vartheta^+(z)\,\rangle^z$ and $\mathcal{Z}$ is a functional given by (41).

The above expressions are entered by the integrals over the radial coordinate

$$I_1 = \int_0^{s_n} J_m^2(k_c s)\, s\, ds\,, \qquad I_2 = \int_0^{s_n} J_m^2(k_c s) \left(\frac{d}{ds} J_m(k_c s)\right)^2 s\, ds$$

which are to be computed numerically and by the integrals over the z coordinate

$$\langle\, f(z)\, f^+(z)\,\rangle^z = \int_{z_B}^{z_T} f(z)\, f^+(z)\, dz\,.$$

Particular integrals needed for evaluation of the coefficients are

$$\begin{aligned}
\langle\, \vartheta(z)\, \omega^+(z)\,\rangle^z &= \frac{1}{2R_c k_c^2} \sum_l c_l \gamma_l\,,\\
\langle\, b(z)\, b^+(z)\,\rangle^z &= -\frac{1}{2} \sum_l \gamma_l \left(\frac{\pi_l}{m^2\Lambda} s_l^\omega - 1\right)\,,\\
\langle\, j(z)\, j^+(z)\,\rangle^z &= -\frac{1}{2m^2\Lambda} \sum_l s_l^j c_l \gamma_l \pi_l\,,\\
\langle\, \vartheta(z)\, \vartheta^+(z)\,\rangle^z &= -\frac{5}{2R_c k_c^2}\,,\\
\langle\, Db(z)\, j^+(z)\,\rangle^z &= -\frac{1}{2} \sum_l \gamma_l^2 \pi_l^2 c_l
\end{aligned}$$

where

$$\begin{aligned}
c_l &= \pi_l^2 + k_c^2 + \lambda\,,\\
\gamma_l^{-1} &= \frac{\pi_l^2}{m^2\Lambda} (\pi_l^2 + k_c^2 + \lambda - 2im\Lambda)^2 + m^2\Lambda(\pi_l^2 + k_c^2)\,,\\
s_l^\omega &= \pi_l (\pi_l^2 + k_c^2 + \lambda - 2im\Lambda)\, c_l \gamma_l\,,\\
s_l^j &= \frac{s_l^\omega}{c_l}
\end{aligned}$$

with $\pi_l = (2l-1)\pi$, $\lambda = i\sigma$ and l equals to 5.

Let us emphasize that the integral I_1 diverges to $+\infty$ whereas I_2 converges as $s_n \to +\infty$. Thus the coefficient β vanishes when s_n tends to $+\infty$. We also notice that the integrals over the z coordinate are entered by functions of $f(z)$ which solve the linearized (eigenvalue) problem and by functions of $f^+(z)$ which solve the homogeneous adjoint problem in Section 3.1.