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Aplikácie metód singulárnych perturbácií v matematickej teórii väzkopružnosti

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Applications of Singular Perturbation Methods in the Mathematical Theory of Viscoelasticity

Thesis

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Contents

Part I

Spurt phenomenon for a non-Newtonian model of shearing motions including diffusion

Part II

Singular limit dynamics of invariant manifolds and vector fields

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Spurt phenomenon

Section 1.

Introduction

A surprising feature of the flow of polymers is associated with a sudden increase in the volumetric flow rate when the pressure gradient is gradually increased beyond a critical value. This striking phenomenon, called "spurt", was apparently first observed by Vinogradov *et al.* [48] in rheological experiments involving the flow through thin capillaries of highly elastic and very viscous non-Newtonian fluids like some synthesized polybutadienes and polyisoprenes. The interested reader is referred to [48, Table 1] for more detailed information about microstructure characteristics of samples. The spurt phenomenon is a kind of a flow instability in pressure driven shear flows of viscoelastic fluids.

Much effort is being spent to explain spurt and related phenomena mathematically. Several authors have considered mathematical models based on differential constitutive equations due to Johnson, Sagelman and Oldroyd exhibiting local extrema of the steady shear stress as a function of steady strain rate (see [25], [26], [27], [29], [30], [37] and [36]). These papers show that the spurt phenomenon is dynamic, and hence cannot be explained in a satisfactory manner by only studying the steady state equations. Dynamical theory can explain phenomena observed in experiments and in numerical simulations, and it can also predict phenomena like latency, shape memory and hysteresis which should be observable in future experiments.

In Part I we modify the models of [25], [37] by adding a diffusion term to the constitutive equation. The resulting system of equations (in dimensionless units) governing planar shear flow has the form

$$\begin{aligned} \alpha v_t &= v_{xx} + \sigma_x + f \\ \sigma_t &= -\sigma + g(v_x) + \nu^2 \sigma_{xx} \end{aligned} \tag{1.1}$$

Results of Part I are contained in the joint paper with P.Brunovský [13] where $v(t,x)$ is the velocity of the planar flow, $\sigma(t,x)$ is the polymer contribution to the shear stress, $g : R \to R$ is a given smooth function, and $f > 0$ is the pressure gradient driving the flow.

Unlike the models investigated in [37], [25], and in the other models in [29], [30], [36], system (1.1) contains the spatial diffusion term $\nu^2\sigma_{xx}$. Spatial diffusion is usually neglected in non-Newtonian models because of the spatial homogeneity of the structure. In the model of [18] (also see [6]), Brownian motion prevents polymer molecules (treated as dumb-bells) from being completely independent of each other giving rise to a diffusion term in constitutive equations. Typical values of ν^2 will be described in Section 6. The structure of steady states of (1.1) is determined by treating $\nu^2 > 0$ as a small parameter, and by applying the singular perturbation theory of [28]. This theory enables us to select steady states that appear to be appropriate for capturing the spurt phenomenon.

System (1.1) with $\nu^2 = 0$ exhibits the same behavior in steady shear as the more realistic models studied in [29], [30] and [36], where the differential constitutive equations also involve normal stresses (in particular, the first normal stress difference) giving rise to a governing system of three quasilinear parabolic-hyperbolic PDE's in place of the two in (1.1). The dimensionless parameter α representing the ratio of Reynolds number to Deborah number is very small. The analytical study in [37], [30] and [36] is based on treating the respective governing equations as singular perturbation problems with α as a singular parameter. Their approach is to determine the complete dynamics when $\alpha = 0$ and then to show that the dynamics of the full system is similar for $\alpha > 0$ sufficiently small. By contrast, our quasilinear system (1.1) with $\nu^2 > 0$ is parabolic, and the theory of parabolic systems can be exploited to determine the global dynamics for $\alpha > 0$ sufficiently small. In particular, the existence of a global compact attractor and an inertial manifold can be established. It should be noted that the feature of mathematical models studied in [37], [30], [36] that makes their qualitative analysis (asymptotic behavior as $t \to \infty$, stability properties, etc.) particularly difficult is that the governing equations posses uncountable many isolated steady states. From this fact one can deduce that these governing equations can admit neither a compact global attractor nor a finite dimensional inertial manifold.

Part I is organized as follows. In Section 2, we use general ideas from [25] to derive a non-Newtonian model of shearing motions incorporating spatial diffusion. Basic properties of the model (existence and long-time behavior of solutions, qualitative properties of steady states) are established in Section 3. It is shown that in case of a generic g, the asymptotic behavior of solutions is very simple - each solution tends to some steady state and the number of steady states is finite. We also prove exponential stability of two particular steady states playing a crucial role in the explanation of spurt. In Sections 4 and 5, spurt and hysteresis phenomena in our mathematical model are established. The phenomenon of spurt is associated with extinction of a stable steady state when the pressure gradient increases beyond a critical (bifurcation) value. The results of numerical simulations for small values of $\alpha, \nu > 0$ are presented in Section 6. We have performed numerical simulations of spurt and hysteresis phenomena for the sample PI-3 (see [48]). Numerical results match the data observed experimentally by Vinogradov *et al.*

Section 2.

A Non-Newtonian model of shearing motions including diffusion

In this section, we derive a mathematical model for shearing motion of a fluid leading to a system of governing equations including a diffusion term in the constitutive equation.

We consider the planar shear flow of a viscoelastic fluid in an infinite narrow strip: $x \in [-h, h]$ and $y \in (-\infty, \infty)$, with the flow directed along the y-axis. We suppose the fluid to be non-Newtonian, incompressible and the motion to take place under isothermal conditions. We restrict ourselves to motions which are symmetric with respect to the centerline. Under our assumptions the flow variables will depend only the transversal variable x. Hence, the velocity vector $\vec{v}$ has the form $\vec{v} = (0, v(t, x))$ with $v(t, x) = v(t, -x)$. It is easy to verify that the mass balance is then automatically satisfied. The equation governing the motion of the fluid is the balance of linear momentum

$$\varrho \left(\frac{\partial \vec{v}}{\partial t} + (\vec{v}, \nabla)\vec{v} \right) = \nabla \vec{S} \tag{2.1}$$

where ϱ is the constant fluid density and $\vec{S}$ is the total stress which can be decomposed as

$$\vec{S} = p.\vec{Id} + \varepsilon.\vec{D} + \vec{\Sigma}. \tag{2.2}$$

Here p is the isotropic pressure of the form $p = p_0(t, x) + f.y$ where f is the pressure gradient driving the flow, ε is the Newtonian viscosity and $\vec{D}$ is the stretching tensor (or rate of deformation), i.e. $\vec{D} = \left(\nabla\vec{v} + (\nabla\vec{v})^\top\right)/2$. According to [25, Section 2] the extra stress $\vec{\Sigma} = \begin{pmatrix} \sigma^{xx} & , \sigma^{xy} \\ \sigma^{yx} & , \sigma^{yy} \end{pmatrix}$ satisfies

$$\begin{aligned} &\sigma^{xy} = \sigma^{yx} = \mathcal{S}_0{}_{s=0}^{\infty} \left[\Lambda_t(s)\right] \\ &\sigma^{xx} - \sigma^{yy} = \mathcal{S}_1{}_{s=0}^{\infty} \left[\Lambda_t(s)\right] \\ &\sigma^{xx} + \sigma^{yy} = 0 \end{aligned} \tag{2.3}$$

where $\mathcal{S}_0, \mathcal{S}_1$ are generally nonlinear operators acting on the relative shearing history

$$\Lambda_t(s) = -\int_{t-s}^{t} v_x(\tau, x)\, d\tau. \tag{2.4}$$

Since we assume the flow to be planar, equation (2.1) reduces to

$$\varrho v_t = \varepsilon v_{xx} + \sigma_x + f \tag{2.5}$$

where $\sigma := \sigma^{xy}$.

We specify the operator $\mathcal{S}_0$ in such a way that it takes into account long-range molecular forces. According to [18], the latter provide the constitutive equations by a diffusion term $\nu^2\sigma_{xx}$ ‡. The first normal stress difference determined by the operator $\mathcal{S}_1$ plays no role in our model.

Let A denote the self-adjoint closure in $L_2(0,h)$ of the operator defined on $C^2_B(0,h)$ by $Au = -u_{xx}$ for any $u \in C^2_B(0,h) := \{u \in C^2(0,h);\ u(0) = u_x(h) = 0\}$, its domain $D(A)$ is the Sobolev space $W^{2,2}_B(0,h) = \{u \in W^{2,2}(0,h);\ u(0) = u_x(h) = 0\}$. Let $\lambda, \nu > 0$ be fixed. Then the operator $-\left(\lambda + \nu^2 A\right)$ generates an analytic semigroup $\exp\ \left(-\left(\lambda + \nu^2 A\right)t\right)$, $t \geq 0$; (see [23, Chapter 1]) †.

Assume that $g : R \to R$ is a bounded Lipschitz continuous function. As usual, we identify g with the Nemitsky operator $g : \mathrm{W}^{1,2}(0,h) \to L_2(0,h)$ defined by $g(u)(x) = g(u(x))$ for a.e. $x \in [0,h]$. Due to the assumptions on g the nonlinear operator g is well defined and Lipchitz continuous.

Let $\tilde{f} \in L_2(0,h)$ be defined as

$$\tilde{f} : x \mapsto f.x, \qquad for\ any\ x \in [0,h] \tag{2.6}$$

We define

$$\mathcal{S}_0\left(\Lambda_t\right) = \int_0^\infty \exp\ \left(-\left(\lambda + \nu^2 A\right)s\right).\left[g\left(-\frac{d}{ds}\Lambda_t(s)\right) + \lambda.\tilde{f}\right] ds - \tilde{f}$$
$$\text{for any}\ \ v \in C\left(R : \mathrm{W}^{1,2}(0,h)\right), \qquad \text{and}\ \ t \geq 0 \tag{2.7}$$

where $\Lambda_t(s)$ is defined by (2.4), i.e. $\Lambda_t(s) = -\int_{t-s}^{t} v_x(\tau,x)\, d\tau$.

Clearly,

$$\mathcal{S}_0\left(\Lambda_t\right) = \int_0^\infty \exp\ \left(-\left(\lambda + \nu^2 A\right)s\right)\left[g\left(v_x(t-s,.)\right) + \lambda\tilde{f}\right] ds - \tilde{f} \tag{2.8}$$

‡ According to the Noll concept of a simple material viscoelastic fluids having spatially nonlocal constitutive equations are sometimes referred to as non-simple fluids

† Some of the properties of sectorial operators and analytic semigroups will be recalled in Section 9.

In case $\nu = 0$, the definition of the functional $\mathcal{S}_0$ coincides with that of [25], formula (5). However, since the operator $\lambda + \nu^2 A,\ \nu > 0$, is a diffusion operator generating an analytic semigroup, the operator $\exp\left(-(\lambda + \nu^2 A)s\right),\ s > 0$, smoothes out solutions, i.e. $\exp\left(-(\lambda + \nu^2 A)s\right)w \in D(A)$ for any $w \in L_2(0,h)$ and $s > 0$ (see [23, Chapter 1]).

Differentiating (2.8) with respect to t and substituting $u := \sigma + \tilde{f} = \mathcal{S}_0\left(\Lambda_t\right) + \tilde{f}$, we obtain the following constitutive equation of rate type

$$u_t + \left(\lambda + \nu^2 A\right) u = g(v_x) + \lambda \tilde{f} \tag{2.9a}$$

with boundary conditions

$$u(t,0) = u_x(t,h) = 0, \tag{2.9b}$$

or equivalently,

$$\sigma_t + \lambda\sigma - \nu^2 \sigma_{xx} = g(v_x) \tag{2.10a}$$

with boundary conditions

$$\sigma(t,0) = 0, \qquad \sigma_x(t,h) = -f, \tag{2.10b}$$

respectively.†

We note that $\sigma_x(t,h) = -f$ implies $v_{xx}(t,h) = 0$ which is the boundary condition appearing in the theory of multipolar fluids (see, [5, Section 3]). The boundary condition $u(t,0) = 0$ ($\sigma(t,0) = 0$) implies that the function $u(t,.)$ ($\sigma(t,.)$) can be extended as an odd function to the interval $[-h,h]$ for all t. It insures the symmetry of the flow about the centerline.

Summarizing, our model leads to the initial-boundary value problem

$$\begin{aligned} \varrho v_t &= \varepsilon v_{xx} + \sigma_x + f \\ \sigma_t &= \nu^2 \sigma_{xx} + g(v_x) - \lambda\sigma \\ v(0,x) = v_0(x); \qquad \sigma(0,x) &= \sigma_0(x) \qquad\quad for\ a.e.\ \ x \in [0,h] \\ v_x(t,0) = v(t,h) = 0; \quad \sigma(t,0) &= 0; \quad \sigma_x(t,h) = -f \quad for\ t \geq 0. \end{aligned} \tag{2.11}$$

To facilitate the discussion, we scale the space variable x by h, time t by λ^{-1}, v by $h\lambda$, σ by $\varepsilon\lambda$, f by $\varepsilon\lambda/h$, ν^2 by $h^2\lambda$ and replace $g(\xi)$ by $\frac{1}{\varepsilon\lambda^2}g(\lambda\xi)$. The resulting system is

$$\begin{aligned} \alpha v_t &= v_{xx} + \sigma_x + f \\ \sigma_t &= \nu^2 \sigma_{xx} + g(v_x) - \sigma \\ for \quad (t,x) &\in [0,\infty] \times [0,1] \end{aligned} \tag{2.12}$$

† Note that the boundary condition $\sigma_x(.,h) = -f$ has no physical justification based on the theory of Johnson-Sagelman-Oldroyd fluids. Nevertheless, the boundary conditions for the extra stress σ can be justified in a satisfactory manner by means of the kinetic theory of fluids (see [6], [18])

with boundary conditions

$$\begin{aligned} v_x(t,0) =& v(t,1) = 0; \\ \sigma(t,0) = 0;\ \ & \sigma_x(t,1) = -f \end{aligned} \tag{2.13}$$

and initial data

$$v(0,x) = v_0(x); \qquad \sigma(0,x) = \sigma_0(x) \qquad\quad for\ a.e.\ \ x \in [0,1] \tag{2.14}$$

There are two dimensionless parameters $\alpha = \frac{\varrho h^2 \lambda}{\varepsilon}$ and $\nu > 0$. According to [18] and [48], the typical values of α and ν are

$$\alpha = O(10^{-9}) \quad \text{and} \quad \nu^2 = O(10^{-4}).$$

Hence, we may treat α and ν as small parameters.

Section 3.

Existence of solutions, asymptotic behavior, steady state solutions and their stability

In this section, we study the problem of existence of solutions, their long time behavior, as well as some qualitative properties of steady states of the system (2.12). Using the abstract theory developed in [23] we establish local and global solvability. For g real analytic we furthermore prove that the asymptotic behavior of the solutions is simple - each trajectory approaches some steady state and the number of steady state solutions is finite. To single out the appropriate stationary solutions, we apply the results of the theory of singularly perturbed boundary value problems of [28].

3.1. Existence of solutions

In terms of the variables v and u the initial boundary value problem (2.12) takes the form

$$\begin{aligned} \alpha v_t &= v_{xx} + u_x \\ u_t &= \nu^2 u_{xx} - u + g(v_x) + fx \end{aligned} \tag{3.1.1}$$

$$v_x(t,0) = v(t,1) = 0; \quad u(t,0) = u_x(t,1) = 0 \quad for\ t \geq 0$$

$$v(0,x) = v_0(x); \quad u(0,x) = u_0(x) \quad for\ x \in [0,1].$$

To facilitate the discussion, let

$$S = v_x + u = v_x + \sigma + \tilde{f}. \tag{3.1.2}$$

Obviously,

$$\alpha S_t = S_{xx} + \alpha u_t. \tag{3.1.3}$$

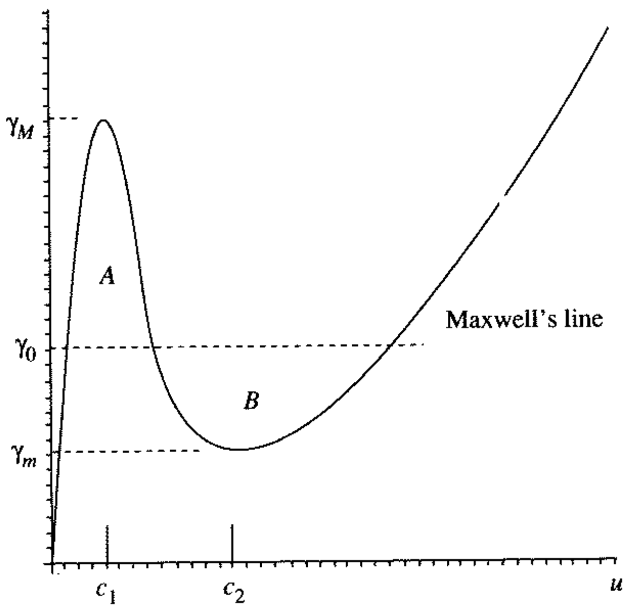

FIG. 1

In terms of S and u, the system (3.1.1) takes the form

$$\begin{aligned} \alpha S_t &= S_{xx} + \alpha\nu^2 u_{xx} + \alpha(g(S-u) + fx - u) \\ u_t &= \nu^2 u_{xx} - u + g(S-u) + fx \end{aligned} \tag{3.1.4}$$

with boundary conditions

$$u(t,0) = u_x(t,1) = 0; \quad S(t,0) = S_x(t,1) = 0$$

and initial data

$$S(0,x) = S_0(x) = v_{0x}(x) + u_0(x); \quad u(0,x) = u_0(x) \;\; for\ x \in [0,1] \tag{3.1.5}$$

Throughout Part I we will assume that α and ν are small parameters. The pressure gradient f is assumed to be positive. Denote

$$h(u) := u + g(u)$$

The function h is assumed to be C^2 with a single loop as shown in Fig.1.

More precisely, we make the following hypotheses:

$$(W) \begin{cases} (i) & g : R \to R \text{ is an odd } C^2 \text{ function with bounded derivatives up to} \\ & \text{the second order satisfying } g(u)u > 0 \text{ for any } u \in R,\ u \neq 0; \\ & \\ (ii) & \text{there exist constants } 0 < c_1 < c_2 \text{ such that} \\ & h'(u) = 1 + g'(u) > 0 \ \text{ on } [0, c_1) \\ & h'(u) = 1 + g'(u) < 0 \ \text{ on } (c_1, c_2) \\ & h'(u) = 1 + g'(u) > 0 \ \text{ on } (c_2, \infty) \end{cases}$$

Under assumptions (W), there exists a $\gamma_0 > 0$ such that

$$\int_{\min h^{-1}(\gamma_0)}^{\max h^{-1}(\gamma_0)} (h(u) - \gamma_0)\ du = 0.$$

The last integral condition is commonly known as *Maxwell's equal area rule* (the area A equals B). In Fig.1 the line $u = \gamma_0$ is called *Maxwell's line.* We also note that the function $h(u) = u + g(u)$ satisfying (W) is sometimes called *van der Walls type curve.*

In what follows, we let X denote the real Hilbert space $L_2(0,1)$ with norm $\|.\|$ and the inner product $(.,.)$. Recall that the operator A defined in the previous section is sectorial and positive in X with domain $D(A) = \{w \in W^{2,2}(0,1);\ w(0) = w_x(1) = 0\}$. Hence, fractional powers of A can be defined. Let $X^\gamma, \gamma \geq 0$, be the Hilbert space consisting of the domain $D(A^\gamma)$ endowed with graph norm $\|w\|_\gamma = \|A^\gamma w\|$ for any $w \in X^\gamma = D(A^\gamma)$. The operator A has a compact resolvent $A^{-1} : X \to X$.

(3.1.6)

Now one can treat the governing equations (3.1.4)-(3.1.5) as an abstract differential equation in the Hilbert space

$$\mathcal{X} = X \times X. \tag{3.1.7}$$

To do so, we let $\Phi = [S, u]$. The system (3.1.4) then becomes

$$\frac{d}{dt}\Phi + L\Phi = F(\Phi); \quad \Phi(0) = \Phi_0 = [S_0, u_0] \tag{3.1.8}$$

where the linear operator L is defined by

$$L\,[S,u] := \left[A(\frac{1}{\alpha}S + \nu^2 u), \nu^2 Au\right] = \left(\begin{pmatrix} \frac{1}{\alpha}A & \nu^2 A \\ 0 & \nu^2 A \end{pmatrix} \begin{bmatrix} S \\ u \end{bmatrix}\right)^\top \tag{3.1.9}$$

on its domain $D(L) = D(A) \times D(A)$. The nonlinearity F is given by

$$F\left([S,u]\right) = \left[g(S-u) - u + fx, g(S-u) - u + fx\right]. \tag{3.1.10}$$

It is routine to verify that $L : D(L) \subset \mathcal{X} \to \mathcal{X}$ is a sectorial operator generating an analytic semigroup $\exp\,(-Lt), t \geq 0$. Since A^{-1} is compact it is easy to show that L has a compact resolvent $L^{-1} : \mathcal{X} \to \mathcal{X}$. The fractional power $L^{1/2}$ is then easily computed as

$$L^{1/2} = \begin{pmatrix} \frac{1}{\sqrt{\alpha}}A^{1/2} & \frac{\sqrt{\alpha}\nu^2}{1+\nu\sqrt{\alpha}}A^{1/2} \\ 0 & \nu A^{1/2} \end{pmatrix}$$

and $D(L^{1/2}) = D(A^{1/2}) \times D(A^{1/2})$. Hence there is an equivalent norm in $\mathcal{X}^{1/2}$ such that

$$\mathcal{X}^{1/2} \cong X^{1/2} \times X^{1/2} \tag{3.1.11}$$

and it can easily be verified that

$$X^{1/2} = \{w \in W^{1,2}(0,1);\ w(0) = 0\}. \tag{3.1.12}$$

Since we have assumed that the first and second derivative of g are bounded, the nonlinearity $\mathcal{F}$ is a C^1 bounded mapping from $\mathcal{X}^{1/2}$ into $\mathcal{X}$.

Now we can apply the general theory of abstract parabolic equations [23]. According to [23, Theorems 3.3.3, 3.3.4, 3.4.1, 3.5.2], for any initial condition $\Phi_0 \in \mathcal{X}^{1/2}$ the abstract equation (3.1.8) has a unique solution $\Phi(t)$ defined on $[0,\infty)$ by the property

$$\Phi \in C_{loc}([0,\infty), \mathcal{X}^{1/2}) \cap C^1_{loc}((0,\infty), \mathcal{X}^{1/2})$$

$$\Phi(t) \in D(L) \text{ for } \ t > 0 \ \text{ and } \ \ \Phi(0) = \Phi_0.$$

Hence, (3.1.8) defines a C^1 semidynamical system $(T(t), t \geq 0)$ in $\mathcal{X}^{1/2}$ defined by

$$T(t)\Phi_0 = \Phi(t, \Phi_0) \ \text{ for any } \ t \geq 0$$

where $\Phi(t, \Phi_0)$ is the solution of (3.1.8) with $\Phi(0) = \Phi_0 \in \mathcal{X}^{1/2}$.

3.2. Asymptotic behavior of solutions

We now turn our attention to the asymptotic behavior of solutions of (3.1.8). First, we will study the set of steady states, i.e. stationary solutions of (3.1.8) which we denote by $\mathcal{E}$. Clearly,

$$\mathcal{E} = \left\{[0,\bar{u}]\,;\ \bar{u} \in D(A) \text{ is a solution of } \ \nu^2 A\bar{u} = -\bar{u} + g(-\bar{u}) + fx\right\}. \tag{3.2.1}$$

In fact, $[0,\bar{u}] \in \mathcal{E}$ iff

$$\bar{u} \in C^4(0,1), \quad \nu^2 \bar{u}_{xx} = \bar{u} + g(\bar{u}) - fx \ \ \bar{u}(0) = \bar{u}_x(1) = 0. \tag{3.2.2}$$

Here we have used the assumption that g is an odd C^2 function.

The system (3.1.8) admits a global Lyapunov function $V : \mathcal{X}^{1/2} \to R$ defined by

$$V\left([S,u]\right) = \frac{1}{2}\left\{\frac{1}{\alpha}\|S\|^2_{1/2} + \nu^2\|S-u\|^2_{1/2} + \|S-u\|^2 + J(S-u)\right\}$$

where

$$J(w) = 2\int_0^1 \int_0^{w(x)} (g(s) + fx)\, ds\, dx. \tag{3.2.3}$$

Indeed, a simple calculation shows that for any solution $[S(t), u(t)]$ the following formula holds

$$\frac{d}{dt}V\left([S(t), u(t)]\right) + \frac{1}{\alpha}\|S(t)\|^2_{1/2} + \frac{1+\alpha\nu^2}{\alpha^2}\|S(t)\|^2_1 = 0 \quad \text{for any } t > 0. \tag{3.2.4}$$

Due to the assumption $g(u)u \geq 0$ for any $u \in R$ it follows that the functional V is bounded from below. From (3.2.2), (3.2.4) it follows that the real valued function $t \mapsto V\left([S(t), u(t)]\right),\ t \geq 0$ is strictly decreasing unless $[S(t), u(t)] \equiv [0, \bar{u}] \in \mathcal{E}$ is a steady state solution of (3.1.8). Then a standard invariancy argument (see, e.g. [49, Theorem 4.1]) enables us to conclude that the omega-limit set

$$\Omega(\Phi_0) := \{\Phi \in \mathcal{X}^{1/2},\ \text{there exists } t_n \to \infty \ \text{ such that } \ T(t_n)\Phi_0 \to \Phi\}$$

satisfies

$$\Omega(\Phi_0) \subseteq \mathcal{E}, \tag{3.2.5}$$

for any $\Phi_0 \in \mathcal{X}^{1/2}$. Since the operator L has a compact resolvent L^{-1}, it follows from [23, Theorems 3.3.6 and 4.3.3] and (3.2.5) that

$$\lim_{t\to\infty} \operatorname{dist}(T(t)\Phi_0, \mathcal{E}) = 0, \tag{3.2.6}$$

where $\operatorname{dist}(\Phi, \mathcal{E}) = \inf\left(\|\Phi - \Psi\|_{\mathcal{X}^{1/2}},\ \Psi \in \mathcal{E}\right)$. In the following simple proposition, we obtain bounds on steady states, and we show for g real analytic that the number of possible steady states is finite.

Proposition 3.2.1. *Let $u_0 \geq c_2$ be such that $h(u_0) \geq f$. Then, $0 \leq u(x) \leq u_0$, for any solution $u(x)$ of (3.2.2). Moreover, there exists a constant $M = M(g, f) > 0$ such that*

$$\nu \sup_{x\in[0,1]} |u_x(x)| + \sup_{x\in[0,1]} |u(x)| \leq M$$

If g is real analytic then the number of solutions of (3.2.2) is finite.

Proof. Let u be an arbitrary solution of (3.2.2). Since $h(u) := u + g(u)$ is nondecreasing on $[u_0, \infty)$ and $h(u_0) \geq f$ it follows that $u(x) \geq u_0$ implies $\nu^2 u_{xx}(x) = h(u(x)) - fx \geq h(u_0) - fx \geq f(1-x)$. Thus the function $u(x)$ is strictly convex whenever $u(x) \geq u_0$. Since $u(0) = 0$, if $u(x_0) > u_0$ for some $x_0 \in (0,1]$ then there exists $x_1 \in (0,1)$ such that $u(x_1) = u_0,\ u(x) > u_0$ and $u_x(x) > 0$ on $(x_1, 1)$. This means that u cannot satisfy $u_x(1) = 0$. Hence $u(x) \leq u_0$ for every $x \in [0,1]$ and $\nu > 0$. The inequality $0 \leq u(x)$ can be obtained in a similar way. The estimate for $\nu u_x(x)$ follows from the well known interpolation inequality

$$\frac{1}{2}\nu \sup_{x\in[0,1]} |u_x(x)| \leq \sup_{x\in[0,1]} |u(x)| + \nu^2 \sup_{x\in[0,1]} |u_{xx}(x)| \text{ for any } u \in C^2([0,1]) \text{ and } \nu > 0.$$

Now we assume that g is real analytic. We fix an $\nu > 0$ and define the map $\mu \mapsto \phi(\mu)$ as $\phi(\mu) = u^\mu_x(1)$ where $u^\mu(x)$ is the solution of the initial-value problem $\nu^2 u_{xx} = u + g(u) -$

$fx, \quad u^\mu(0) = 0,\ u^\mu_x(0) = \mu$. Since g is Lipschitz continuous and analytic the function $\phi(\mu)$ is well defined and analytic on R. Furthermore, $\phi(\mu) = 0$ if and only if $u^\mu(x)$ is a solution of the BVP (3.2.2). Suppose to the contrary the existence of infinitely many solutions of BVP (3.2.2). Then the set $\{\mu \in [-M/\nu, M/\nu];\ \phi(\mu) = 0\}$ must have an accumulation point. Because of analyticity of ϕ we have $\phi \equiv 0$ on R. Hence, there is a solution $u^\mu(x)$ of BVP (3.2.2) for $\mu > M/\nu$ which is inconsistent with $u^\mu_x(0) = \mu$. $\diamondsuit$

The omega-limit set $\Omega(\Phi_0)$ is non-empty and connected ([23, Theorem 4.3.3]). Thus, by (3.2.5), $\Omega(\Phi_0)$ is a singleton whenever $\mathcal{E}$ is finite. We have thus established the following

THEOREM 3.2.2. *Assume the hypotheses (W). Then, for any initial condition $\Phi_0 \in \mathcal{X}^{1/2}$, the evolution problem (3.1.8) has the unique solution $\Phi = \Phi(t, \Phi_0), t \geq 0$, its omega-limit set $\Omega(\Phi_0)$ being contained in the set of steady state solutions $\mathcal{E}$. If, in addition, g is real analytic then each trajectory tends to a single steady state.*

3.3. Steady state solutions

We now examine steady state solutions of (3.1.8). Recall that $[\bar{S}, \bar{u}]$ is a steady state if and only if $\bar{S} \equiv 0$ and $\bar{u} \in C^4(0,1)$ is a solution of the BVP

$$\nu^2 u_{xx} = u + g(u) - fx$$

$$u(0) = u_x(1) = 0. \tag{3.3.1}$$

The steady state velocity profile $\bar{v}$ is then calculated as $\bar{v}(x) = \int_x^1 \bar{u}(\xi)\, d\xi$. Since ν is assumed to be small, the problem (3.3.1) can be viewed as a singular perturbation of the reduced problem

$$0 = u + g(u) - fx. \tag{3.3.2}$$

From now on, we assume

$$f \in [f_{min}, f_{max}],$$

where $0 < f_{min} < \gamma_m$ and $\gamma_M < f_{max} < \infty$. From Fig.1 it is clear that the problem (3.3.2) has a unique C^1 solution $u = \phi_1(x), x \in [0,1]$, whenever $f \in [f_{min}, \gamma_m)$. When $f \in (\gamma_M, f_{max}]$ there exist C^1 functions $\phi_i(x)$ defined on two overlapping intervals I_i contained in $[0,1]$, where $0 \in I_1,\ 1 \in I_2,\ i = 1,2$, and such that $h(\phi_i(x)) - fx = 0,\ x \in I_i$, and $\phi_2(x) > \phi_1(x)$ on $I_1 \cap I_2$. Hence there also exist discontinuous solutions of (3.3.2). Indeed, any function $u = u(x)$ where $u = \phi_1(x)$, on $[0,1]\backslash I_2,\ u(x) \in \{\phi_1(x), \phi_2(x)\}$ on $I_1 \cap I_2$ and $u = \phi_2(x)$, on $[0,1]\backslash I_1$ is the solution of (3.3.2); the number of discontinuities of u is unlimited. Inevitably, each solution of (3.3.2) is discontinuous whenever $f \in (\gamma_M, f_{max}]$. In case $f \in (\gamma_0, f_{max}]$ and ν small we expect the existence of a solution of (3.3.1) having an abrupt transition at some interior point $x_0 \in (0,1)$. When ϕ_1 is defined on the whole interval $[0,1]$ we also expect that (3.3.1) has a solution which is close to ϕ_1 on $[0,1]$ for ν small.

To make the above discussion precise, we employ general results of singularly perturbed equations due to Lin [28]. To this end, let us consider (3.3.1) as the equivalent 2×2 system

$$\begin{aligned} \nu u_x &= w \\ \nu w_x &= u + g(u) - fx \\ u(0) &= w(1) = 0 \end{aligned} \tag{3.3.3}$$

In case $f \in [f_{min}, \gamma_M)$ the piecewise continuous function

$$\bar{U}_\nu^{(1)} = \begin{cases} (0,0), & x \in [0, \nu^{1/2}) \\ (\phi_1(x), 0), & x \in [\nu^{1/2}, 1-\nu^{1/2}) \\ (\phi_1(1), 0), & x \in [1-\nu^{1/2}, 1] \end{cases} \tag{3.3.4}$$

is a formal approximation of the system (3.3.3) in the sense of [28, Theorem 2.1]. When $f \in (\gamma_0, f_{max}]$, ($\gamma_0$ is determined by Maxwell's equal area rule) there is another formal approximation of (3.3.3) given by

$$\bar{U}_\nu^{(2)} = \begin{cases} (0,0), & x \in [0, \nu^{1/2}) \\ (\phi_1(x), 0), & x \in [\nu^{1/2}, x_0 - \nu^{1/2}] \\ \left(z\left(\frac{x-x_0}{\nu}\right), z'\left(\frac{x-x_0}{\nu}\right)\right), & x \in (x_0 - \nu^{1/2}, x_0 + \nu^{1/2}) \\ (\phi_2(x), 0), & x \in [x_0 + \nu^{1/2}, 1-\nu^{1/2}) \\ (\phi_2(1), 0), & x \in [1-\nu^{1/2}, 1] \end{cases} \tag{3.3.5}$$

Here $x_0 \in (0,1)$ is determined by $fx_0 = \gamma_0$ and $z = z(\tau)$ is the heteroclinic solution of the second order autonomous ODE

$$z" = z + g(z) - \gamma_0 \tag{3.3.6}$$

such that $\lim_{\tau\to-\infty} z(\tau) = \phi_1(x_0)$, $\lim_{\tau\to\infty} z(\tau) = \phi_2(x_0)$, $z > 0$ and $z' > 0$. The existence of such a solution follows (by phase-plane analysis) from the fact that (due to the hypothesis (W)) $\phi_1(x_0)$ and $\phi_2(x_0)$ lie on the same level curve of an integral for the system (3.3.3). We note that $\phi_1(x_0) = \min h^{-1}(\gamma_0), \phi_2(x_0) = \max h^{-1}(\gamma_0)$ for any $f \in [\gamma_0, f_{max}]$ and hence the solution z does not depend on f.

It is now easy to verify that the formal approximations $\bar{U}_\nu^{(1)}$ and $\bar{U}_\nu^{(2)}$ satisfy the hypotheses (H1)-(H3) of [28]. We omit this detail. Then the main result of [28] adapted to the BVP (3.3.1) reads

Theorem 3.3.1 ([28, Theorem 2.2]). *Let $\bar{U}_\nu$ be a formal approximation of (3.3.1) given by (3.3.4) or (3.3.5). Then there exists $\nu_0 > 0$ and $\delta_0 > 0$ such that for $0 < \nu \le \nu_0$ there exists a unique true solution $u = u_\nu(x)$ of (3.3.1) with $r := \sup_{x\in[0,1]} |U_\nu(x) - \bar{U}(x)| \le \delta_0$, where $U_\nu(x) = (u(x), \nu u_x(x))$. The remainder r is of order $O(\nu^{1/2})$ when $\nu \to 0^+$.*

Remark 3.3.2. Theorem 2.2 of [28], however, does not specify the explicit dependence of the remainder r on the coefficients of the equation (3.3.1). The decay of the remainder r

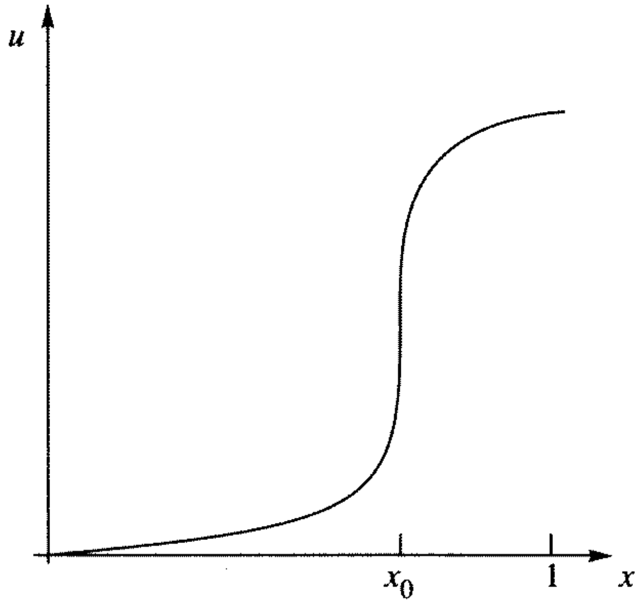

FIG. 2

the parameter f. Nevertheless, for any fixed $\eta > 0$ small enough, using the implicit function theorem and following the lines of the proofs of [28, Theorem 2.2, 4.3 and 4.4], one can show that the remainder $r = r(\nu, f)$ for the formal approximation $\bar{U}_\nu^{(1)}$ $(\bar{U}_\nu^{(2)})$ is $O(\nu^{1/2})$ uniformly with respect to $f \in [f_{min}, \gamma_M - \eta]$ and $f \in [\gamma_0 + \eta, f_{max}]$, respectively, when $\nu \to 0^+$.

For $f \in [f_{min}, \gamma_M)$, Theorem 3.3.1 asserts the existence of a true solution $u_\nu^{(1)}$ of (3.3.1) approximating the given formal approximation $\bar{U}_\nu^{(1)}$. We have

$$u_\nu^{(1)}(x) \xrightarrow{unif} \phi_1(x) \quad \text{and } v_\nu^{(1)}(x) \xrightarrow{unif} \int_x^1 \phi_1(\xi)\, d\xi \quad \text{for any } x \in [0,1] \quad \text{as} \quad \nu \to 0^+. \tag{3.3.7}$$

Again, by Theorem 3.3.1, for any $f \in (\gamma_0, f_{max}]$, there exists a solution $u_\nu^{(2)}$ of (3.3.1) such that

$$\begin{aligned} \lim_{\nu \to 0^+} u_\nu^{(2)}(x) &= \phi_1(x) \qquad \text{for any } x \in [0, x_0) \\ \lim_{\nu \to 0^+} u_\nu^{(2)}(x) &= \phi_2(x) \qquad \text{for any } x \in (x_0, 1] \end{aligned} \tag{3.3.8}$$

Hence, for small $\nu > 0$ the solution $u_\nu^{(2)}$ has a graph as in Fig.2.

By the Lebesgue dominated convergence theorem we have the uniform convergence

$$v_\nu^{(2)} \xrightarrow{unif} v_0^{(2)} \equiv \begin{cases} \int_x^1 \phi_2(\xi)\, d\xi & x \in [x_0, 1] \\ \int_x^{x_0} \phi_1(\xi)\, d\xi + \int_{x_0}^1 \phi_2(\xi)\, d\xi & x \in [0, x_0] \end{cases}$$

when $\nu \to 0^+$. Hence, the family $\left(v_\nu^{(2)}\right)_{\nu>0}$ converges uniformly to the velocity profile $v_0^{(2)}$ with a kink located at x_0 as shown in Fig.3.

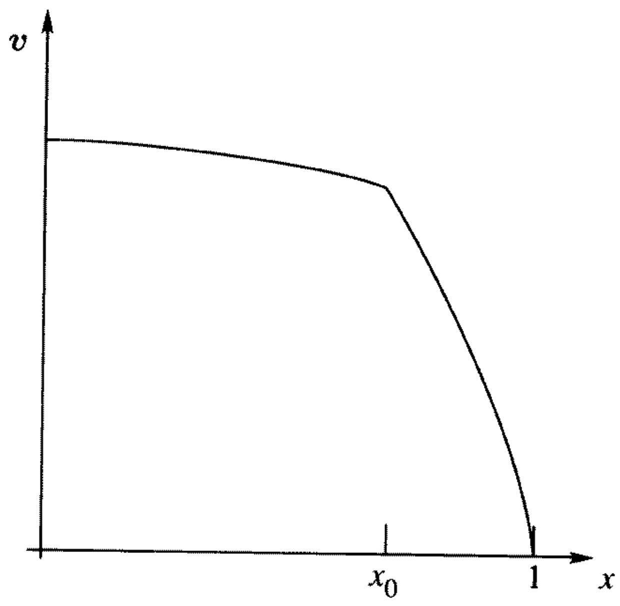

FIG. 3

It is now clear that given a pressure gradient $f \in (\gamma_0, \gamma_M)$, for any ν sufficiently small there exist at least two solutions $u_\nu^{(1)}, u_\nu^{(2)}$ of (3.3.1) satisfying (3.3.7) and (3.3.8), respectively.

Integrating the velocity $\bar{v}$ with respect to x, yields the steady state flow rate per cross-section

$$Q = 2\int_0^1 \bar{v}(x)\,dx. \tag{3.3.9}$$

Denote $Q_\nu^{(i)}$ the volumetric flow rate corresponding to the velocity $v_\nu^{(i)}$ given by (3.3.7) and (3.3.8), respectively. Clearly, for any $\eta > 0$ there is $d = d(g, \eta) > 0$ such that

$$Q_\nu^{(2)} - Q_\nu^{(1)} \geq d \quad \text{for any} \quad f \in [\gamma_0 + \eta, \gamma_M) \text{ and } \nu > 0 \text{ sufficiently small.} \tag{3.3.10}$$

We conclude this section by discussing the stability of steady states. We first show that linearized stability of a solution $\bar{u}$ of (3.3.1) extends to that of the steady state solution $[0, \bar{u}]$ of (3.1.8).

LEMMA 3.3.3. *Let $0 < \alpha < 1/\sup_{u\in R} |g'(u)|$. A steady state solution $[0, \bar{u}]$ of (3.1.8) is exponentially asymptotically stable with respect to small perturbations of initial data in the phase space $\mathcal{X}^{1/2} = X^{1/2} \times X^{1/2}$, provided that the principal eigenvalue μ_0 of the linearized*

Sturm-Liouville problem $B_1[u] = \nu^2 u_{xx} - u - g'(-\bar{u}(x))u = \mu u, \quad u(0) = u_x(1) = 0$ *is negative.*

PROOF. Let $[0, \bar{u}]$ be an arbitrary steady state solution of (3.1.8). The linearization of (3.1.8) at $[0, \bar{u}]$ has the form

$$\frac{d}{dt}[S, u] = B[S, u]$$

where the linear operator B is given by

$$B[S, u] = \left[\frac{1}{\alpha} S_{xx} + \nu^2 u_{xx} - u + g'(-\bar{u}(x))(S - u), \quad \nu^2 u_{xx} - u + g'(-\bar{u}(x))(S - u)\right], \tag{3.3.11}$$

its domain being

$$D(B) = \left\{[S, u],\ S, u \in W^{2,2}(0,1);\ S(0) = S_x(1) = u(0) = u_x(1) = 0\right\} \subset (L_2(0,1))^2$$

Denote B_1 the Sturm-Liouville operator

$$B_1[u] = \nu^2 u_{xx} - u - g'(-\bar{u}(x))u \tag{3.3.12}$$

on its domain $D(B_1) = \{w \in W^{2,2}(0,1);\ w(0) = w_x(1) = 0\} \subset L_2(0,1)$. †

Assume that the principal eigenvalue μ_0 of the linear problem $B_1[u] = \mu u,\ u \in D(B_1)$ is negative. Since B_1 is a self-adjoint Sturm-Liouville operator we have

$$\frac{(B_1[u], u)}{\|u\|^2} \leq \mu_0 < 0, \tag{3.3.13}$$

for any $u \in D(B_1), u \neq 0$. Moreover, B_1 is invertible and $B_1^{-1} : L_2 \to L_2$ is compact. Hence, the operator B is also invertible and

$$B^{-1}[\phi, \psi] = \left[\alpha A^{-1}(\psi - \phi), \quad B_1^{-1}\left(\psi - \alpha g'(-\bar{u}(.))A^{-1}(\psi - \phi)\right)\right]$$

where the linear operator A was defined in Section 2. Since, by (3.1.6), $A^{-1} : L_2 \to L_2$ is compact $B^{-1} : \mathcal{X} \to \mathcal{X}$ is compact as well. Therefore the spectrum $\sigma(B)$ consists of eigenvalues.

We will show that $Re\lambda < 0$ for any $\lambda \in \sigma(B) = \sigma_P(B)$. Suppose to the contrary that there exists an eigenvalue $\lambda \in \sigma(B)$ such that $Re\lambda \geq 0$. Let $[S, u]$ denote the eigenvector of the linear problem

$$B[S, u] = \lambda[S, u]. \tag{3.3.14}$$

Subtracting the equations for S and u we obtain $\frac{1}{\alpha} S_{xx} = \lambda(S - u)$. Thus,

$$S_x(x) = -\alpha\lambda \int_x^1 (S - u)(\xi)\, d\xi. \tag{3.3.15}$$

† When dealing with spectrum we operate with the canonical complexification of the real Hilbert space $L_2(0,1)$

Taking the inner product of (3.3.15) with $-\int_x^1 (S-u)(\xi)\, d\xi$ we obtain

$$-\|S-u\|^2 - (u, S-u) = \alpha\lambda \| \int_x^1 (S-u)(\xi)\, d\xi \|^2.$$

Since $Re\lambda \geq 0$ we have $\|S-u\|^2 \leq -Re(u, S-u) \leq \|u\|\|S-u\|$ and hence,

$$\|S-u\| \leq \|u\|. \tag{3.3.16}$$

From (3.3.15) we have $S(x) = -\alpha\lambda \int_0^x \int_r^1 (S-u)(\xi)\, d\xi\, dr$. Thus $S = \alpha\lambda J(S-u)$ where $J : L_2 \to L_2$ is a linear bounded operator with $\|J\| \leq 1$. Therefore, u satisfies the equation

$$B_1[u] + \alpha\lambda g'(-\bar{u}(.))J(S-u) = \lambda u. \tag{3.3.17}$$

Take the inner product of (3.3.17) with u to obtain

$$(B_1[u], u) = \lambda \left(\|u\|^2 - \alpha \left(g'(-\bar{u}(.))J(S-u), u \right) \right).$$

Since B_1 is self-adjoint we have $Im \left(\lambda - \alpha\lambda \left(g'(-\bar{u}(.))J(S-u), u \right) / \|u\|^2 \right) = 0$ and

$$\mu_0 \geq \frac{(B_1[u], u)}{\|u\|^2} = \lambda \left(1 - \alpha \frac{(g'(-\bar{u}(.))J(S-u), u)}{\|u\|^2} \right).$$

According to (3.3.16) we have

$$\alpha \left| \frac{(g'(-\bar{u}(.))J(S-u), u)}{\|u\|^2} \right| \leq \alpha \sup_{s\in R} |g'(s)| \frac{\|J(S-u)\|\|u\|}{\|u\|^2} \leq \alpha \sup_{s\in R} |g'(s)| < 1,$$

because $\|J\| \leq 1$. Therefore,

$$\mu_0 \geq \lambda \left(1 - \alpha \frac{(g'(-\bar{u}(.))J(S-u), u)}{\|u\|^2} \right) \geq 0,$$

a contradiction. Hence, $Re\lambda < 0$ for any $\lambda \in \sigma(B)$. By [23, Theorem 5.1.1], the steady state solution $[0, \bar{u}]$ of (3.1.8) is exponentially asymptotically stable with respect to small perturbations of initial data in the phase space $\mathcal{X}^{1/2} = X^{1/2} \times X^{1/2}$.

$\diamondsuit$

Using Lemma 3.3.3 we are able to prove the theorem below establishing stability of the solutions $\left[0, u_\nu^{(i)}\right]$, $i = 1, 2$ as well as their uniqueness for certain parameter values.

THEOREM 3.3.4. *Assume that* $0 < \alpha < 1/\sup_{u\in R}|g'(u)|$ *and* g *satisfies the hypotheses (W).*

a) *If* $f \in [f_{min}, \gamma_M)$ *and* $\nu > 0$ *is sufficiently small then the principal eigenvalue* μ_0 *of the linearized Sturm-Liouville problem* $B_1[u] = \mu u$ *at* $u_\nu^{(1)}$ *is negative. Consequently, the steady state solution* $\left[0, u_\nu^{(1)}\right]$ *of (3.1.8) is exponentially asymptotically stable with respect to small perturbations of initial data in the phase space* $\mathcal{X}^{1/2} = X^{1/2} \times X^{1/2}$

b) *If* $f \in (\gamma_0, f_{max}]$ *and* $\nu > 0$ *is sufficiently small then the principal eigenvalue* μ_0 *of the linearized Sturm-Liouville problem* $B_1[u] = \mu u$ *at* $u_\nu^{(2)}$ *is negative. Consequently, the steady state solution* $\left[0, u_\nu^{(2)}\right]$ *of (3.1.8) is exponentially asymptotically stable with respect to small perturbations of initial data in the phase space* $\mathcal{X}^{1/2} = X^{1/2} \times X^{1/2}$

c) *there exists a unique steady state solution of (3.1.8) whenever* $f \in [f_{min}, \gamma_m)$ *or* $f \in (\gamma_M, f_{max}]$ *and* $\nu > 0$ *sufficiently small.*

Proof. a) For any $u \in D(B_1), u \neq 0$ we have

$$\frac{(B_1[u], u)}{\|u\|^2} = \frac{1}{\|u\|^2}\left(-\nu^2 \int_0^1 u_x^2(x)\,dx - \int_0^1 h'(u_\nu^{(1)}(x))u^2(x)\,dx\right) \leq$$

$$\leq -\frac{1}{\|u\|^2}\int_0^1 h'(u_\nu^{(1)}(x))u^2(x)\,dx \tag{3.3.18}$$

We have $h'(\phi_1(x)) > 0$ for $x \in [0,1]$. Therefore

$$h'(u_\nu^{(1)}(x)) > 0$$

for any $x \in [0,1]$ and ν small. Hence the principal eigenvalue μ_0 of B_1 satisfies

$$\mu_0 = \sup_{u\in D(B_1), u\neq 0} \frac{(B_1[u], u)}{\|u\|^2} < 0. \tag{3.3.19}$$

b) Let us now consider the solution $u_\nu^{(2)}$ of (3.3.1) having an abrupt transition at the point $x_0 = \gamma_0/f \in (0,1)$.

First we prove that $u_\nu^{(2)}$ is increasing on $[0,1)$. The curve $h(u) - fx = 0$ splits the first quadrant into two parts (Fig.A-1).

The function $u_\nu^{(2)}$ is convex or concave at x depending on whether the point $(x, u_\nu^{(2)}(x))$ belongs to the left hand or to the right hand component labeled by +, -, respectively. According to Theorem 3.3.1 we have

$$\sup\{|u_\nu^{(2)}(x) - \phi_1(x)|, \quad x \in [0, x_0 - \nu^{1/2}]\} = O(\nu^{1/2})$$

$$\sup\{|u_\nu^{(2)}(x) - \phi_2(x)|, \quad x \in [x_0 + \nu^{1/2}, 1]\} = O(\nu^{1/2})$$

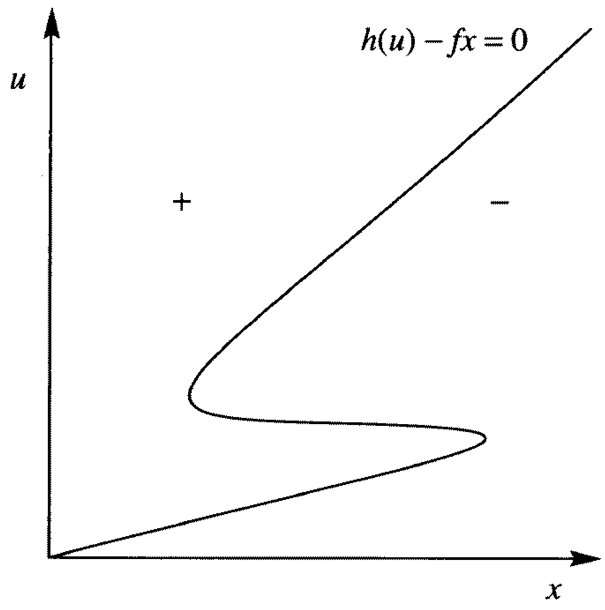

FIG. A1

as $\nu \to 0^+$. Since $u_\nu^{(2)}$ is a solution of (3.3.1) and $0 \leq u_\nu^{(2)}$ (by Proposition 3.2.1) we have $\frac{d}{dx}u_\nu^{(2)}(0) > 0$. Indeed, $\frac{d}{dx}u_\nu^{(2)}(0) \leq 0$ would imply

$$\frac{d^3}{dx^3}u_\nu^{(2)}(0) = \frac{1}{\nu^2}\left(h'(u_\nu^{(2)}(0))\frac{d}{dx}u_\nu^{(2)}(0) - f\right) < 0.$$

Since $u_\nu^{(2)}(0) = \frac{d^2}{dx^2}u_\nu^{(2)}(0) = 0$, we have $u_\nu^{(2)}(x) < 0$ for some $x > 0$, a contradiction. By an obvious indirect argument, one can show that $\frac{d}{dx}u_\nu^{(2)}(x)$ cannot become negative in $[0, x_0 - \nu^{1/2}] \cup [x_0 + \nu^{1/2}, 1]$. To prove that $\frac{d}{dx}u_\nu^{(2)}$ is positive in $(x_0 - \nu^{1/2}, x_0 + \nu^{1/2})$ suppose the contrary. Since $u_\nu^{(2)}$ convex in + and concave in - this is possible only if there exists an $\bar{x} \in (x_0 - \nu^{1/2}, x_0 + \nu^{1/2})$ such that $\frac{d}{dx}u_\nu^{(2)}(\bar{x}) < 0$ and $u_\nu^{(2)}(\bar{x}) = \phi_3(\bar{x})$, ϕ_3 being the middle branch solution of $h(u) - fx = 0$ as shown in Fig.A-2.

Let us introduce the "fast-time" variable $\tau = (x - x_0)/\nu$ for $x \in (x_0 - \nu^{1/2}, x_0 + \nu^{1/2})$ and put $u(\tau) = u_\nu^{(2)}(x_0 + \nu\tau)$. Then $\frac{d}{d\tau}u(\tau) = \nu\frac{d}{dx}u_\nu^{(2)}(x_0 + \nu\tau)$. According to Theorem 3.3.1 we have

$$\sup_{\tau \in (-\nu^{-1/2}, \nu^{-1/2})} \left|\frac{d}{d\tau}(u(\tau) - z(\tau))\right| = O(\nu^{1/2}) \quad as \quad \nu \to 0^+,$$

z being the heteroclinic solution of the problem (3.3.3). Since $\bar{x} - x_0 = O(\nu^{1/2})$ we have $|\phi_3(\bar{x}) - \phi_3(x_0)| = O(\nu^{1/2})$ as $\nu \to 0^+$. Therefore $\frac{d}{dx}u_\nu^{(2)}(\bar{x}) = \nu\frac{d}{d\tau}u((\bar{x} - x_0)/\nu)$ must have the same sign as $\frac{d}{d\tau}z((\bar{x} - x_0)/\nu)$ for any ν small. Hence $\frac{d}{dx}u_\nu^{(2)}(\bar{x}) > 0$, a contradiction.

Knowing that for any $f \in (\gamma_0, f_{max}]$ $u_\nu^{(2)}$ is increasing in $[0, 1)$ for ν small we return to the linearized eigenvalue problem $B_1[u] = \mu u$ where $B_1[u] = \nu^2 u_{xx} - h'(u_\nu^{(2)}(x))u$, $u(0) = u_x(1) = 0$. First we prove the following useful lemma.

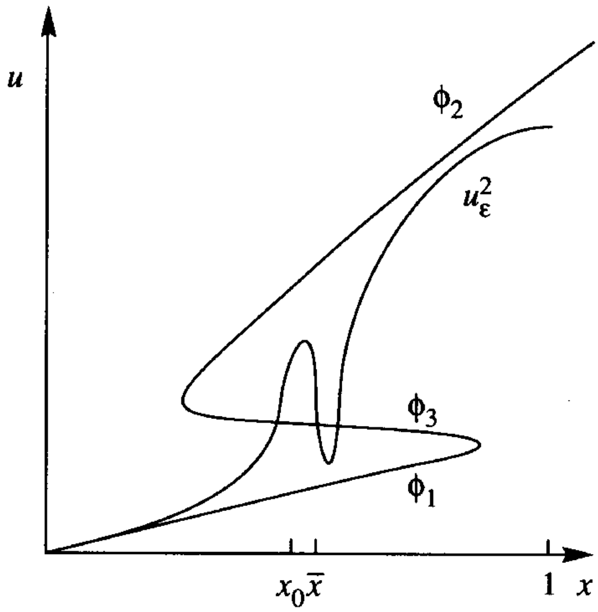

FIG. A2

LEMMA 3.3.5. *Assume $f \in [f_{min}, f_{max}]$. Let $\bar{u}$ be any nondecreasing solution of (3.3.1) such that $|h(\bar{u}(1)) - f| < (1-a)f$ and $h'(\bar{u}(x)) \geq 0$ on $[a,1]$ for some $a \in (0,1)$. Then the principal eigenvalue μ_0 of the linear operator $B_1[w] = \nu^2 w_{xx} - h'(\bar{u}(x))w, \quad w \in D(B_1)$, is negative.*

PROOF. Denote $\phi(x) = \frac{d}{dx}\bar{u}(x)$. Then ϕ satisfies

$$\nu^2 \phi_{xx} - h'(\bar{u}(x))\phi = -f; \quad \phi_x(0) = \phi(1) = 0 \tag{3.3.20}$$

and $\phi > 0$ on $[0,1)$. Let w be a solution of

$$B_1[w] = \nu^2 w_{xx} - h'(\bar{u}(x))w = \mu_0 w; \quad w(0) = w_x(1) = 0 \tag{3.3.21}$$

corresponding to the principal eigenvalue μ_0 of B_1. Since (3.3.21) is a Sturm-Liouville problem there exists w satisfying (3.3.21) such that $w > 0$ on $(0,1)$ and $\int_0^1 w(x)\,dx = 1$. If we multiply (3.3.21) by ϕ and integrate over $[0,1]$ we obtain

$$\mu_0 \int_0^1 w(x)\phi(x)\,dx = \nu^2(w_x\phi - w\phi_x)|_0^1 - f\int_0^1 w(x)\,dx \leq$$

$$\text{(because } w_x(0)\phi(0) \geq 0\text{)}$$

$$\leq -w(1)(h(\bar{u}(1) - f) - f \leq w(1)|h(\bar{u}(1)) - f| - f. \tag{3.3.22}$$

Now suppose to the contrary that $\mu_0 \geq 0$. Since $w > 0$ on $(0,1)$, $w_x(1) = 0$ we have $\nu^2 w_{xx} = h'(\bar{u}(x))w + \mu_0 w \geq 0$ on $[a,1]$. Hence $w(x) \geq w(1)$ on $[a,1]$ and, consequently,

$$1 = \int_0^1 w(x)\,dx \geq \int_a^1 w(x)\,dx \geq (1-a)w(1).$$

From (3.3.22) we obtain

$$\mu_0 \int_0^1 w(x)\phi(x)\,dx < 0.$$

Since $w \geq 0, \phi \geq 0$, we have $\mu_0 < 0$, a contradiction.

$\diamondsuit$

Now it is easy to complete the proof of part b) of Theorem 3.3.4. We fix an $a > x_0$. Then, by Theorem 3.3.1, $\sup\{|u_\nu^{(2)}(x) - \phi_2(x)|, \quad x \in [a,1]\} = O(\nu^{1/2})$ as $\nu \to 0^+$. Therefore, $|h(u_\nu^{(2)}(1)) - f| < (1-a)f$ and $h'(u_\nu^{(2)}(x)) > 0$ on $[a,1]$ for any $\nu > 0$ sufficiently small. Lemma 3.3.5 completes the proof.

Note that for certain singularly perturbed problems an asymptotic estimate of the form $\mu_0(\nu) = O(\nu)$ as $\nu \to 0^+$ is proved in [2].

c) Our next goal is to prove uniqueness of solutions of (3.3.1) for $f \in [f_{min}, \gamma_m) \cup (\gamma_M, f_{max}]$ and ν small. Let us consider the case $f \in (\gamma_M, f_{max}]$. First, we show linearized stability of an arbitrary nondecreasing solution $\bar{u}$ of (3.3.1). By Lemma 3.3.5 it is sufficient to prove that $|h(\bar{u}(1)) - f| < (1-a)f$ and $h'(\bar{u}(x)) \geq 0$ on $[a,1]$ for some $a \in (0,1)$. To this end, we recall first that according to Proposition 3.2.1 there exists a $M > 0$ such that

$$\nu \sup_{x\in[0,1]} |\bar{u}_x(x)| + \sup_{x\in[0,1]} |\bar{u}(x)| \leq M \tag{3.3.23}$$

for any solution $\bar{u}$ of (3.3.1) and $\nu > 0$.

Let $\bar{u}$ be a nondecreasing solution of (3.3.1). Let $1 > \tilde{a} > \gamma_M/f$. Then for any $x \in [\tilde{a}, 1]$ we have $fx > \gamma_M$, so $\bar{u}$ is concave on $[\tilde{a}, 1]$. Thus, by (3.3.23)

$$0 \leq \bar{u}_x(x) \leq \int_{\tilde{a}}^x \bar{u}_x(\xi)\,d\xi . \frac{1}{x - \tilde{a}} \leq \frac{4M}{1 - \tilde{a}} \tag{3.3.24}$$

for any $x \in [a,1]$ where $a = (\tilde{a}+1)/2$. Therefore, there exists an constant $M_1 > 0$ such that

$$0 \leq fx - h(\bar{u}(x)) \leq f\xi - h(\bar{u}(\xi)) + M_1(\xi - x) \tag{3.3.25}$$

for any $\xi, x \in [a,1], x \leq \xi$. Thus, by (3.3.24),(3.3.25)

$$0 \leq \nu^{1/2}\,(fx - h(\bar{u}(x))) \leq \int_x^{x+\nu^{1/2}} (f\xi - h(\bar{u}(\xi)) + M_1(\xi - x))\,d\xi =$$

$$= -\nu^2 \int_x^{x+\nu^{1/2}} \bar{u}_{xx}(\xi)\,d\xi + \frac{M_1\nu}{2} \leq \left(2M + \frac{M_1}{2}\right)\nu =: M_2\nu.$$

Hence $|fx - h(\bar{u}(x))| \leq M_2\nu^{1/2}$ for any $x \in [a,1]$, $\nu > 0$ and any nondecreasing solution $\bar{u}$ of (3.3.1).

For $\nu \leq ((fa-\gamma_M)/M_2)^2$ we have

$$h(\bar{u}(x)) \geq fx - |fx - h(\bar{u}(x))| \geq fa - |fa-\gamma_M| = \gamma_M \text{ for any } \ x \in [a,1].$$

Since $h(u) \leq \gamma_M$ for $u \leq c_2$ (see Fig.1), we have $\bar{u}(x) \geq c_2$ on $[a,1]$, hence $h'(\bar{u}(x)) \geq 0$ for $x \in [a,1]$. By Lemma 3.3.5, the principal eigenvalue μ_0 of the problem $B_1[w] = \nu^2 w_{xx} - h'(\bar{u}(x))w = \mu w,\ w \in D(B_1)$, is negative.

Now, consider the parabolic equation

$$u_\tau = \nu^2 u_{xx} - h(u) + fx$$

$$u(\tau,0) = u_x(\tau,1) = 0; \quad \tau \geq 0, \quad u(0,x) = u_0(x), \quad x \in [0,1]$$

This equation generates a gradient-like semidynamical system $\mathcal{S}(\tau), \tau \geq 0$, in the Hilbert space $X^{1/2} = \{u \in W^{1,2}(0,1),\ u(0)=0\}$ defined by $\mathcal{S}(\tau)u_0 = u(\tau,.)$, where $u(0,.) = u_0(.)$ (see [23, Chapter 4]). The set $\mathcal{K} = \{u \in X^{1/2}, u_x(x) \geq 0, \text{ a.e. on } [0,1]\}$ is a closed convex cone in $X^{1/2}$. Moreover, $\mathcal{K}$ is invariant under $\mathcal{S}$, i.e.

$$u(\tau,.) \in \mathcal{K} \ \text{ whenever } \ u(0,.) \in \mathcal{K} \ \text{ for any } \ \tau \geq 0$$

Indeed, the function $w(\tau,x) = \begin{cases} -u_x(\tau,x), & x \in [0,1],\ \tau \geq 0 \\ -u_x(\tau,-x), & x \in [-1,0],\ \tau \geq 0 \end{cases}$ is the solution of the scalar parabolic equation

$$w_\tau = \nu^2 w_{xx} - h'(u(x))w - f$$

$$w(\tau,-1) = w(\tau,1) = 0.$$

Therefore $w(\tau,x) \leq 0$ whenever $w(0,x) \leq 0$ by the Maximum Principle (see [41]). Hence $\mathcal{S}$ is a semidynamical system on the complete metric space $\mathcal{K}$ with the topology induced by $X^{1/2}$.

To complete the proof we argue similarly as in [2,Theorem 4]. Since $\mathcal{K}$ is invariant, it is the union of (disjoint) attraction domains of the nondecreasing stationary solutions of (3.3.24). Because those solutions are asymptotically stable, these attraction domains are open in $\mathcal{K}$. Since the set $\mathcal{K}$ is connected, it cannot be a union of two non-empty disjoint open sets, hence $u_\nu^{(2)}$ is the unique stationary solution in $\mathcal{K}$.

Now, let $\bar{u}$ be arbitrary solution of (3.3.1) (not necessarily nondecreasing). By Proposition 3.2.1, $\bar{u}$ is bounded and $\bar{u} \geq 0$. Then there exist $\bar{u}^-, \bar{u}^+ \in \mathcal{K} \cap D(A)$ such that $\bar{u}^-(x) \leq \bar{u}(x) \leq \bar{u}^+(x),\ x \in [0,1]$. With regard to the Maximum Principle ([41, Chapter 3, Theorem 3]) we obtain $\mathcal{S}(\tau)\bar{u}^-(x) \leq \bar{u}(x) \leq \mathcal{S}(\tau)\bar{u}^+(x)$ for any $\tau \geq 0$ and $x \in [0,1]$. Since $\mathcal{S}(\tau)\bar{u}^\pm \in \mathcal{K}$, for any $\tau \geq 0$, we have $\mathcal{S}(\tau)\bar{u}^\pm \to u_\nu^{(2)}$ as $\tau \to \infty$. Thus, $\bar{u} = u_\nu^{(2)}$.

Hence, the solution $u_\nu^{(2)}$ is unique, provided ν is small and $f \in (\gamma_M, f_{max}]$. The proof of uniqueness of solutions of (3.3.1) for $f \in [f_{min}, \gamma_m)$ is similar. It completes the proof of Theorem 3.3.4.

$\diamondsuit$

Section 4.

Spurt

Having developed the mathematical background we are in a position to explain the occurrence of spurt for a fluid governed by the system of equations (3.1.8).

Suppose that we are loading the pressure gradient quasi-statically from f_{min} to f_{max} allowing the system to settle down to its equilibrium state at each step.

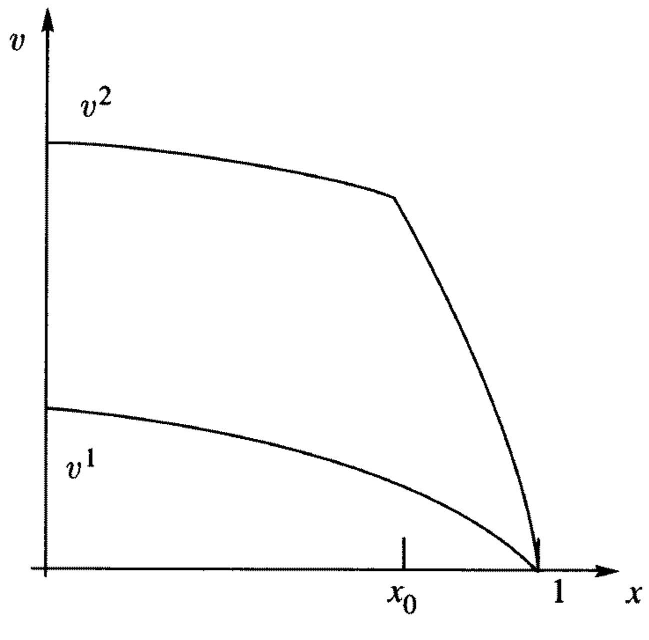

Fig. 4

Since $v_\nu^{(1)} = v_\nu^{(1)}(f)$ depends continuously on f, the volumetric flow rate $Q_\nu^{(1)} = Q_\nu^{(1)}(f)$ of the steady state velocity $v_\nu^{(1)} = v_\nu^{(1)}(f)$ for $f < \gamma_M$ forms a continuous curve. At each step of the "loading-stabilization" procedure, the volumetric flow rate corresponding to the velocity $v(T)$ is close to $Q_\nu^{(1)} = Q_\nu^{(1)}(f)$ when T is large enough.

The situation changes dramatically when the pressure gradient f passes γ_M. For $f > \gamma_M$ the solution has no other possibility than to settle down to the unique steady state solution $\left[0, u_\nu^{(2)}(., f)\right]$ of system (3.1.8) which is globally asymptotically stable by Theorem 3.3.4. Hence, by (3.3.10), this small change of the pressure gradient causes a jump of size $d > 0$ in the volumetric flow rate as shown in Fig.4. This jump is equal to the area between the two equilibrium solutions $v_\nu^{(1)}$ and $v_\nu^{(2)}$.

For f varying in the interval $(\gamma_M, f_{max}]$, the "loading-stabilization" can be repeated. The corresponding volumetric flow rates are close to the continuous curve $f \mapsto Q_\nu^{(2)}(f)$ of the steady state volumetric flow rates in Fig.5.

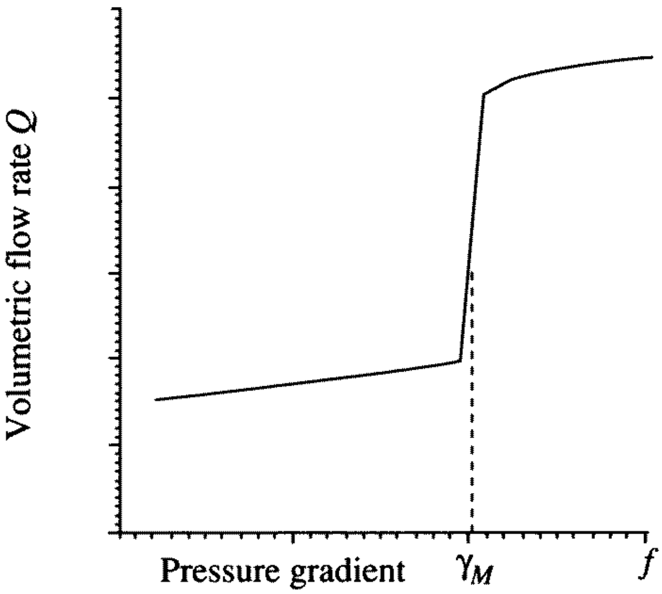

FIG. 5

Let us note that earlier models that did not include the diffusion terms in their constitutive relations also captured the spurt phenomenon [29], [30], [36]. For $f > \gamma_M$ the principal difference between our explanation of spurt and that of papers mentioned is: the change in volumetric flow rate as f passes through the critical value γ_M on loading, is much more drastic in our model than the earlier ones; here the "kink" develops at the point $0 < \gamma_0/\gamma_M < 1$ very suddenly, and then moves slowly with a definite speed towards the centerline. In [29], [30], the kink develops at the wall; for $f > \gamma_M$, the layer position is $x^* = \gamma_M/f$. The phenomenon of latency that occurs on loading described in [29], [30] is not discussed here.

Section 5.

Hysteresis

We now consider the loading - unloading cyclic process. The behavior of the volumetric flow rate during the loading period has been described in the previous section. Recall that the volumetric flow rate increased rapidly when the pressure gradient passed the value γ_M. Now let us unload the pressure gradient starting from $f = f_{max}$. By convention, as long as f stays larger than γ_0, the solution still settles down on $\left[v_\nu^{(2)}(.,f), u_\nu^{(2)}(.,f)\right]$. On the other hand, for any $f < \gamma_m$ there exists the unique solution $\left[v_\nu^{(1)}(.,f), u_\nu^{(1)}(.,f)\right]$. Therefore the solution $\left[v_\nu^{(2)}(.,f), u_\nu^{(2)}(.,f)\right]$ ceases to exist at some critical value near γ_0. The figure below shows two branches of the bifurcation diagram corresponding to the stable steady states $\left[v_\nu^{(i)}(.,f), u_\nu^{(i)}(.,f)\right]$, $i = 1, 2$.

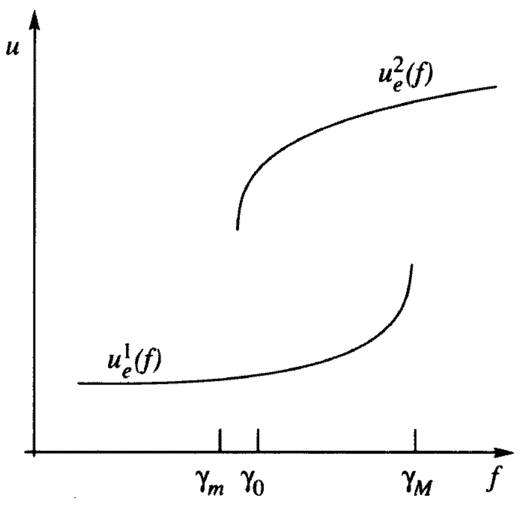

Fig. 6

By (3.3.10), $Q_\nu^{(2)}(f) - Q_\nu^{(1)}(f) \geq d(\eta) > 0$ for any $f \in [\gamma_0 + \eta, \gamma_M)$ where $\eta > 0$ is fixed. Hence there is a hysteresis loop as shown in Fig.7.

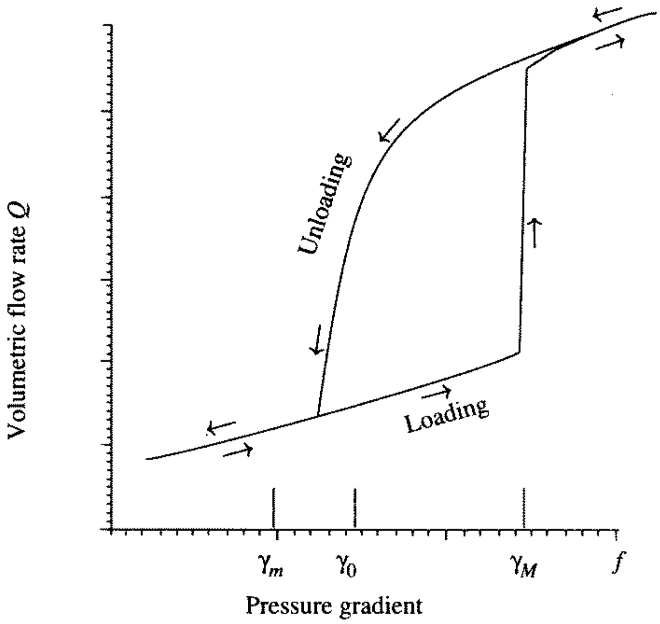
Volumetric flow rate Q
Unloading
Loading
γ_m
γ_0
γ_M
f
Pressure gradient

FIG. 7

Section 6.

Numerical simulations

In this section we present some numerical results exhibiting spurt and hysteresis. Recall that our model leads to the system of governing equations

$$\begin{aligned} \varrho v_t &= \varepsilon v_{xx} + \sigma_x + f \\ \sigma_t &= \nu^2 \sigma_{xx} + g(v_x) - \lambda\sigma \\ for \quad & (t,x) \in [0,\infty] \times [0, r_{cap}] \end{aligned} \tag{6.1}$$

with boundary conditions

$$\begin{aligned} v_x(t,0) =& v(t, r_{cap}) = 0; \\ \sigma(t,0) = 0;& \;\; \sigma_x(t, r_{cap}) = -f \end{aligned}$$

and initial data

$$v(0,x) = v_0(x); \qquad \sigma(0,x) = \sigma_0(x) \qquad for\ a.e.\ \ x \in [0, r_{cap}] \tag{6.2}$$

We will consider an analytic function g of a particular form

$$g(u) = \mu \frac{u}{1 + \frac{1-a^2}{\lambda^2} u^2} \tag{6.3}$$

where $\mu > 0$ is the elastic modulus, a is the dimensionless slip parameter and λ is the relaxation time of the polymer. The particular choice of the function g is taken from [30, Section 3].

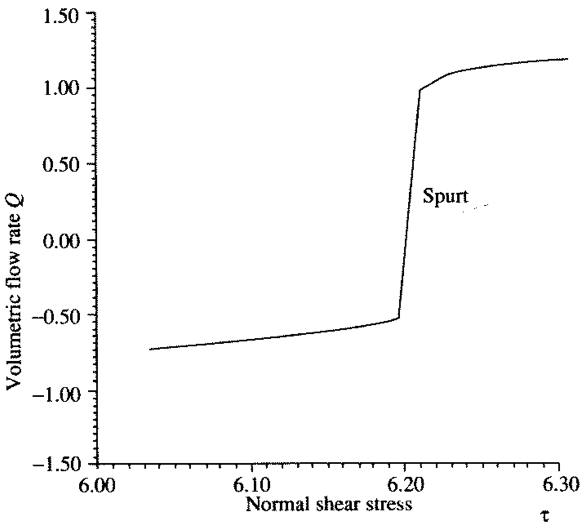

Fig. 8a

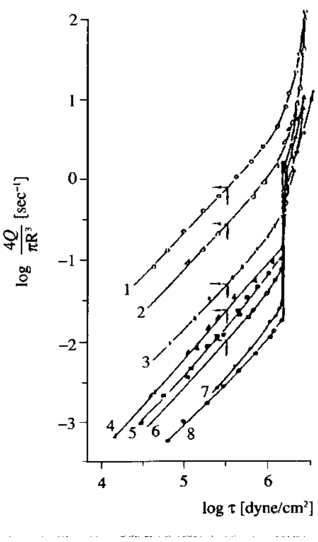

Fig. 8b

First, we determine the magnitude of the coefficient $\nu > 0$ in (6.1). Following [18]

$$\nu^2 \approx \frac{k.\theta}{2\xi} \tag{6.4}$$

where θ is the absolute temperature, k is the Boltzman constant, ξ is the hydrodynamic resistance of one dumbbell bead (assumed to be constant). If we take typical values of $\theta \approx 10^2$K, $\xi \approx 10^{-9}$kg.s^{-1} and recall that $k \approx 10^{-23}$J.K^{-1} we obtain $\nu^2 \approx 10^{-12}$m^2.s^{-1}. In our numerical simulations we have chosen the fixed value

$$\nu^2 = 4.10^{-12}\text{m}^2.\text{s}^{-1} \tag{6.5}$$

We next turn to the Vinogradov *et al.* rheological data. In all experiments, the radius of the capillary was

$$r_{cap} = 0,48.10^{-3}\text{m}.$$

The elastic modulus μ and the density ϱ have been taken constant for all samples and equal to

$$\mu = 6.10^4\text{Pa} \ \ ; \varrho = 10^3\text{kg.m}^{-3}, \tag{6.6}$$

respectively.

Numerical experiments were performed for the polyisoprene PI-3 which was the first sample for which spurt was observed ([48, Fig.3b]). According to [48] and [27, p.323] we have

$$\lambda = 0,1\text{s}^{-1}, \quad \varepsilon = 0,01484.\frac{\mu}{\lambda} = 8,9.10^3\text{Pa.s}^{-1} \quad \text{and} \quad a = 0,98 \tag{6.7}$$

We see that the constants $\alpha = \frac{\varrho r_{cap}^2 \lambda}{\varepsilon} = 2,58.10^{-9}$ and $\frac{\nu^2}{r_{cap}^2\lambda} = 10^{-4}$ introduced in Section 2 can be treated as small parameters. It is easy to verify that the real analytic function

$$h(u) = \lambda u + \frac{\mu}{\varepsilon}.\frac{u}{1 + \frac{1-a^2}{\varepsilon^2\lambda^2}u^2}$$

is of van der Walls type (see the hypothesis (W)).

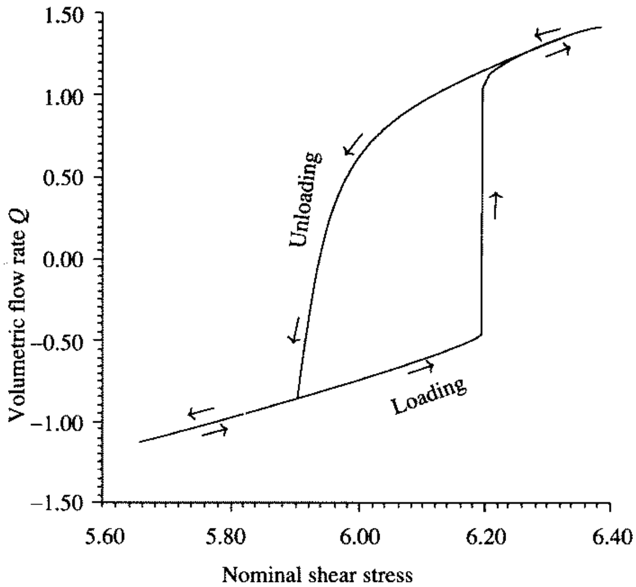

FIG. 9

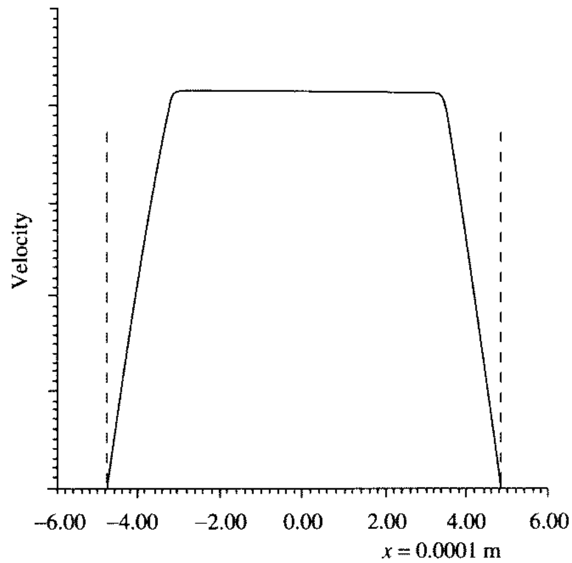

FIG. 10

As our first numerical experiment, we simulated spurt. In S.I. units, we choose

$$f_{min} = 9,3.10^7 \text{ kg.m}^{-2}.\text{s}^{-2}, \quad f_{max} = 51,2.10^7 \text{ kg.m}^{-2}.\text{s}^{-2}, \quad \Delta f = 1,8.10^7 \text{ kg.m}^{-2}.\text{s}^{-2}.$$

The startup initial condition (for $f = f_{min}$) was chosen $(v_0, u_0) = (0, 0)$. At each loading step, the solutions were followed for a sufficiently long time $T_{max} = 10$ s to allow them to settle down. Since $\alpha > 0$ was very small, we could use the Crank-Nicholson implicit time-space discretization scheme. The spatial mesh contained a total of 200 nodes.† The time step was chosen as $\Delta t = 0.0005$ s.

Fig.8 shows the results obtained (8a) and compares them with Vinogradov *et al*'s experimental data (8b, the flow curve for PI-3 is labeled by 3). Following [48] *c-g-s* units are employed and axes are in the logarithmic scale. The nominal shear stress τ is defined by $\tau = r_{cap} f$ (see (48) [27]). Since we have considered a planar flow instead of a capillary flow the corresponding definition of a volumetric flow rate is

$$Q = \frac{3}{r_{cap}^2} \int_0^{r_{cap}} v(x)\, dx$$

(see (47) [27]).

Finally, we have performed numerical simulations of a loading-unloading cycle. The hysteresis loop under the cyclic load is displayed in Fig.9. Fig.10 shows the steady, kinked velocity profile for the spurt value of the nominal shear stress $\tau = 1.61.10^6$ dyne.cm^{-2} ($\log \tau = 6.21$).

† The spatial mesh should contain 200-500 nodes in order to compare the distance between grids with a typical lenght of a polymer molecule ($\approx 10^{-7}$ m) [35]

Section 6.

Discussion

We have proposed a modification of the mathematical model of shearing motions leading to a system of governing equations including a diffusion term $\nu^2\sigma_{xx}$ in the constitutive equation. In addition, we have described the asymptotic behavior of solutions which is simple in typical situations - each solution tends to some steady state and the number of steady states is finite.

The diffusion term makes the system of governing equations parabolic. As a consequence of the resulting parabolic smoothing effect the system will admit a finite dimensional invariant manifold as well as a compact global attractor. The existence of invariant manifolds and their singular limit dynamics for $\alpha \to 0^+$ will be discussed in Part II of this thesis.

Singular limit dynamics of invariant manifolds

Section 8.

Foreword

In this part we will treat the qualitative properties of semiflows generated by the following system of abstract evolution equations

$$\begin{aligned} u' + A_\alpha u &= g(u, w) \\ \alpha w' + B_\alpha w &= f(u) \end{aligned} \tag{8.1$_\alpha$}$$

where $\alpha \in (0, \alpha_1]$, $\{A_\alpha\}_{\alpha \geq 0}$ and $\{B_\alpha\}_{\alpha \geq 0}$ are continuously depending families of sectorial operators in the Banach spaces X and Y, respectively, $g : X^\gamma \times Y^\beta \to X$; $f : X^\gamma \to Y$ are nonlinear C^1 functions for some $\gamma, \beta \in [0, 1)$. Hereafter X^γ and Y^β will denote the fractional power spaces with respect to the sectorial operators A_0 and B_0, respectively (cf. [23, Chapter 1]).

The goal of this part is to establish the existence of a finite dimensional invariant C^1 manifold $\mathcal{M}_\alpha$ for the semiflow $\mathcal{S}_\alpha(t), t \geq 0$, generated by system $(8.1)_\alpha$. We furthermore prove that both $\mathcal{M}_\alpha$ and the vector field on $\mathcal{M}_\alpha$ converge in the C^1 topology towards the ones corresponding to $\alpha = 0$ (Theorem 10.2.7). By combining this result with the well-known theory of Morse-Smale vector fields (cf. [39]) one can prove topological equivalence of vector fields on $\mathcal{M}_\alpha$ and $\mathcal{M}_0$ whenever the vector field on $\mathcal{M}_0$ is Morse-Smale.

The techniques used in the proof of Theorem 10.2.7 are similar, in spirit, to those developed by Mora and Solà-Morales [34]. The construction of an invariant manifold for (8.1) is based on the well-known method of integral equations due to Lyapunov and Perron. In this method the substantial role is played by the choice of functional spaces we will operate with. For the proof of the existence of $\mathcal{M}_\alpha$, we notice that the usual choice would be the Banach space consisting of all continuous functions on $(-\infty, 0]$ with values in $X^\gamma \times Y^\beta$ equipped with some exponentially weighted sup or integral norm. Then one can look for $\mathcal{M}_\alpha$ as the union of all solutions of (8.1) belonging to this functional space. We

Results of Part II are contained in the paper [45] refer to [15], [31], [33], [34] for details. However, it turns out that such a setting does not capture the singular limit behavior of the derivative of the vector field on $\mathcal{M}_\alpha$ as $\alpha \to 0^+$. In order to overcome this difficulty, by contrast to the approach of [34], we will operate with Banach spaces consisting of all Hölder continuous functions which grow exponentially at $-\infty$. In the proof of Theorem 10.2.7 an important tool is a slightly modified version of the two parameter contraction theorem due to Mora and Solà-Morales covering differentiability and continuity of a family of nonlinear contractive mappings operating between a pair of Banach spaces.

Part II is organized as follows. Section 9 is devoted to preliminaries. Perturbations of sectorial operators are investigated in Section 9.1. The existence of solutions of (8.1) is established in Section 9.2. In Section 9.3 we introduce functional spaces we will work with. The core of Part II is contained in Section 10. First, we prove the existence of a family of invariant manifolds $\{\mathcal{M}_\alpha,\}_{\alpha\geq 0}$ for system (8.1). The singular limit dynamics of $\mathcal{M}_\alpha$, $\alpha \to 0^+$, is investigated in Section 10.2. The main results are summarized in Theorem 10.2.7. Section 11 is focused on some applications of Theorem 10.2.7. In Section 11.1 we treat the non-Newtonian model of shearing motions of a fluid introduced in Section 2. The aim is twofold: 1) using the fairly standard method of *à priori* estimates we show the existence of a compact global attractor $\mathcal{A}_\alpha$ for $(3.1.1)_\alpha$; 2) with regard to the Morse-Smale structure of the limiting equation $(3.1.1)_0$ we establish topological equivalence of the semiflow on the attractor of $(3.1.1)_\alpha$, $\alpha > 0$ small and that of $(3.3.1)_0$. Finally, Section 11.2 illustrates an application of obtained results to some abstract second order evolution equations arising in the mathematical theory of elastic systems with strong dissipation.

Section 9.

Preliminaries

9.1. Properties of a family of sectorial operators

The goal of this section is to establish perturbation results for a family of closed densely defined operators. First, we recall the definition of a sectorial operator. Let $\mathcal{X}$ be a Banach space, $L : D(L) \subset \mathcal{X} \to \mathcal{X}$ be a closed densely defined operator. The operator L is called sectorial if for some $M \geq 1, a \in R$ and $\phi \in (0, \pi/2)$ the sector

$$S_{a,\phi} := \{\lambda \in C;\ \phi \leq \arg \mid \lambda - a \mid \leq \pi; \lambda \neq a\}$$

is contained in the resolvent set $\varrho(L)$ and

$$\|(\lambda - L)^{-1}\| \leq M/ \mid \lambda - a \mid, \quad \text{for any } \lambda \in S_{a,\phi} \tag{9.1.1}$$

(cf. [23, Def. 1.3.1]). It is well known (see, [23, Th. 1.3.4]) that if L is sectorial then the operator $-L$ generates an analytic semigroup $\exp\,(-Lt), t \geq 0$, and

$$\exp\,(-Lt) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(\lambda + L)^{-1}\, d\lambda \quad t > 0 \tag{9.1.2}$$

where Γ is a contour in $\varrho(-L)$ such that $\arg\lambda \to \pm\theta$ as $\mid \lambda \mid \to \infty$ for some $\theta \in (\pi/2, \pi)$.

Consider a family $\{L_\alpha\}$ of closed densely defined operators in a Banach space $\mathcal{X}$ satisfying the following hypothesis

$$(H1)\begin{cases} 1) & D(L_0) = D(L_\alpha) \\ 2) & 0 \in \varrho(L_\alpha) \text{ and } L_0 L_\alpha^{-1} \to I \text{ as } \alpha \to 0^+ \text{ in } L(\mathcal{X}, \mathcal{X}) \\ 3) & L_0^{-1} L_\alpha^{-1} = L_\alpha^{-1} L_0^{-1} \\ 4) & \text{Re } \sigma(L_0) > \omega > 0 \ \text{ (i.e. Re } \lambda > \omega > 0 \text{ for any } \lambda \in \sigma(L_0)) \text{ and} \\ & L_0 \text{ is a sectorial operator in } \mathcal{X} \text{ its sector being } S_{a,\phi} \text{ and} \\ & \|(\lambda - L_0)^{-1}\| \le M/ \mid \lambda - a \mid, \text{ for any } \lambda \in S_{a,\phi} \end{cases}$$

for any $\alpha \in [0, \alpha_0]$.

Lemma 9.1.1. *Assume that the hypothesis (H1) is satisfied. Then*

a) $(\lambda - L_0)^{-1}$ *commutes with* $(\mu - L_\alpha)^{-1}$ *for any* $\alpha \in [0, \alpha_0]$, $\lambda \in \varrho(L_0)$ *and* $\mu \in \varrho(L_\alpha)$

There exists $0 < \alpha_1 \le \alpha_0$ *such that*

b) $L_\alpha L_0^{-1} \in L(\mathcal{X}, \mathcal{X})$ *for any* $\alpha \in [0, \alpha_1]$ *and* $L_\alpha L_0^{-1} \to I$ *as* $\alpha \to 0^+$

c) for any $\alpha \in [0, \alpha_1]$, *the operator* L_α *is sectorial in* $\mathcal{X}$ *with the sector* $S_{a,\phi}$ *and* $\|(\lambda - L_\alpha)^{-1}\| \le 2M/ \mid \lambda - a \mid$, *for any* $\lambda \in S_{a,\phi}$

Proof. The part a) is obvious because L_α^{-1} commutes with L_0^{-1}. b) Take $\alpha_1 > 0$ such that $\|I - L_0 L_\alpha^{-1}\| < 1$ for any $\alpha \in [0, \alpha_1]$. Then $L_\alpha L_0^{-1} = \left(L_0 L_\alpha^{-1}\right)^{-1} = \sum_{n=0}^\infty \left(I - L_0 L_\alpha^{-1}\right)^n$. Thus, $\|L_\alpha L_0^{-1}\| \le 1/\left(1 - \|I - L_0 L_\alpha^{-1}\|\right)$ and $\|L_\alpha L_0^{-1} - I\| \le \|I - L_0 L_\alpha^{-1}\|/\left(1 - \|I - L_0 L_\alpha^{-1}\|\right)$

c) Similarly as in the proof of [23, Th. 1.3.2], for any $\lambda \in S_{a,\phi}$ we obtain

$$\|(\lambda - L_\alpha)^{-1}\| = \|(\lambda - L_0)^{-1} \left[I + \left(L_\alpha L_0^{-1} - I\right)\left(I - \lambda(\lambda - L_0)^{-1}\right)\right]^{-1} \| \le$$

$$\le \frac{M}{\mid \lambda - a \mid \left(1 - \|L_\alpha L_0^{-1} - I\| \left(1 + \frac{|\lambda| M}{|\lambda - a|}\right)\right)} \le \frac{2M}{\mid \lambda - a \mid}$$

for any $\alpha \in [0, \alpha_1]$ provided α_1 is small enough.

$\diamondsuit$

Let L be a closed densely defined operator in a Banach space $\mathcal{X}$. Suppose that $\sigma(L) = \sigma_1 \cup \sigma_2$ where σ_1, σ_2 are disjoint spectral sets and σ_1 is bounded in C. Recall that the projector $P : \mathcal{X} \to \mathcal{X}$ associated with the operator L and the spectral set σ_1 is defined by

$$P := \frac{1}{2\pi i} \int_{\Gamma_1} (\lambda - L)^{-1} \, d\lambda \tag{9.1.3}$$

where Γ_1 is a closed Jordan curve such that $\sigma_1 \subset \text{int} < \Gamma_1 >$ and $\sigma_2 \subset \text{ext} < \Gamma_1 >$. Denote

$$Q := I - P, \quad \mathcal{X}_1 := P\mathcal{X} \quad \text{and} \quad \mathcal{X}_2 := Q\mathcal{X}$$

Besides (H1) we also make the following hypothesis

$$(H2) \begin{cases} 1) & L_0^{-1} \text{ is a compact linear operator on } \mathcal{X} \\ 2) & \text{there are } 0 < \lambda_- < \lambda_+ < \infty \text{ such that } \sigma(L_0) = \sigma_1^0 \cup \sigma_2^0 \text{ where} \\ & \sigma_1^0 = \{\lambda \in \sigma(L_0);\ \ \text{Re}\,\lambda < \lambda_-\} \text{ and } \sigma_2^0 = \{\lambda \in \sigma(L_0);\ \ \text{Re}\,\lambda > \lambda_+\} \end{cases}$$

Notice that the condition $(\text{H2})_1$ implies that σ_1^0 is finite and $\dim \mathcal{X}_{1,0} < \infty$ where $\mathcal{X}_{1,0} := P_0\mathcal{X}$, P_0 is the projector in $\mathcal{X}$ associated with L_0 and σ_1^0.

Concerning continuity properties of projectors and spectral sets we have

Lemma 9.1.2. *Assume that the hypotheses (H1) and (H2) are satisfied. Then there is $\alpha_1 > 0$ sufficiently small and such that for any $\alpha \in [0, \alpha_1]$*

a) $\sigma(L_\alpha) = \sigma_1^\alpha \cup \sigma_2^\alpha$ where
$\sigma_1^\alpha = \{\lambda \in \sigma(L_\alpha);\ \ Re\,\lambda < \lambda_-\}$ and $\sigma_2^\alpha = \{\lambda \in \sigma(L_\alpha);\ \ Re\,\lambda > \lambda_+\}$

b) $P_\alpha \to P_0$ in $L(\mathcal{X}, \mathcal{X})$ as $\alpha \to 0^+$ where P_α is the projector associated with L_α and σ_1^α. Furthermore, $P_0P_\alpha = P_\alpha P_0$.

c) $P_\alpha\,|_{\mathcal{X}_{1,0}}: \mathcal{X}_{1,0} \to \mathcal{X}_{1,\alpha}$ is a linear isomorphism where $\mathcal{X}_{1,\alpha} := P_\alpha\mathcal{X}$. Moreover, $\dim \mathcal{X}_{1,0} = \dim \mathcal{X}_{1,\alpha} < \infty$

Proof. Since the spectrum $\sigma(L_0)$ is contained in the angle $\{\lambda \in C;\ \arg \mid \lambda - a \mid \leq \phi\}$ and $L_\alpha^{-1}L_0^{-1} = L_0^{-1}L_\alpha^{-1}$ the proof of the part a) follows from the inequality $\text{dist}_H\left(\sigma(L_\alpha^{-1}), \sigma(L_0^{-1})\right) \leq \|L_\alpha^{-1} - L_0^{-1}\| \to 0$ as $\alpha \to 0^+$ (cf. [46, Ex.3, p.287]). Here dist_H denotes the Haussdorff set distance in the complex plane. The proof of the part b) follows from Lemma 9.1.1 a), (9.1.3) and the fact that

$$\|(\lambda - L_\alpha)^{-1} - (\lambda - L_0)^{-1}\| = \|(L_\alpha L_0^{-1} - I)L_0(\lambda - L_0)^{-1}(\lambda - L_\alpha)^{-1}\| \leq$$

$$\leq \|L_\alpha L_0^{-1} - I\|\|L_0(\lambda - L_0)^{-1}\|\|(\lambda - L_\alpha)^{-1}\| \to 0 \text{ as } \alpha \to 0^+$$

uniformly with respect to $\lambda \in \Gamma_1$ where Γ_1 is a closed Jordan curve such that $\sigma_1^\alpha \subset \text{int} < \Gamma_1 >$ and $\sigma_2^\alpha \subset \text{ext} < \Gamma_1 >$ for any $\alpha \in [0, \alpha_1]$.

c) Since L_0^{-1} is compact we have that σ_1^0 is finite and $\dim \mathcal{X}_{1,0} < \infty$. It is obvious that

$$P_\alpha\,|_{\mathcal{X}_{1,0}}: \mathcal{X}_{1,0} \to \mathcal{X}_{1,\alpha} \quad \text{and} \quad P_0\,|_{\mathcal{X}_{1,\alpha}}: \mathcal{X}_{1,\alpha} \to \mathcal{X}_{1,0}$$

are one-to-one linear operators, provided that $\|P_\alpha - P_0\| < 1$. Hence $P_\alpha\,|_{\mathcal{X}_{1,0}}: \mathcal{X}_{1,0} \to \mathcal{X}_{1,\alpha}$ is a linear isomorphism and $\dim \mathcal{X}_{1,0} = \dim \mathcal{X}_{1,\alpha} < \infty$ for any $\alpha \in [0, \alpha_1]$ where α_1 is small enough. $\diamondsuit$

REMARK. For any $\alpha \geq 0$ small enough, we denote

$$P_\alpha^{(-1)} := \left(P_\alpha \mid_{\mathcal{X}_{1,0}}\right)^{-1} : \mathcal{X}_{1,\alpha} \to \mathcal{X}_{1,0} \tag{9.1.4}$$

the inverse operator of $P_\alpha \mid_{\mathcal{X}_{1,0}}: \mathcal{X}_{1,0} \to \mathcal{X}_{1,\alpha}$. Since $P_\alpha \to P_0$ as $\alpha \to 0^+$ the linear operator $P_\alpha^{(-1)} P_\alpha$ converges to P_0 in the space $L(\mathcal{X}, \mathcal{X})$.

If L_0 is a sectorial operator in $\mathcal{X}$ with $\text{Re}\ \sigma(L_0) > \omega > 0$ then the fractional powers $L_0^\gamma, \gamma \in R$, can be defined (see, [23, Def. 1.4.7]). Under the hypothesis (H1) we have shown that $-L_\alpha$ generates an analytic semigroup $\exp\,(-L_\alpha t), t \geq 0$. In the following lemma we give some estimates on the growth of $\exp\,(-L_\alpha t)$.

LEMMA 9.1.3. *Assume that the hypothesis (H1) is satisfied. Then there is a $C > 0$ such that for any $\alpha \in [0, \alpha_1]$ and $\gamma \geq 0$, the following estimates hold*

a) $\|exp\,(-L_\alpha t)\| \leq Ce^{-\omega t}; \quad t \geq 0$

b) $\|L_0^\gamma\,(exp\,(-L_\alpha t) - exp\,(-L_0 t))\,\| \leq C\|L_0 L_\alpha^{-1} - I\| t^{-\gamma} e^{-\omega t}; \quad t > 0$

c) $\|L_0^\gamma exp\,(-L_\alpha t)\| \leq Ct^{-\gamma} e^{-\omega t}; \quad t > 0$

PROOF. Using the translation $L_\alpha - \omega I$ it is sufficient to prove Lemma with $\omega = 0$. The proof of a) immediately follows from Lemma 9.1.1 and [23, Th. 1.3.4].

To show b), we make use of the integral representation of $\exp\,(-L_\alpha t)$. We obtain, for $t > 0$

$$L_0^\gamma\,(\exp\,(-L_\alpha t) - \exp\,(-L_0 t)) = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} L_0^\gamma (\lambda + L_0)^{-1} (L_0 L_\alpha^{-1} - I) L_\alpha (\lambda + L_\alpha)^{-1}\, d\lambda =$$

$$= \frac{1}{2\pi i} \int_\Gamma e^{\mu} L_0^\gamma (\mu/t + L_0)^{-1} (L_0 L_\alpha^{-1} - I) L_\alpha (\mu/t + L_\alpha)^{-1} \frac{d\mu}{t}.$$

Since $\text{Re}\ \sigma(L_\alpha) > \omega > 0$ one can choose a contour Γ with the property: $\text{Re}\ \lambda < 0$ for any $\lambda \in \Gamma$. By [23, Th. 1.4.4], there is a $M' > 0$ such that $\|L_0^\gamma(\lambda + L_0)^{-1}\| \leq M' \mid \lambda \mid^{\gamma - 1}$ for any $\lambda \in \Gamma$. Furthermore, $\|L_\alpha(\lambda + L_\alpha)^{-1}\| \leq 1 + \mid \lambda \mid \frac{2M}{|\lambda|} = 1 + 2M$ for any $\lambda \in \Gamma$ Hence,

$$\|L_0^\gamma\,(\exp\,(-L_\alpha t) - \exp\,(-L_0 t))\,\| \leq C\|L_0 L_\alpha^{-1} - I\| t^{-\gamma}; \quad t > 0$$

Because of the well known estimate $\|L_0^\gamma \exp\,(-L_0 t)\| \leq Ct^{-\gamma},\ t > 0$ ([23, Th. 1.4.3]), it is clear that c) follows from b).

$\diamondsuit$

Assume that a family $\{L_\alpha,\ \alpha \in [0, \alpha_1]\}$ satisfies (H1) and (H2). For any $\alpha \in [0, \alpha_1]$, we denote $Q_\alpha := I - P_\alpha$ and let

$$L_{1,\alpha} := P_\alpha L_\alpha = L_\alpha P_\alpha; \quad L_{2,\alpha} := Q_\alpha L_\alpha = L_\alpha Q_\alpha \quad \mathcal{X}_{1,\alpha} := P_\alpha \mathcal{X}; \quad \mathcal{X}_{2,\alpha} := Q_\alpha \mathcal{X} \tag{9.1.5}$$

Then $L_{1,\alpha}$ is a bounded linear operator, $\text{Re}\ \sigma(L_{1,\alpha}) < \lambda_-$ in $\mathcal{X}$, and $L_{2,\alpha}$ is a sectorial operator in $\mathcal{X}$, $\text{Re}\ \sigma(L_{2,\alpha}) > \lambda_+$,. Moreover, $\|L_{1,0}^\gamma\| \leq C\lambda_-^\gamma,\ \gamma \in [0, 1)$. Applying Lemma 9.1.3 to the operators $\lambda_- - L_{1,\alpha}$ and $L_{2,\alpha} - \lambda_+$, respectively, one obtains

LEMMA 9.1.4. *Assume that the hypotheses (H1) and (H2) are satisfied. Then there is a $C > 0$ such that, for any $\alpha \in [0, \alpha_1]$ and $\gamma \geq 0$ the following estimates are true*

a) $\|L_0^\gamma \exp(-L_{1,\alpha}t)P_\alpha\| \leq C\lambda_-^\gamma e^{-\lambda_- t}; \quad t \leq 0$

b) $\|L_0^\gamma (\exp(-L_{1,\alpha}t)P_\alpha - \exp(-L_{1,0}t)P_0)\| \leq C\lambda_-^\gamma \|L_0 L_\alpha^{-1} - I\| e^{-\lambda_- t}; \quad t \leq 0$

c) $\|L_0^\gamma \exp(-L_\alpha t)Q_\alpha\| \leq Ct^{-\gamma} e^{-\lambda_+ t}; \quad t > 0$

d) $\|L_0^\gamma (\exp(-L_\alpha t)Q_\alpha - \exp(-L_0 t)Q_0)\| \leq Ct^{-\gamma}\|L_0 L_\alpha^{-1} - I\| e^{-\lambda_+ t}; \quad t > 0$

In what follows, by C we will always denote the positive constant existence of which is ensured by Lemmas 9.1.3 and 9.1.4.

We finish this section by a useful lemma referring to Hölder continuity of the exponential mapping $t \mapsto \exp(-Lt)$.

LEMMA 9.1.5. *Assume that the hypotheses (H1) and (H2) are satisfied. Suppose that $\mu \in (\lambda_-, \lambda_+)$. Then, for any $\gamma \in [0, 1)$, $\varrho \in (0, 1-\gamma)$ and $\alpha \in [0, \alpha_1]$ the following estimates are true*

a) $\|L_0^\gamma [\exp((\mu - L_{1,\alpha})r) - \exp((\mu - L_{1,\alpha})(r-h))] P_\alpha x\| \leq$
$\leq Ch\lambda_-^{1+\gamma} e^{(\mu-\lambda_-)r} \|(\mu - L_\alpha)L_0^{-1}\|\|x\|$
for any $h > 0$, $r \leq 0$ *and* $x \in \mathcal{X}$

b) $\|L_0^\gamma [\exp(-(L_\alpha - \mu)(r+h)) - \exp(-(L_\alpha - \mu)r)] Q_\alpha x\| \leq$
$\leq C^2 h^{(1-\gamma+\varrho)/2} r^{-(1+\gamma+\varrho)/2} e^{-(\lambda_+ - \mu)r} \frac{2}{1-\gamma+\varrho} \|(\mu - L_\alpha)L_0^{-1}\|\|x\|$
for any $h > 0$, $r \geq 0$ *and* $x \in \mathcal{X}$

PROOF. a) Clearly, for any $r \leq 0$ and $h > 0$

$$I_1 := L_0^\gamma [\exp((\mu - L_{1,\alpha})r) - \exp((\mu - L_{1,\alpha})(r-h))] P_\alpha x =$$

$$= (\mu - L_{1,\alpha})L_0^{-1} \int_{r-h}^{r} L_0^{1+\gamma} \exp((\mu - L_{1,\alpha})\xi)\, d\xi P_\alpha x$$

By Lemma 9.1.4 a),

$$\|I_1\| \leq \|(\mu - L_\alpha)L_0^{-1}\|\ C\lambda_-^{1+\gamma} \int_{r-h}^{r} e^{(\mu-\lambda_-)\xi}\, d\xi \|x\| \leq h\ C\lambda_-^{1+\gamma} e^{(\mu-\lambda_-)r} \|(\mu - L_\alpha)L_0^{-1}\|\|x\|$$

To show b) we will argue similarly as above. We have, for any $r \geq 0$ and $h > 0$

$$I_2 := L_0^\gamma [\exp(-(L_\alpha - \mu)(r+h)) - \exp(-(L_\alpha - \mu)r)] Q_\alpha x =$$

$$= (\mu - L_\alpha)L_0^{-1} \int_0^h L_0^{(1+\gamma-\varrho)/2} \exp(-(L_\alpha - \mu)\xi)Q_\alpha\, d\xi\ L_0^{(1+\gamma+\varrho)/2} \exp(-(L_\alpha - \mu)r)Q_\alpha x$$

Hence,

$$\|I_2\| \le \|(\mu - L_\alpha)L_0^{-1}\| \; C \int_0^h \xi^{-(1+\gamma-\varrho)/2} e^{-(\lambda_+-\mu)\xi} d\xi \; C r^{-(1+\gamma+\varrho)/2} e^{-(\lambda_+-\mu)r} \|x\| \le$$

$$\le C^2 h^{(1-\gamma+\varrho)/2} r^{-(1+\gamma+\varrho)/2} e^{-(\lambda_+-\mu)r} \frac{2}{1-\gamma+\varrho} \|(\mu - L_\alpha)L_0^{-1}\| \; \|x\|$$

for any $r \ge 0,\ h > 0$ and $x \in \mathcal{X}$.

$\diamondsuit$

9.2. Existence of solutions of the system of abstract equations

In this section, the aim is to show local and global solvability of a family of abstract equations

$$\begin{aligned} u' + A_\alpha u &= g(u,w) \\ \alpha w' + B_\alpha w &= f(u) \end{aligned} \qquad \alpha \in (0, \alpha_1] \tag{9.2.1$_\alpha$}$$

and

$$u' + A_0 u = g(u, B_0^{-1} f(u)) \tag{9.2.1$_0$}$$

where the families $\{A_\alpha, \alpha \in [0,\alpha_1]\}$ and $\{B_\alpha, \alpha \in [0,\alpha_1]\}$, $\alpha_1 > 0$, small enough, fulfill the hypotheses (H1)-(H2) and (H1) on the Banach spaces X and Y, respectively.

Denote

$$X^\gamma := [D(A_0^\gamma)]; \quad Y^\beta := [D(B_0^\beta)]; \quad \gamma, \beta \ge 0 \tag{9.2.2}$$

the fractional power spaces A_0^γ and B_0^β, respectively, with graph norms, i.e. $\|u\|_\gamma := \|A_0^\gamma u\|$ and $\|w\|_\beta := \|B_0^\beta w\|$.

By a globally defined solution of $(9.2.1)_\alpha$ with initial data $(u_0, w_0) \in X^\gamma \times Y^\beta$ we understand a function

$$t \mapsto (u(t), w(t)) \in C([0,T]; X^\gamma \times Y^\beta) \cap C^1((0,T); X \times Y) \quad \text{for any } T > 0 \tag{9.2.3$_\alpha$}$$

such that $(u(0), w(0)) = (u_0, w_0)$; $(u(t), w(t)) \in D(A) \times D(B)$ for $t > 0$ and $(u(.), w(.))$ satisfies $(9.2.1)_\alpha$ for any $t > 0$.

By a globally defined solution of $(9.2.1)_0$ with initial data $u_0 \in X^\gamma$ we understand a function

$$t \mapsto u(t) \in C([0,T]; X^\gamma) \cap C^1((0,T); X) \quad \text{for any } T > 0 \tag{9.2.3$_0$}$$

such that $u(0) = u_0; u(t) \in D(A)$ for $t > 0$ and $u(.)$ satisfies $(9.2.1)_0$ for any $t > 0$.

As usual, for Banach spaces E_1, E_2 and $\eta \in (0,1]$ we denote $C^1_{bdd}(E_1, E_2)$ the Banach space consisting of the mappings $F : E_1 \to E_2$ which are Fréchet differentiable and such that F and DF are bounded and uniformly continuous, the norm being given by $\|F\|_1 := \sup \mid F \mid + \sup \mid DF \mid$. $C^{1+\eta}_{bdd}(E_1, E_2)$ will denote the Banach space consisting of the mappings $F \in C^1_{bdd}(E_1, E_2)$ such that DF is η- Hölder continuous, the norm being given by $\|F\|_{1,\eta} := \|F\|_1 + \sup_{\substack{x \neq y \\ x,y \in E_1}} \frac{\|DF(x)-DF(y)\|}{\|x-y\|^\eta}$.

Concerning functions g and f we will assume

$$(H3) \left\{ \begin{array}{l} g \in C^1_{bdd}(X^\gamma \times Y^\beta; X),\ f \in C^{1+\eta}_{bdd}(X^\gamma; Y^\xi) \\ \text{for some } \gamma, \beta \in [0,1),\ \beta > \xi > \beta - 1 \text{ and } \ \eta \in (0,1]. \end{array} \right.$$

First, we will consider the case $\alpha > 0$. According to Lemmas 9.1.1 and 9.1.3 the operator A_α (B_α) is sectorial in X (Y). In Lemma 9.1.4 we have shown the estimates

$$\begin{array}{ll} \|A_0^\gamma \exp\,(-A_\alpha t)x\| \leq C t^{-\gamma} e^{-\omega t}\|x\| & \\ \|B_0^\beta \exp\,(-B_\alpha t)y\| \leq C t^{-(\beta-\xi)} e^{-\omega t}\|B_0^\xi y\| & x \in X, y \in Y^\xi, t > 0 \end{array} \tag{9.2.4}$$

With help of these inequalities one can easily adapt the proofs of [23, Theorems 3.3.3 and 3.3.4] to establish local and global existence of solutions of $(9.2.1)_\alpha,\ \alpha \in (0, \alpha_1]$, for initial data belonging to the phase-space $X^\gamma \times Y^\beta$. Local and global existence of solutions of $(9.2.1)_0$ with initial data from X^γ follows from [23, Theorems. 3.3.3 and 3.3.4].

This way we have shown that system $(9.2.1)_\alpha,\ \alpha \in (0, \alpha_1]$, generates a semiflow $\mathcal{S}_\alpha(t), t \geq 0$, on $X^\gamma \times Y^\beta$ defined by $\mathcal{S}_\alpha(t)(u(0), w(0)) := (u(t), w(t))$. Similarly, system $(9.2.1)_0$ generates a semiflow $\tilde{\mathcal{S}}_0(t), t \geq 0$, on X^γ.

9.3. Banach spaces with exponentially weighted norms

Let $\mathcal{X}$ be a Banach space and $\mu \in R$. Following the notation of [15] and [34] we denote

$$C^-_\mu(\mathcal{X}) := \left\{ u : (-\infty, 0] \to \mathcal{X}, u \text{ is continuous and } \sup_{t \leq 0} e^{\mu t}\|u(t)\|_{\mathcal{X}} < \infty \right\}$$

and

$$\|u\|_{C^-_\mu(\mathcal{X})} := \sup_{t \leq 0} e^{\mu t}\|u(t)\|_{\mathcal{X}} \tag{9.3.1}$$

The linear space $C^-_\mu(\mathcal{X})$ endowed with the norm $\|.\|_{C^-_\mu(\mathcal{X})}$ is a Banach space. If $\mu \leq \nu$ then embedding $C^-_\mu(\mathcal{X}) \hookrightarrow C^-_\nu(\mathcal{X})$ is continuous with an embedding constant equal to 1.

Let X, Y be Banach spaces and $F : X \to Y$ be a bounded and Lipschitz continuous mapping. Denote

$$\tilde{F} : C^-_\mu(X) \to C^-_\mu(Y) \tag{9.3.2}$$

a mapping defined as $\tilde{F}(u)(t) := F(u(t))$ for any $t \leq 0$. By [34, Lemma 5.1], for every $\mu \geq 0$, the mapping $\tilde{F}$ is bounded and Lipschitzian with $\sup \mid \tilde{F} \mid \leq \sup \mid F \mid$ and $\mathrm{Lip} \mid \tilde{F} \mid \leq \mathrm{Lip} \mid F \mid$. If $F : X \to Y$ is Fréchet differentiable then $\tilde{F} : C^-_\mu(X) \to C^-_\mu(Y)$ need not be necessarily differentiable. Nevertheless, the following result holds

Lemma 9.3.1. [47] *If $F : X \to Y$ is Fréchet differentiable with $DF : X \to L(X,Y)$ bounded and uniformly continuous, then, for every $\nu > \mu, \nu > 0$, the mapping $\tilde{F} : C^-_\mu(X) \to C^-_\nu(Y)$ is Fréchet differentiable, its derivative being given by $D\tilde{F}(u)h = DF(u(.))h(.)$ and $D\tilde{F} : C^-_\mu(X) \to L(C^-_\mu(X), C^-_\nu(Y))$ is bounded and uniformly continuous.*

We now recall a notion of uniform equicontinuity of a subset of $C^-_\mu(X)$ (see, [34]). By definition, a subset $\mathcal{F} \subset C^-_\mu(X)$ is called C^-_μ - uniformly equicontinuous if and only if the set of functions $\{f_\mu, f \in \mathcal{F}\}$, where $f_\mu(t) := e^{\mu t} f(t)$ is equicontinuous, i.e. for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\sup_{f \in \mathcal{F}} \sup_{\substack{t,s \leq 0 \\ |t-s|<\delta}} \|e^{\mu t} f(t) - e^{\mu s} f(s)\| < \varepsilon \tag{9.3.3}$$

For any $\varrho \in (0,1],\ a \in (0,1]$ and $\mu \geq 0$ we furthermore denote

$$C^-_{\mu,\varrho,a}(\mathcal{X}) := \left\{u \in C^-_\mu(\mathcal{X});\ [u]_{\mu,\varrho,a} < \infty\right\} \tag{9.3.4}$$

where

$$[u]_{\mu,\varrho,a} := \sup_{\substack{t \leq 0 \\ h \in (0,a]}} \frac{\|e^{\mu t} u(t) - e^{\mu(t-h)} u(t-h)\|}{h^\varrho} \tag{9.3.5}$$

and let

$$\|u\|_{C^-_{\mu,\varrho,a}(\mathcal{X})} := \|u\|_{C^-_\mu(\mathcal{X})} + [u]_{\mu,\varrho,a} \quad \text{for any } u \in C^-_{\mu,\varrho,a}(\mathcal{X}) \tag{9.3.6}$$

The space $C^-_{\mu,\varrho,a}(\mathcal{X})$ endowed with the norm $\|.\|_{C^-_{\mu,\varrho,a}}$ is a Banach space continuously embedded into $C^-_\mu(\mathcal{X})$ with an embedding constant equal to 1. Furthermore, the space $C^-_{\mu,\varrho,a}(\mathcal{X})$ is continuously embedded into $C^-_{\nu,\varrho,a}(\mathcal{X})$ for any $0 \leq \mu \leq \nu$ and $\varrho \in (0,1]$. Indeed, for any $u \in C^-_{\mu,\varrho,a}(\mathcal{X}),\ t \leq 0$ and $h \in (0,a]$, we have

$$\|e^{\nu t} u(t) - e^{\nu(t-h)} u(t-h)\| \leq \| \left(e^{(\nu-\mu)t} - e^{(\nu-\mu)(t-h)}\right) e^{\mu t} u(t)\| +$$

$$+ \|e^{(\nu-\mu)(t-h)} \left(e^{\mu t} u(t) - e^{\mu(t-h)} u(t-h)\right)\| \leq \|u\|_{C^-_\mu(\mathcal{X})} (\nu - \mu) h + [u]_{\mu,\varrho,a} h^\varrho$$

Thus $u \in C^-_{\nu,\varrho,a}(\mathcal{X})$ and the embedding $C^-_{\mu,\varrho,a}(\mathcal{X}) \hookrightarrow C^-_{\nu,\varrho,a}(\mathcal{X})$ is continuous, its embedding constant being less or equal to $\max\{1, (\nu-\mu) a^{1-\varrho}\}$.

For any $K > 0$, the set

$$\mathcal{F}_C := \left\{u \in C^-_{\mu,\varrho,a}(\mathcal{X});\ \|u\|_{C^-_{\mu,\varrho,a}} \leq K\right\} \tag{9.3.7}$$

is a C^-_μ - uniformly equicontinuous and bounded subset of $C^-_\mu(\mathcal{X})$.

Since $C^-_{\mu,\varrho,a}(\mathcal{X})$ is continuously embedded into $C^-_\mu(\mathcal{X})$ we obtain the following consequence of Lemma 9.3.1

LEMMA 9.3.2. *Let $F : X \to Y$ be as in Lemma 9.3.1. Suppose that $\nu > \mu, \nu > 0$ and $\varrho \in (0,1]$. Then the mapping $\tilde{F} : C^-_{\mu,\varrho,a}(X) \to C^-_\nu(Y)$ is Fréchet differentiable, its derivative $D\tilde{F} : C^-_{\mu,\varrho,a}(X) \to L(C^-_{\mu,\varrho,a}(X), C^-_\nu(Y))$ being bounded and uniformly continuous.*

SECTION 10.

Invariant manifolds

10.1. Construction of a family of invariant manifolds

In this section, we establish the existence of a one-parameter family of invariant manifolds for semiflows generated by abstract singularly perturbed equations $(9.2.1)_\alpha,\ \alpha \geq 0$ small enough.

First, we will deal with solutions of

$$\alpha w' + B_\alpha w = f \tag*{$(10.1.1)_\alpha$}$$

existing on R and satisfying a growth condition of an exponential type when $t \to -\infty$. We will also consider the "limiting equation"

$$B_0 w = f \tag*{$(10.1.1)_0$}$$

Henceforth, we will assume that a family $\{B_\alpha, \alpha \in [0, \alpha_1]\}$ satisfies the hypothesis (H1). From Lemma 9.1.1 we know that B_α is sectorial and $\operatorname{Re}\sigma(B_\alpha) > \omega > 0$ for any $\alpha \in [0, \alpha_1],\ \alpha_1$ small. Moreover, we choose $\alpha_1 > 0$ such that

$$\omega > \nu\alpha_1 > 0 \tag{10.1.2}$$

where $\nu > 0$ is given. Now, it is routine to verify that $(10.1.1)_\alpha,\ \alpha \in (0, \alpha_1]$ has a unique solution $w \in C^-_\nu(Y^\beta),\ \beta \in [0, 1)$, for any $f \in C^-_\nu(Y^\xi),\ \xi > \beta - 1$. This solution is given by

$$w(t) := \frac{1}{\alpha}\int_{-\infty}^{t} \exp\ (-B_\alpha(t-s)/\alpha)\, f(s)\, ds =: C_\alpha f(t);\quad t \le 0 \tag{10.1.3}$$

The unique solution of $(10.1.1)_0$ is determined by

$$w := B_0^{-1} f =: C_0 f \tag{10.1.4}$$

Concerning the boundedness and limiting behavior of the linear operators

$$C_\alpha : C_\nu^-(Y^\xi) \to C_\nu^-(Y^\beta);\ \alpha \in [0,\alpha_1] \tag{10.1.5}$$

we claim

LEMMA 10.1.1. *Assume that the family $\{B_\alpha; \alpha \in [0,\alpha_1]\}$ fulfills the hypothesis (H1). Let $\beta \in [0,1)$, $\beta > \xi > \beta - 1$ and $0 < \nu\alpha_1 < \omega$.Then*

a) *there is a $C > 0$ such that*

$$\|C_\alpha\|_{L(C_\nu^-(Y^\xi), C_\nu^-(Y^\beta))} \le C\Gamma(1-\beta+\xi)(\omega - \nu\alpha_1)^{\beta-\xi-1} \quad \textit{for any} \quad \alpha \in [0,\alpha_1]$$

where Γ is the Gamma function

$$\Gamma(\theta) := \int_0^\infty r^{\theta-1} e^{-r}\, dr \ \textit{for}\ \theta > 0 \tag{10.1.6}$$

b) *$C_\alpha f \to C_0 f$ as $\alpha \to 0^+$ uniformly with respect to $f \in \mathcal{F}$ where $\mathcal{F}$ is a C_ν^- - uniformly equicontinuous and bounded subset of $C_\nu^-(Y^\xi)$.*

c) *$C_\alpha \to C_0$ as $\alpha \to 0^+$ in the norm topology of the space $L(C^-_{\nu,\varrho,a}(Y^\xi), C_\nu^-(Y^\beta))$ for any $\varrho \in (0,1]$, $a \in (0,1]$.*

PROOF. Denote $w := C_\alpha f$ for $f \in C_\nu^-(Y^\xi)$. With regard to Lemma 9.1.3 we obtain, for any $t \le 0$ and $\alpha \in (0,\alpha_1]$,

$$e^{\nu t}\|w(t)\|_\beta \le \frac{1}{\alpha}\int_{-\infty}^{t} \|B_0^{\beta-\xi} \exp\ (-(B_\alpha - \nu\alpha)(t-s)/\alpha)\,\| e^{\nu s}\|B_0^\xi f(s)\|\, ds \le$$

$$\le \frac{C}{\alpha}\int_{-\infty}^{t} ((t-s)/\alpha)^{-(\beta-\xi)} e^{-(\omega-\nu\alpha)(t-s)/\alpha}\, ds \|f\|_{C_\nu^-(Y^\xi)} \le$$

$$\le C\Gamma(1-\beta+\xi)(\omega-\nu\alpha_1)^{\beta-\xi-1}\|f\|_{C_\nu^-(Y^\xi)}$$

For $\alpha = 0$ we have

$$e^{\nu t}\|w(t)\|_\beta \le \|B_0^{\beta-\xi-1}\|\|f\|_{C_\nu^-(Y^\xi)} \le C\omega^{\beta-\xi-1}\|f\|_{C_\nu^-(Y^\xi)}.$$

b) Because Re $\sigma(B_0) > \omega > 0$, we have the following integral representation of B_0^{-1}

$$B_0^{-1} = \frac{1}{\alpha}\int_{-\infty}^{t} \exp\ (-B_0(t-s)/\alpha)\ ds \quad \text{for any} \quad t \le 0, \quad \alpha > 0, \tag{10.1.7}$$

Let $t \le 0$ and $f \in \mathcal{F}$ be arbitrary. Using Lemma 9.1.3 we obtain

$$e^{\nu t}\|C_\alpha f(t) - C_0 f(t)\|_\beta \le$$

$$\le \frac{e^{\nu t}}{\alpha}\int_{-\infty}^{t} \|B_0^{\beta-\xi}\Big(\exp\ (-B_\alpha(t-s)/\alpha)\, B_0^{\xi} f(s) - \exp\ (-B_0(t-s)/\alpha)\, B_0^{\xi} f(t)\Big)\|\, ds \le$$

$$\le \frac{1}{\alpha}\int_{-\infty}^{t} \|B_0^{\beta-\xi}\exp\ (-B_\alpha(t-s)/\alpha)\, B_0^{\xi}(f(s)-f(t))e^{\nu t}\|\, ds +$$

$$+\frac{1}{\alpha}\int_{-\infty}^{t} \|B_0^{\beta-\xi}\,(\exp\ (-B_\alpha(t-s)/\alpha) - \exp\ (-B_0(t-s)/\alpha))\,\|\, ds\|f\|_{C_\nu^-(Y^\xi)} \le$$

$$\le \frac{C}{\alpha}\int_{-\infty}^{t} ((t-s)/\alpha)^{-(\beta-\xi)} e^{-\omega(t-s)/\alpha} e^{\nu t}\|f(s)-f(t)\|_\xi\, ds +$$

$$+\frac{C}{\alpha}\|B_0 B_\alpha^{-1} - I\|\int_{-\infty}^{t} ((t-s)/\alpha)^{-(\beta-\xi)} e^{-\omega(t-s)/\alpha}\, ds\|f\|_{C_\nu^-(Y^\xi)} =: I_1 + I_2$$

As it is usual in integral with singular kernels (see, e.g. [34]) we decompose the first integral into two parts $I_1 = \int_{-\infty}^{t-\tau} + \int_{t-\tau}^{t} =: I_{1,1} + I_{1,2}$ where $\tau > 0$ will be determined later. Clearly,

$$e^{\nu t}\|f(s)-f(t)\|_\xi \le 2e^{\nu(t-s)}\|f\|_{C_\nu^-(Y^\xi)}, \quad \text{for any} \ -\infty < s \le t \le 0.$$

Then

$$I_{1,1} := \frac{C}{\alpha}\int_{-\infty}^{t-\tau} ((t-s)/\alpha)^{-(\beta-\xi)} e^{-\omega(t-s)/\alpha} e^{\nu t}\|f(s)-f(t)\|_\xi\, ds \le$$

$$\le \frac{2C\tau^{-(\beta-\xi)}\alpha^{\beta-\xi}}{\omega - \nu\alpha_1}\|f\|_{C_\nu^-(Y^\xi)}$$

On the other hand, for any $s \in [t-\tau, t]$, we have

$$e^{\nu t}\|f(s)-f(t)\|_\xi \le \|e^{\nu s} f(s) - e^{\nu t} f(t)\|_\xi + (e^{\nu(t-s)} - 1)\|f\|_{C_\nu^-(Y^\xi)} \le$$

$$\le e^{\nu(t-s)}\Big(\ \mathrm{osc}(f_\nu,\tau) + (1-e^{-\nu\tau})\|f\|_{C_\nu^-(Y^\xi)}\Big)$$

where $\mathrm{osc}(f_\nu,\tau) := \sup\limits_{\substack{t,s\le 0\\ |t-s|<\tau}} \|e^{\nu t} f(t) - e^{\nu s} f(s)\|_\xi$. Hence,

$$I_{1,2} := \frac{C}{\alpha}\int_{t-\tau}^{t} ((t-s)/\alpha)^{-(\beta-\xi)} e^{-\omega(t-s)/\alpha} e^{\nu t}\|f(s)-f(t)\|\, ds \le$$

$$\leq C\Gamma(1-\beta+\xi)(\omega-\nu\alpha_1)^{\beta-\xi-1}\Big(\operatorname{osc}(f_\nu,\tau)+(1-e^{-\nu\tau})\|f\|_{C_\nu^-(Y^\xi)}\Big)$$

Finally, we have

$$I_2 \leq C\Gamma(1-\beta+\xi)\omega^{\beta-\xi-1}\|B_0 B_\alpha^{-1}-I\|\|f\|_{C_\nu^-(Y^\xi)}.$$

Since the set $\mathcal{F} \subset C_\nu^-(Y^\xi)$ is assumed to be C_ν^- - uniformly equicontinuous and bounded we have

$$\operatorname{osc}(f_\nu,\tau)+(1-e^{-\nu\tau})\|f\|_{C_\nu^-(Y^\xi)} \to 0^+ \quad \text{as} \quad \tau \to 0^+$$

uniformly with respect to $f \in \mathcal{F}$. Now, it is easy to see that $C_\alpha f \to C_0 f$ in $C_\nu^-(Y^\beta)$ when $\alpha \to 0^+$ uniformly for $f \in \mathcal{F}$.

Finally, by (9.3.7), the set $\mathcal{F}_1 := \Big\{\phi \in C^-_{\nu,\varrho,a}(Y^\xi);\ \|\phi\|_{C^-_{\nu,\varrho,a}} \leq 1\Big\}$ is a C_ν^- - uniformly equicontinuous and bounded subset of $C_\nu^-(Y^\xi)$. Hence, by b), $C_\alpha \to C_0$ as $\alpha \to 0^+$ in the topology of the space $L(C^-_{\nu,\varrho,a}(Y^\xi), C_\nu^-(Y^\beta))$.

$\diamondsuit$

We now turn our attention to the construction of an invariant manifold $\mathcal{M}_\alpha$ for a semiflow generated by the system

$$\begin{aligned} u' + A_\alpha u &= g(u,w) \\ \alpha w' + B_\alpha w &= f(u) \end{aligned} \tag{10.1.8}$$

where $\alpha \in [0,\alpha_1]$. From now on we will assume that the hypothesis

$$(H)\begin{cases} 1) & \text{the family } \{A_\alpha, \alpha \in [0,\alpha_1]\} \text{ satisfies (H1)-(H2) on a Banach space } X \\ 2) & \text{the family } \{B_\alpha, \alpha \in [0,\alpha_1]\} \text{ satisfies (H1) on a Banach space } Y \\ 3) & \text{the functions } g \text{ and } f \text{ satisfy (H3) for some } \gamma,\beta \in [0,1) \text{ and } \beta > \xi > \beta - 1 \end{cases}$$

holds.

The idea of the construction of an invariant manifold $\mathcal{M}_\alpha$ for (10.1.8) is fairly standard and is based on the well-known method of integral equations due to Lyapunov and Perron. According to this method, $\mathcal{M}_\alpha$ contains all solutions $(u(.), w(.)) \in X^\gamma \times Y^\beta$ of (10.1.8) existing on R and satisfying an exponential growth condition of the form

$$\|u(t)\|_\gamma + \|w(t)\|_\beta = O(e^{-\mu t}) \quad \text{as } t \to -\infty \tag{10.1.9}$$

where $\mu > 0$ is fixed. In our case, we will take the advantage of the particular form of (10.1.8). With regard to Lemma 10.1.1, for a given $u \in C_\mu^-(X^\gamma)$ we have $\tilde{f}(u) \in C_\mu^-(Y^\xi)$ ($\tilde{f}$ defined in (9.3.2)) and hence $w := C_\alpha \tilde{f}(u)$ is the unique solution of $(10.1.1)_\alpha$ belonging to $C_\mu^-(Y^\beta)$. Roughly speaking, the w - variable of the semiflow $\mathcal{S}_\alpha$ on an invariant manifold $\mathcal{M}_\alpha$ (if it exists) is governed by the u - variable. More precisely, as usual (see, e.g. [15], [33], [22]), we will construct $\mathcal{M}_\alpha$ as the union of curves $(u, C_\alpha \tilde{f}(u))$ where $u \in C_\mu^-(X^\gamma)$ are fixed points of the mapping

$$T_\alpha(x,.) : C_\mu^-(X^\gamma) \to C_\mu^-(X^\gamma) \tag{10.1.10}$$

$\alpha \in [0, \alpha_1],\ x \in X_{1,0} := P_0 X$ and, for any $u \in C_\mu^-(X^\gamma)$

$$T_\alpha(x,u) := \mathcal{K}_\alpha x + \mathcal{T}_\alpha(\mathcal{G}_\alpha(u)) \tag{10.1.11}$$

The linear operators $\mathcal{K}_\alpha : X_{1,0} \to C_\mu^-(X^\gamma)$; $\mathcal{T}_\alpha : C_\mu^-(X) \to C_\mu^-(X^\gamma)$ are given by

$$\mathcal{K}_\alpha x := \exp\,(-A_{1,\alpha} t) P_\alpha x; \quad \text{for any } x \in X_{1,0}, \tag{10.1.12}$$

$$\mathcal{T}_\alpha(g)(t) := \int_0^t \exp\,(-A_{1,\alpha}(t-s)) P_\alpha g(s)\, ds +$$

$$+ \int_{-\infty}^t \exp\,(-A_\alpha(t-s)) Q_\alpha g(s)\, ds \quad \text{for any} \quad g \in C_\mu^-(X) \tag{10.1.13}$$

and the nonlinearity $\mathcal{G}_\alpha : C_\mu^-(X^\gamma) \to C_\mu^-(X)$ is given by

$$\mathcal{G}_\alpha(u)(t) := g(u(t), C_\alpha \tilde{f}(u)(t)); \quad \text{for any } u \in C_\mu^-(X^\gamma) \tag{10.1.14}$$

By means of the Banach fixed point theorem, we will show that the operator $T_\alpha(x,.)$ has a fixed point $Y_\alpha(x) \in C_\mu^-(X^\gamma)$. To do this, we first establish estimates of norms of $\mathcal{T}_\alpha$ and $\mathcal{K}_\alpha$ and the Lipschitz constant of $\mathcal{G}_\alpha$.

LEMMA 10.1.2. *Assume that* $\mu \in (\lambda_-, \lambda_+)$. *Then, for any* $\alpha \in [0, \alpha_1]$

a) $\mathcal{K}_\alpha \in L(X_{1,0}, C_\mu^-(X^\gamma))$; $\|\mathcal{K}_\alpha\|_{L(X_{1,0}, C_\mu^-(X^\gamma))} \le C \lambda_-^\gamma$

$\|\mathcal{K}_\alpha - \mathcal{K}_0\|_{L(X_{1,0}, C_\mu^-(X^\gamma))} \le C \lambda_-^\gamma \|A_0 A_\alpha^{-1} - I\|$

b) $\mathcal{T}_\alpha \in L(C_\mu^-(X), C_\mu^-(X^\gamma))$; $\|\mathcal{T}_\alpha\|_{L(C_\mu^-(X), C_\mu^-(X^\gamma))} \le C K(\lambda_-, \lambda_+, \mu, \gamma)$

$\|\mathcal{T}_\alpha - \mathcal{T}_0\|_{L(C_\mu^-(X), C_\mu^-(X^\gamma))} \le C \|A_0 A_\alpha^{-1} - I\| K(\lambda_-, \lambda_+, \mu, \gamma)$

where

$$K(\lambda_-, \lambda_+, \mu, \gamma) := \frac{\lambda_-^\gamma}{\mu - \lambda_-} + \frac{2-\gamma}{1-\gamma}(\lambda_+ - \mu)^{\gamma - 1} \tag{10.1.15}$$

In addition, if $\varrho \in (0, 1-\gamma)$ *then there is a constant* $a = a(\lambda_-, \lambda_+, \mu, \gamma, \varrho, C) > 0$ *such that*

c) $\mathcal{K}_\alpha \in L(X_{1,0}, C_{\mu,\varrho,a}^-(X^\gamma))$; $\|\mathcal{K}_\alpha\|_{L(X_{1,0}, C_{\mu,\varrho,a}^-(X^\gamma))} \le 2C\lambda_-^\gamma$

$\mathcal{T}_\alpha \in L(C_\mu^-(X), C_{\mu,\varrho,a}^-(X^\gamma))$; $\|\mathcal{T}_\alpha\|_{L(C_\mu^-(X), C_{\mu,\varrho,a}^-(X^\gamma))} \le 2C K(\lambda_-, \lambda_+, \mu, \gamma)$

d) $\mathcal{K}_\alpha \to \mathcal{K}_0$ *and* $\mathcal{T}_\alpha \to \mathcal{T}_0$ *as* $\alpha \to 0^+$ *in* $L(X_{1,0}, C^-_{\mu,\varrho,a}(X^\gamma))$ *and* $L(C^-_\mu(X), C^-_{\mu,\varrho,a}(X^\gamma))$, *respectively.*

Proof. Using the estimates from Lemma 9.1.4 the proof of a) is obvious. Again, with help of Lemma 9.1.4, the proof of b) is an immediate adaptation of that of [15, Lemma 10.1].

In order to prove c), we make use of Lemma 9.1.5. Applying Lemma 9.1.5, part a), we obtain

$$[\mathcal{K}_\alpha x]_{\mu,\varrho,a} \le a^{1-\varrho} C \lambda_-^{1+\gamma} \|(\mu - A_\alpha)A_0^{-1}\| \|x\| \quad \text{for any} \quad x \in X_{1,0}$$

Further, by definition (10.1.13),

$$P_\alpha \mathcal{T}_\alpha(g)(t) = \int_0^t \exp\,(-A_{1,\alpha}(t-s)) P_\alpha g(s)\, ds$$

$$Q_\alpha \mathcal{T}_\alpha(g)(t) = \int_{-\infty}^t \exp\,(-A_\alpha(t-s)) Q_\alpha g(s)\, ds$$

for any $g \in C^-_\mu(X)$. Hence

$$I_1 := P_\alpha \mathcal{T}_\alpha(g)(t) e^{\mu t} - P_\alpha \mathcal{T}_\alpha(g)(t-h) e^{\mu(t-h)} =$$

$$= \int_0^t \left[\exp\,((\mu - A_{1,\alpha})(t-s)) - \exp\,((\mu - A_{1,\alpha})(t-h-s))\right] P_\alpha e^{\mu s} g(s)\, ds +$$

$$+ \int_{t-h}^t \exp\,((\mu - A_{1,\alpha})(t-h-s)) P_\alpha e^{\mu s} g(s)\, ds$$

By taking norms and using Lemmas 9.1.4 and 9.1.5 a), we obtain

$$\|A_0^\gamma I_1\| \le \{h\, C \|(\mu - A_\alpha)A_0^{-1}\| \lambda_-^{1+\gamma} \int_0^t e^{(\mu-\lambda_-)(t-s)} ds +$$

$$+ C\lambda_-^\gamma \int_{t-h}^t e^{(\mu-\lambda_-)(t-h-s)} ds\} \|g\|_{C^-_\mu(X)} \le h\, C\lambda_-^\gamma \left\{1 + \frac{\|(\mu - A_\alpha)A_0^{-1}\|\lambda_-}{\mu - \lambda_-}\right\} \|g\|_{C^-_\mu(X)}$$

Thus

$$[P_\alpha \mathcal{T}_\alpha(g)]_{\mu,\varrho,a} \le a^{1-\varrho} C \lambda_-^\gamma \left\{1 + \frac{\|(\mu - A_\alpha)A_0^{-1}\|\lambda_-}{\mu - \lambda_-}\right\} \|g\|_{C^-_\mu(X)}$$

Acting similarly as above, we deduce that

$$I_2 := Q_\alpha \mathcal{T}_\alpha(g)(t) e^{\mu t} - Q_\alpha \mathcal{T}_\alpha(g)(t-h) e^{\mu(t-h)} =$$

$$= \int_{-\infty}^{t-h} \left[\exp\,(-(A_\alpha - \mu)(t-s)) - \exp\,(-(A_\alpha - \mu)(t-h-s))\right] Q_\alpha e^{\mu s} g(s)\, ds +$$

$$+\int_{t-h}^{t} \exp\,(-(A_\alpha-\mu)(t-s))Q_\alpha e^{\mu s} g(s)\,ds$$

Again, by Lemmas 9.1.4 and 9.1.5 b),

$$\|A_0^\gamma I_2\| \le \Big\{\frac{2C^2 h^{(1-\gamma+\varrho)/2}\|(\mu-A_\alpha)A_0^{-1}\|}{1-\gamma+\varrho}\int_{-\infty}^{t-h}(t-h-s)^{-(1+\gamma+\varrho)/2}e^{-(\lambda_+-\mu)(t-h-s)}ds+$$

$$+C\int_{t-h}^{t}(t-s)^{-\gamma}e^{-(\lambda_+-\mu)(t-s)}ds\Big\}\|g\|_{C_\mu^-(X)} \le$$

$$h^\varrho\left\{\frac{2a^{(1-\gamma-\varrho)/2}C^2\|(\mu-A_\alpha)A_0^{-1}\|}{(1-\gamma+\varrho)(\lambda_+-\mu)^{(1-\gamma-\varrho)/2}}\Gamma\left((1-\gamma-\varrho)/2\right)+\frac{Ca^{1-\gamma-\varrho}}{1-\gamma}\right\}\|g\|_{C_\mu^-(X)}$$

This way we have shown that there exists a constant $k = k(\lambda_-, \lambda_+, \mu, \gamma, \varrho, C) > 0$ such that

$$[\mathcal{K}_\alpha x]_{\mu,\varrho,a} \le a^{1-\varrho}k\|x\| \quad \text{and} \quad [\mathcal{T}_\alpha g]_{\mu,\varrho,a} \le a^{(1-\gamma-\varrho)/2}k\|g\|_{C_\mu^-(X)} \quad \text{for any } 0 < a \le 1$$

Hence, by taking $a = a(\lambda_-, \lambda_+, \mu, \gamma, \varrho, C) > 0$ sufficiently small and using the statements a) and b) it follows that

$$\|\mathcal{K}_\alpha\|_{L(X_{1,0},C^-_{\mu,\varrho,a}(X^\gamma))} \le 2C\lambda_-^\gamma \quad \text{and} \quad \|\mathcal{T}_\alpha\|_{L(C_\mu^-(X),C^-_{\mu,\varrho,a}(X^\gamma))} \le 2CK(\lambda_-, \lambda_+, \mu, \gamma)$$

Finally, we will prove that

$$[\mathcal{K}_\alpha x - \mathcal{K}_0 x]_{\mu,\varrho,a} \to 0 \quad \text{and} \quad [\mathcal{T}_\alpha g - \mathcal{T}_0 g]_{\mu,\varrho,a} \to 0 \quad \text{as } \alpha \to 0^+$$

uniformly with respect to $\|x\| \le 1$ and $\|g\|_{C_\mu^-(X)} \le 1$, respectively.

Denote

$$U_\alpha(t) := \exp\,((\mu - A_{1,\alpha})t)P_\alpha - \exp\,((\mu - A_{1,0})t)P_0$$

$$V_\alpha(t) := \exp\,(-(A_\alpha - \mu)t)Q_\alpha - \exp\,(-(A_0 - \mu)t)Q_0$$

We have the following integral representation of $U_\alpha(t)$

$$\text{for any} \quad r \le 0, h > 0, \quad U_\alpha(r) - U_\alpha(r-h) = \int_{r-h}^{r}\frac{d}{d\xi}U_\alpha(\xi)\,d\xi =$$

$$= \int_{r-h}^{r}\left[(\mu - A_{1,\alpha})\exp\,((\mu - A_{1,\alpha})\xi)P_\alpha - (\mu - A_{1,0})\exp\,((\mu - A_{1,0})\xi)P_0\right]\,d\xi =$$

$$= (A_{1,0} - A_{1,\alpha})\int_{r-h}^{r}\exp\,((\mu - A_{1,\alpha})\xi)P_\alpha\,d\xi + (\mu - A_{1,0})\int_{r-h}^{r}U_\alpha(\xi)\,d\xi$$

Using the above expression and Lemma 9.1.4 one can proceed similarly as in the proof of Lemma 9.1.5 a). One obtains

$$\|A_0^\gamma\left(U_\alpha(r) - U_\alpha(r-h)\right)x\| \le C_1 h e^{(\mu-\lambda_-)r}\|I - A_\alpha A_0^{-1}\|\|x\|; \quad r \le 0,\ h > 0$$

where $C_1 > 0$ is a constant. Analogously, one also deduces that for any $r \geq 0,\ h > 0$

$$\|A_0^\gamma\left(V_\alpha(r+h) - V_\alpha(r)\right)x\| \leq C_1 h^{(1-\gamma+\varrho)/2} r^{-(1+\gamma+\varrho)/2} e^{-(\lambda_+-\mu)r}\|I - A_\alpha A_0^{-1}\|\|x\|$$

With help of these estimates, the statement d) can be readily proved by repeating the lines of the proof of c) but now operating with $U_\alpha(t)$ and $V_\alpha(t)$ instead of $\exp\left((\mu - A_{1,\alpha})t\right)P_\alpha$ and $\exp\left(-(A_\alpha - \mu)t\right)Q_\alpha$, respectively.

$\diamondsuit$

Since $g : X^\gamma \times Y^\beta \to X$ and $f : Y^\beta \to Y^\xi$ are bounded and Lipschitzian we have that

$$\mathcal{G}_\alpha : C_\mu^-(X^\gamma) \to C_\mu^-(X)$$

$$\text{is bounded uniformly with respect to} \quad \alpha \in [0, \alpha_1] \tag{10.1.16}$$

and, moreover,

$$\|\mathcal{G}_\alpha(u_1) - \mathcal{G}_\alpha(u_2)\|_{C_\mu^-(X)} \leq \ \mathrm{Lip}(g)\left(1 + \|C_\alpha\|\ \mathrm{Lip}(f)\right)\|u_1 - u_2\|_{C_\mu^-(X^\gamma)}$$

Hence, by Lemma 10.1.1, we obtain

$$\mathrm{Lip}(\mathcal{G}_\alpha) \leq \ \mathrm{Lip}(g)\left(1 + C\Gamma(1-\beta+\xi)(\omega - \mu\alpha_1)^{\beta-\xi-1}\ \mathrm{Lip}(f)\right) \tag{10.1.17}$$

With this we have establish the following

Lemma 10.1.3. *Let $\mu \in (\lambda_-, \lambda_+)$. Assume that the hypothesis (H) holds. Then the operator $T_\alpha(x, .) : C_\mu^-(X^\gamma) \to C_\mu^-(X^\gamma)$ is a uniform contraction with respect to $\alpha \in [0, \alpha_1]$ and $x \in X_{1,0}$, provided that the following inequality is satisfied*

$$\theta := CK(\lambda_-, \lambda_+, \mu, \gamma)\ Lip(g)\left(1 + C\Gamma(1-\beta+\xi)(\omega - \mu\alpha_1)^{\beta-\xi-1}\ Lip(f)\right) < 1 \tag{10.1.18}$$

According to the previous lemma, if (10.1.18) is satisfied then, by the Banach fixed point theorem, there is a family $Y_\alpha(x), \alpha \in [0, \alpha_1], x \in X_{1,0}$, of fixed points of $T_\alpha(x, .)$. Because

$$\|T_\alpha(x_1, u) - T_\alpha(x_2, u)\| = \|\mathcal{K}_\alpha(x_1 - x_2)\| \leq C\lambda_-^\gamma\|x_1 - x_2\|$$

we furthermore have

$$\|Y_\alpha(x_1) - Y_\alpha(x_2)\| \leq C\lambda_-^\gamma(1-\theta)^{-1}\|x_1 - x_2\| \tag{10.1.19}$$

i.e $Y_\alpha(.)$ are Lipschitz continuous uniformly with respect to $\alpha \in [0, \alpha_1]$.

Now, we can define a set $\mathcal{M}_\alpha$ as follows

$$\mathcal{M}_\alpha := \left\{\left(Y_\alpha(x)(0), C_\alpha\tilde{f}(Y_\alpha(x))(0)\right);\quad x \in X_{1,0}\right\}, \quad \alpha \in (0, \alpha_1] \tag{10.1.20}$$

In order to show invariancy of $\mathcal{M}_\alpha$ under the semiflow $\mathcal{S}_\alpha(t), t \geq 0$, generated by (10.1.8) it suffices to prove that

$$\mathcal{M}_\alpha = \left\{(u(\tau), w(\tau)) \in X^\gamma \times Y^\beta,\ \tau \in R,\ (u,w) \in C^-_\mu(X^\gamma) \times C^-_\mu(Y^\beta) \ \text{ solves } (10.1.8)\ \right\} \tag{10.1.21}$$

Indeed, let us consider an arbitrary solution $(u(.), w(.))$, belonging to the right hand side of (10.1.21). Take a $\tau \in R$ and put $\bar{u}(t) := u(t+\tau), \bar{w}(t) := w(t+\tau)$. Then $(\bar{u}, \bar{w})$ is a solution of (10.1.8) as well and $(\bar{u}(.), \bar{w}(.)) \in C^-_\mu(X^\gamma) \times C^-_\mu(Y^\beta)$. By Lemma 10.1.1, we have $\bar{w} = C_\alpha \tilde{f}(\bar{u})$ and $\bar{u}$ is therefore a solution of

$$\bar{u}'(t) + A_\alpha \bar{u}(t) = g(\bar{u}(t), C_\alpha \tilde{f}(\bar{u})(t)) = \mathcal{G}_\alpha(\bar{u})(t)$$

According to [15, Lemma 4.2] $\bar{u}$ is a solution of

$$\bar{u}(t) = \exp\,(-A_{1,\alpha}t) P_\alpha \bar{u}(0) + \mathcal{T}_\alpha(\mathcal{G}_\alpha(\bar{u}))(t), \quad t \leq 0$$

By Lemma 9.1.2 and 9.1.4 , $P_\alpha\ |_{\mathcal{X}_{1,0}}\colon \mathcal{X}_{1,0} \to \mathcal{X}_{1,\alpha}$ is a linear isomorphism. Therefore there exists $x \in X_{1,0}$ such that $P_\alpha x = P_\alpha \bar{u}(0)$. Thus, $\bar{u}$ solves the operator equation $T_\alpha(x, \bar{u}) = \bar{u}$. By uniqueness of a fixed point of $T_\alpha(x, .)$ we have $\bar{u} = Y_\alpha(x)$ and hence

$$(u(\tau), w(\tau)) = (\bar{u}(0), \bar{w}(0)) = \Big(Y_\alpha(x)(0), C_\alpha \tilde{f}(Y_\alpha(x))(0)\Big) \in \mathcal{M}_\alpha$$

On the other hand, take an arbitrary $x \in X_{1,0}$. Then $\Big(Y_\alpha(x)(.), C_\alpha \tilde{f}(Y_\alpha(x))(.)\Big) \in C^-_\mu(X^\gamma) \times C^-_\mu(Y^\beta)$ is a solution of (10.1.8) which can be extended to a solution existing globally on R. Hence $\Big(Y_\alpha(x)(0), C_\alpha \tilde{f}(Y_\alpha(x))(0)\Big)$ belongs to the right hand side of (10.1.21). This way we have shown (10.1.21).

For $\alpha = 0$, we put

$$\tilde{\mathcal{M}}_0 := \{Y_0(x)(0);\quad x \in X_{1,0}\} \tag{10.1.22}$$

With regard to [15, Th. 4.4], $\tilde{\mathcal{M}}_0 \subset X^\gamma$ is an invariant manifold for the semiflow $\tilde{\mathcal{S}}_0$ generated by $(9.2.1)_0$. This manifold can be naturally embedded into a manifold $\mathcal{M}_0 \subset X^\gamma \times Y^\beta$ defined as

$$\mathcal{M}_0 := \Big\{\big(u, B_0^{-1} f(u)\big)\,;\ u \in \tilde{\mathcal{M}}_0\Big\} \tag{10.1.23}$$

We note that the manifolds $\mathcal{M}_\alpha, \alpha \in [0, \alpha_1]$, are Lipschitz continuous submanifolds of $X^\gamma \times Y^\beta$ (see (10.1.19)) and $\dim \mathcal{M}_\alpha = \dim X_{1,0} < \infty$ for any $\alpha \in [0, \alpha_1]$.

Denote

$$\Phi_\alpha := Y_\alpha(x)(0) \quad \text{and} \quad \Psi_\alpha := C_\alpha \tilde{f}(Y_\alpha(x))(0)$$
$$\text{for any} \ \ \alpha \in [0, \alpha_1] \tag{10.1.24}$$

The mapping $X_{1,0} \ni x \mapsto (\Phi_\alpha(x), \Psi_\alpha(x)) \in X^\gamma \times Y^\beta$ is Lipschitz continuous, its Lipschitz constant being independent of $\alpha \in [0, \alpha_1]$.

In terms of Φ_α and Ψ_α, the manifold $\mathcal{M}_\alpha$ is given by

$$\mathcal{M}_\alpha = \{(\Phi_\alpha(x), \Psi_\alpha(x)),\ x \in X_{1,0}\} \quad \alpha \in [0, \alpha_1] \tag{10.1.25}$$

and the semiflow $\mathcal{S}_\alpha$ ($\tilde{\mathcal{S}}_\alpha$) on $\mathcal{M}_\alpha$ ($\tilde{\mathcal{M}}_0$) is determined by solutions of its inertial form. By definition (see [22, Chapter 2.1]), an inertial form for (10.1.8) is an ordinary differential equation in a finite dimensional space $X_{1,0}$ given by

$$p' + P_\alpha^{(-1)} A_{1,\alpha} P_\alpha p = P_\alpha^{(-1)} P_\alpha g(\Phi_\alpha(p), \Psi_\alpha(p)) \qquad (10.1.26)_\alpha$$

where the linear operator $P_\alpha^{(-1)}$ was defined in (9.1.4). Indeed, any solution (u, w) of (10.1.8), $\alpha \in [0, \alpha_1]$, belonging to $\mathcal{M}_\alpha$ can be written as

$$(u(t), w(t)) = (\Phi_\alpha(p(t)), \Psi_\alpha(p(t)))$$

where $p(.)$ is a solution of (10.1.26) and *vice versa*.

By definition, a compact subset $\mathcal{A}$ of a Banach space $\mathcal{X}$ is called a compact global attractor for a semiflow $\mathcal{S}(t), t \geq 0$ on $\mathcal{X}$, if it is invariant under $\mathcal{S}(t)$ and

$$\lim_{t\to\infty} \text{dist}\ (\mathcal{S}(t)\mathcal{B}, \mathcal{A}) = 0 \quad \text{for any bounded subset} \ \ \mathcal{B} \subset \mathcal{X} \tag{10.1.29}$$

where dist $(\mathcal{B}, \mathcal{A}) := \sup_{b\in\mathcal{B}} \inf_{a\in\mathcal{A}} \|b - a\|$.

Remark 10.1.4. Suppose that system $(10.1.8)_\alpha$ admits a compact global attractor $\mathcal{A}_\alpha$. Since $\mathcal{A}_\alpha$ consists of all globally defined trajectories which are bounded in $X^\gamma \times Y^\beta$ we have $\mathcal{A}_\alpha \subset \mathcal{M}_\alpha$.

Remark 10.1.5. Assume that $\theta \ll 1$ is sufficiently small. Then, following the lines of the proof of [15, Th. 5.1], one can also prove exponential attractivity of $\mathcal{M}_\alpha$. It means that, for any $(u, w) \in X^\gamma \times Y^\beta$, there is a unique $(u^*, w^*) \in \mathcal{M}_\alpha$ such that $\|\mathcal{S}_\alpha(t)(u, w) - \mathcal{S}_\alpha(t)(u^*, w^*)\| = O(e^{-\mu t})$ as $t \to \infty$. Hence, $\mathcal{M}_\alpha$ is an inertial manifold for the semiflow $\mathcal{S}_\alpha$ in the sense of [22].

10.2. The singular limit dynamics of invariant manifolds

In this section, our objective is to study singular limit dynamics of invariant manifolds $\mathcal{M}_\alpha$ when $\alpha \to 0^+$. The main purpose is to show

$$(\Phi_\alpha, \Psi_\alpha) \to (\Phi_0, \Psi_0) \text{ as } \alpha \to 0^+ \tag{10.2.1}$$

in the topology of the space $C^1_{bdd}(B, X^\gamma \times Y^\beta)$ where B is an arbitrary bounded and open subset of $X_{1,0}$.

The proof uses abstract results due to Mora and Solà-Morales regarding the limiting behavior of fixed points of a two-parameter family of nonlinear mappings. With regard to Lemma 9.3.1, we note that the mapping $T_\alpha(x,.) : C^-_\mu(X^\gamma) \to C^-_\mu(X^\gamma)$ need not be generally C^1 differentiable. One can, however, expect that T_α is a C^1 mapping when considering $T_\alpha(x,.)$ as a mapping from $C^-_\mu(X^\gamma)$ into $C^-_\nu(X^\gamma)$ for some $\nu > \mu$. Therefore, we need a version of a contraction theorem covering the case in which differentiability involves a pair of Banach spaces.

First, we recall the assumptions of [34, Th. 5.1]. Let $\mathcal{X}, U$ be Banach spaces, $\alpha_1 > 0$. Let $T_\alpha,\ \alpha \in [0,\alpha_1]$ be a family of mappings from $\mathcal{X} \times U$ into U such that

$$
(T)\begin{cases}
1) & \text{there is } \bar\theta < 1 \text{ such that } \|T_\alpha(x,u_1) - T_\alpha(x,u_2)\|_U \le \bar\theta \|u_1 - u_2\|_U \\
 & \text{for any } x \in \mathcal{X}, u_1, u_2 \in U \text{ and } \alpha \in [0,\alpha_1]. \\[1ex]
2) & \text{there is a } Q < \infty \text{ such that } \|T_\alpha(x_1,u) - T_\alpha(x_2,u)\|_U \le Q\|x_1 - x_2\|_{\mathcal{X}} \\
 & \text{for any } x_1, x_2 \in \mathcal{X}, u \in U \text{ and } \alpha \in [0,\alpha_1]. \\[1ex]
3) & \text{for any } B \subset \mathcal{X} \text{ bounded and open} \\
 & \sup_{x\in B} \|T_\alpha(x, Y_0(x)) - T_0(x, Y_0(x))\|_U \to 0 \text{ as } \alpha \to 0^+ \\
 & \text{where } Y_\alpha(x),\ x \in \mathcal{X},\ \alpha \in [0,\alpha_1] \text{ is the unique fixed point of } T_\alpha(x,Y) = Y
\end{cases}
$$

Remark 10.2.1. Note that, by the Banach fixed point theorem, $(T)_1$ and $(T)_2$ ensure the existence of a family of fixed points $Y_\alpha(x)$ of $T_\alpha(x,.)$ such that the mapping $x \mapsto Y_\alpha(x)$ is Lipschitzian, its Lipschitz constant being $Q(1-\bar\theta)^{-1}$. Furthermore, $(T)_3$ implies $Y_\alpha(x) \to Y_0(x)$ as $\alpha \to 0^+$ uniformly with respect to $x \in B$, B is an arbitrary bounded and open subset of $\mathcal{X}$.

We assume that the space U is continuously embedded into a Banach space $\bar U$ through a linear embedding operator J. We also denote $\bar T_\alpha := JT_\alpha$ and $\bar Y_\alpha := JY_\alpha$.

We are now in a position to state a slightly modified version of [34, Th. 5.1]

Theorem 10.2.2. ([34, Th. 5.1]) *Besides the hypothesis (T) we also assume that the mappings* $\bar T_\alpha : \mathcal{X} \times U \to \bar U, \alpha \in [0,\alpha_1]$ *satisfy the following conditions:*

1) *for any* $\alpha \in [0,\alpha_1]$, $\bar T_\alpha$ *is Fréchet differentiable with* $D\bar T_\alpha : \mathcal{X} \times U \to L(\mathcal{X} \times U, \bar U)$ *bounded and uniformly continuous and there exist mappings*

$$d_u T_\alpha : \mathcal{X} \times U \to L(U,U);\ \bar d_u T_\alpha : \mathcal{X} \times U \to L(\bar U, \bar U);\ d_x T_\alpha : \mathcal{X} \times U \to L(\mathcal{X}, U)$$

such that

$$D_u \bar T_\alpha(x,u) = J d_u T_\alpha(x,u) = \bar d_u T_\alpha(x,u) J$$
$$D_x \bar T_\alpha(x,u) = J d_x T_\alpha(x,u)$$
$$\|d_u T_\alpha(x,u)\|_{L(U,U)} \le \bar\theta$$

$$\|\bar{d}_u T_\alpha(x,u)\|_{L(\bar{U},\bar{U})} \leq \bar{\theta}$$
$$\|d_x T_\alpha(x,u)\|_{L(\mathcal{X},U)} \leq Q$$

2) *for any B bounded and open subset of $\mathcal{X}$,*

$$D\bar{T}_\alpha(x,u) \to D\bar{T}_0(x,u) \quad \text{as} \quad \alpha \to 0^+$$

uniformly with respect to $(x,u) \in B \times \mathcal{F}_B$ where

$$\mathcal{F}_B := \{Y_\alpha(x) \in U;\ x \in B, \alpha \in [0,\alpha_1]\} \tag{10.2.2}$$

Then the mappings $\bar{Y}_\alpha : \mathcal{X} \to \bar{U}$ have the following properties

a) *for any $\alpha \in [0,\alpha_1]$; $\bar{Y}_\alpha : \mathcal{X} \to \bar{U}$ is Fréchet differentiable, with $D\bar{Y}_\alpha : \mathcal{X} \to L(\mathcal{X},\bar{U})$ bounded and uniformly continuous*

b) *for any B bounded and open subset of $\mathcal{X}$,*

$$D\bar{Y}_\alpha(x) \to D\bar{Y}_0(x) \quad \text{as} \quad \alpha \to 0^+$$

uniformly with respect to $x \in B$.

Proof. The only difference between the assumptions of the above theorem and those made in [34, Th. 5.1] resides in the part 2). Hence, the proof of the part 1) remains the same as that of [34, Th. 5.1, part K1)].

Recall that in [34, Th 5.1] they required a uniform convergence of $D\bar{T}_\alpha \to D\bar{T}_0$ instead of 2). Nevertheless, they have shown the estimate

$$\|D\bar{Y}_\alpha(x) - D\bar{Y}_0(x)\|_{L(\bar{U},\bar{U})} \leq \frac{1}{1-\bar{\theta}}\Big\{\frac{Q}{1-\bar{\theta}}\|D_u\bar{T}_\alpha(x,Y_\alpha(x)) - D_u\bar{T}_0(x,Y_0(x))\|_{L(\bar{U},\bar{U})}+$$

$$+\|D_x\bar{T}_\alpha(x,Y_\alpha(x)) - D_x\bar{T}_0(x,Y_0(x))\|_{L(\mathcal{X},\bar{U})}\Big\}$$

Furthermore,

$$\|D_i\bar{T}_\alpha(x,Y_\alpha(x)) - D_i\bar{T}_0(x,Y_0(x))\| \leq$$
$$\leq \|D_i\bar{T}_\alpha(x,Y_\alpha(x)) - D_i\bar{T}_0(x,Y_\alpha(x))\| + \omega_i\left(\|Y_\alpha(x) - Y_0(x)\|\right)$$

where i stands either for u or x and ω_i denotes the modulus of continuity of $D_i\bar{T}_0$. Hence the assumption 2) is sufficient for the proof of the local uniform convergence $D\bar{Y}_\alpha \to D\bar{Y}_0$ as stated in b).

♢

Henceforth, we will assume that

$$\varrho \in (0, 1-\gamma), \quad \text{and} \quad \lambda_- < \mu < (1+\eta)\mu \leq \kappa < \bar{\mu} < \lambda_+ \tag{10.2.3}$$

In order to apply Theorem 10.2.2 to fixed points $Y_\alpha(x)$ of the nonlinear operator $T_\alpha(x,.)$ defined in (10.1.11) we choose the following Banach spaces

$$\mathcal{U} := C^-_{\mu,\varrho,a}(X^\gamma) \quad \text{and } \bar{\mathcal{U}} := C^-_{\bar{\mu},\varrho,a}(X^\gamma), \tag{10.2.4}$$

and denote

$$J : C^-_{\mu,\varrho,a}(X^\gamma) \to C^-_{\bar{\mu},\varrho,a}(X^\gamma) \tag{10.2.5}$$

a linear embedding operator. A constant $0 < a \ll 1$ will be determined later. Before proving that the family of mappings $T_\alpha, \alpha \in [0,\alpha_1]$, fulfills the assumptions of Theorem 10.2.2 we need several auxiliary lemmas each of which is under the hypothesis (H) and (10.2.3). First, we introduce a notation.

In the following, with regard to Lemma 10.1.2, c), d), the mappings $\mathcal{K}_\alpha$ and $\mathcal{T}_\alpha$ will be considered as bounded linear operators acting on

$$\mathcal{K}_\alpha : X_{1,0} \to \mathcal{U} \quad \mathcal{T}_\alpha : C^-_\mu(X) \to \mathcal{U}$$

We also denote

$$\bar{\mathcal{K}}_\alpha : X_{1,0} \to \bar{\mathcal{U}} \quad \bar{\mathcal{T}}_\alpha : C^-_{\bar{\mu}}(X) \to \bar{\mathcal{U}} \tag{10.2.6}$$

the bounded linear operators analogous to $\mathcal{K}_\alpha$ and $\mathcal{T}_\alpha$, respectively, but operating on exponentially weighted spaces with an weight $e^{\bar{\mu}t}$. We remind that the boundedness of $\mathcal{K}_\alpha, \mathcal{T}_\alpha, \bar{\mathcal{K}}_\alpha, \bar{\mathcal{T}}_\alpha$ follows from Lemma 10.2.1, parts c) and d). Because Rank $T_\alpha \subseteq \mathcal{U}$ (see Lemma 10.1.2), we obtain

$$Y_\alpha(x) = T_\alpha(x, Y_\alpha(x)) \in \mathcal{U} \;\text{ for any } x \in X_{1,0} \;\text{ and } \alpha \in [0,\alpha_1] \tag{10.2.7}$$

Moreover, we have

Lemma 10.2.3. *Let B a bounded subset of $X_{1,0}$. Then the set*

$$\mathcal{F}_B := \{Y_\alpha(x) \in U;\; x \in B, \alpha \in [0,\alpha_1]\} \tag{10.2.8}$$

is a bounded subset of $\mathcal{U}$. At the same time, $\mathcal{F}_B$ is a C^-_μ - uniformly equicontinuous and bounded subset of $C^-_\mu(X^\gamma)$.

Proof. Since $Y_\alpha(x) = T_\alpha(x, Y_\alpha(x)) = \mathcal{K}_\alpha x + \mathcal{T}_\alpha(\mathcal{G}_\alpha(Y_\alpha(x)))$ and $\mathcal{G}_\alpha$ is bounded the proof follows from Lemma 10.1.2 c) and (9.3.7).

$\diamondsuit$

Because of the assumption $f \in C^{1+\eta}_{bdd}(X^\gamma, Y^\xi)$ we have that

$$\tilde{f} : C^-_\mu(X^\gamma) \to C^-_\mu(Y^\xi); \quad \tilde{f}(u)(t) := f(u(t))$$

is bounded and Lipschitz continuous. Recall that the space $\mathcal{U}$ is continuously embedded into $C_\mu^-(X^\gamma)$. Hence the mapping

$$H_\alpha : \mathcal{U} \to C_\mu^-(Y^\beta); \quad H_\alpha(u) := C_\alpha \tilde{f}(u) \tag{10.2.9}$$

is bounded and Lipschitz continuous. We also denote $\bar{C}_\alpha$ the linear operator defined in (10.1.5) and operating from $C_\kappa^-(Y^\xi) \to C_\kappa^-(Y^\beta)$. Let

$$\bar{H}_\alpha := J' H_\alpha : \mathcal{U} \to C_\kappa^-(Y^\beta) \tag{10.2.10}$$

Here

$$J' : C_\mu^-(Y^\beta) \to C_\kappa^-(Y^\beta) \tag{10.2.11}$$

is a linear embedding operator.

LEMMA 10.2.4.

a) $\bar{H}_\alpha \in C^1_{bdd}(\mathcal{U}, C_\kappa^-(Y^\beta))$

b) *there is an operator* $dH_\alpha : \mathcal{U} \to L(\mathcal{U}, C_\mu^-(Y^\beta))$ *such that* $D\bar{H}_\alpha = J' dH_\alpha$

PROOF. From Lemma 9.3.2 we have $\tilde{f} \in C^1_{bdd}(\mathcal{U}, C_\kappa^-(Y^\xi))$ and, by Lemma 10.1.1, $\bar{C}_\alpha \in L(C_\kappa^-(Y^\xi), C_\kappa^-(Y^\beta))$. Hence $\bar{H}_\alpha \in C^1_{bdd}(\mathcal{U}, C_\kappa^-(Y^\beta))$ and $D\bar{H}_\alpha = \bar{C}_\alpha D\tilde{f}$.

b) Since $Df : X^\gamma \to L(X^\gamma, Y^\xi)$ is bounded we obtain that the operator

$$df : \mathcal{U} \to L(\mathcal{U}, C_\mu^-(Y^\xi)), \quad df(u) := Df(u(.)) \tag{10.2.12}$$

is well defined and bounded. By Lemmas 9.3.1 and 9.3.2, the derivative $D\tilde{f}$ is given by $D\tilde{f} = J' df$. Denote

$$dH_\alpha := C_\alpha df \tag{10.2.14}$$

Then $D\bar{H}_\alpha = \bar{C}_\alpha D\tilde{f} = J' C_\alpha df = J' dH_\alpha$ [†].

$\diamondsuit$

LEMMA 10.2.5. *Let* $\mathcal{F}$ *be a bounded subset of* $\mathcal{U}$. *Then*

$H_\alpha(u) \to H_0(u)$ *as* $\alpha \to 0^+$ *in* $C_\mu^-(Y^\beta)$ *and*

$D\bar{H}_\alpha(u) \to D\bar{H}_0(u)$ *as* $\alpha \to 0^+$ *in* $L(\mathcal{U}, C_\kappa^-(Y^\beta))$ *uniformly with respect to* $u \in \mathcal{F}$.

PROOF. a) Because both f and Df are assumed to be bounded, one can show that the set

$$\mathcal{F}_0 := \{\tilde{f}(u); \quad u \in \mathcal{F}\}$$

[†] Here we have used the operator identity $\bar{C}_\alpha J' = J' C_\alpha$ which follows from the uniqueness of solutions of $(10.1.1)_\alpha$ in the space $C_\kappa^-(Y^\beta)$

is a C^-_μ - uniformly equicontinuous and bounded subset of $C^-_\mu(Y^\xi)$. By Lemma 10.1.1, we obtain

$$H_\alpha(u) = C_\alpha \tilde{f}(u) \to C_0 \tilde{f}(u) = H_0(u) \quad \text{as } \alpha \to 0^+$$

in the space $C^-_\mu(Y^\beta)$ uniformly for $u \in \mathcal{F}$.

b) From (10.2.3) we have $(\kappa - \mu)/\eta \geq \mu$. Since we have assumed that the mapping $X^\gamma \ni u \mapsto Df(u) \in L(X^\gamma, Y^\xi)$ is η-Hölder continuous one easily verifies that the set

$$\mathcal{F}_1 := \{df(u); \quad u \in \mathcal{F}\}$$

is a $C^-_{\kappa-\mu}$-uniformly equicontinuous and bounded subset of $C^-_{\kappa-\mu}(L(X^\gamma, Y^\xi))$. Then, by [34, Lemma 5.4, d)] and (9.3.7), the set

$$\mathcal{F}_2 := \{df(u)h; \ \|h\|_{C^-_{\mu,\varrho,a}} \leq 1, \ u \in \mathcal{F}\}$$

is a C^-_κ-uniformly equicontinuous and bounded subset of $C^-_\kappa(Y^\xi)$. Again, by Lemma 10.1.1, b), we obtain

$$D\bar{H}_\alpha(u)h = \bar{C}_\alpha df(u)h \to \bar{C}_0 df(u)h = D\bar{H}_0(u)h \quad \text{as } \alpha \to 0^+$$

uniformly for $\|h\|_{C^-_{\mu,\varrho,a}} \leq 1$ and $u \in \mathcal{F}$.

$\diamondsuit$

It follows from Lemma 9.3.1 and (H3) that

$$\tilde{g} \in C^1_{bdd}(C^-_i(X^\gamma) \times C^-_i(Y^\beta), C^-_{\bar{\mu}}(X)) \tag{10.2.15}$$

where i stands either for μ or κ. Define the operators

$$d_u g : C^-_\mu(X^\gamma) \times C^-_\mu(Y^\beta) \to L(C^-_\mu(X^\gamma), C^-_\mu(X))$$

and

$$d_w g : C^-_\mu(X^\gamma) \times C^-_\mu(Y^\beta) \to L(C^-_\mu(Y^\beta), C^-_\mu(X))$$

as follows

$$d_u g(u, w) := D_u g(u(.), w(.)) \quad \text{and} \quad d_w g(u, w) := D_w g(u(.), w(.)) \tag{10.2.16}$$

As Dg is bounded, the mappings $d_u g$ and $d_w g$ are bounded as well. Further, the derivative $D\tilde{g}$ when restricted to $C^-_\mu(X^\gamma) \times C^-_\mu(Y^\beta)$ can be expressed as

$$D_u \tilde{g} = J'' d_u g \qquad D_w \tilde{g} = J'' d_w g \tag{10.2.17}$$

where $J'' : C^-_\mu(X) \to C^-_{\bar{\mu}}(X)$ is a linear embedding operator.

Lemma 10.2.6. *Let $\mathcal{F}$ be a bounded subset of $\mathcal{U}$. Then*

a) $\bar{\mathcal{G}}_\alpha \in C^1_{bdd}(\mathcal{U}, C^-_{\bar{\mu}}(X))$ *where* $\bar{\mathcal{G}}_\alpha := J'' \mathcal{G}_\alpha$

b) there is a mapping $d\mathcal{G}_\alpha : \mathcal{U} \to L(\mathcal{U}, C^-_\mu(X))$ *such that* $D\bar{\mathcal{G}}_\alpha = J'' d\mathcal{G}_\alpha$

c) $\mathcal{G}_\alpha(u) \to \mathcal{G}_0(u)$ *in* $C^-_\mu(X)$ *and* $D\bar{\mathcal{G}}_\alpha(u) \to D\bar{\mathcal{G}}_0(u)$ *as* $\alpha \to 0^+$ *in* $L(\mathcal{U}, C^-_{\bar{\mu}}(X))$ *uniformly with respect to* $u \in \mathcal{F}$.

Proof. The proof of the statement a) follows from Lemma 10.2.4 and (10.2.15). Let us define $d\mathcal{G}_\alpha$ as follows

$$d\mathcal{G}_\alpha := d_u g + d_w g \; dH_\alpha \tag{10.2.18}$$

By Lemmas 9.3.1, 10.2.4 and (10.2.17),

$$D\bar{\mathcal{G}}_\alpha = D_u\tilde{g} + D_w\tilde{g}D\bar{H}_\alpha = J'' d_u g + J'' d_w g \; dH_\alpha = J'' d\mathcal{G}_\alpha$$

Since $\mathcal{G}_\alpha(u) = g(u(.), C_\alpha\tilde{f}(u)(.)) = g(u(.), H_\alpha(u)(.))$, the first part of the statement c) follows from Lemma 10.2.5. As $D\tilde{g}_w$ is bounded, the second part is a consequence of Lemmas 9.3.1, 9.3.2 and 10.2.5.

◊

We are now in a position to apply Theorem 10.2.2 to the family of nonlinear operators $\{T_\alpha\}$, introduced in Section 10.1.

Since the mapping $\mathcal{G}_\alpha : \mathcal{U} \to C^-_\mu(X)$ is Lipschitz continuous, its Lipschitz constant being estimated by the right hand side of (10.1.17), using Lemma 10.1.2, c), d)†, we obtain that the family $T_\alpha(x, .)$ satisfies the hypotheses T_1 and T_2 in the Banach space $\mathcal{U}$ with constants

$$\bar{\theta} := 2\theta \qquad \bar{Q} := 2C\lambda_-^\gamma \tag{10.2.19}$$

where the constant $\theta > 0$ was introduced in (10.1.18). Furthermore, according to Lemmas 10.1.2 and 10.2.6, the assumption T_3 is also fulfilled. Let us define the operators

$$\begin{aligned} d_u T_\alpha &: X_{1,0} \times \mathcal{U} \to L(\mathcal{U}, \mathcal{U}) \\ d_x T_\alpha &: X_{1,0} \times \mathcal{U} \to L(X_{1,0}, \mathcal{U}) \\ \bar{d}_u T_\alpha &: X_{1,0} \times \mathcal{U} \to L(\bar{\mathcal{U}}, \bar{\mathcal{U}}) \end{aligned} \tag{10.2.20}$$

as follows

$$d_u T_\alpha(x, u) := \mathcal{T}_\alpha d\mathcal{G}_\alpha(u); \quad d_x T_\alpha(x, u) := \mathcal{K}_\alpha; \quad \bar{d}_u T_\alpha(x, u) := \bar{\mathcal{T}}_\alpha \bar{d}\mathcal{G}_\alpha(u) \tag{10.2.21}$$

where $\bar{d}\mathcal{G}_\alpha$ is defined in the same way as the operator $d\mathcal{G}_\alpha$ but operating from $\mathcal{U}$ to $L(\bar{\mathcal{U}}, C^-_{\bar{\mu}}(X))$. Denote

$$\bar{T}_\alpha := JT_\alpha : X_{1,0} \times \mathcal{U} \to \bar{\mathcal{U}} \quad \text{and} \quad \bar{Y}_\alpha := JY_\alpha \tag{10.2.22}$$

† In order to ensure the assumptions of Lemma 10.1.2 we now choose a suficiently small constant $a > 0$ appearing in the definition of the spaces $\mathcal{U}, \tilde{\mathcal{U}}$.

By Lemma 10.2.6, $\bar{T}_\alpha \in C^1_{bdd}(X_{1,0} \times \mathcal{U}, \bar{\mathcal{U}})$. Moreover, from Lemmas 10.1.2 and 10.2.3 we obtain $D\bar{T}_\alpha(x,u) \to D\bar{T}_0(x,u)$ as $\alpha \to 0^+$ uniformly with respect to $(x,u) \in X_{1,0} \times \mathcal{F}_B$ for every B bounded and open subset of $X_{1,0}$.

Hence we have shown that the family $T_\alpha(x,.)$ and the mappings $d_u T_\alpha, d_x T_\alpha, \bar{d}_u T_\alpha$ satisfy the assumptions of Theorem 10.2.2, provided that the constant $\bar{\theta} > 0$ defined by (10.2.19) is less than 1. In the case $\bar{\theta} < 1$, by Theorem 10.2.2 we obtain

$$\bar{Y}_\alpha \to \bar{Y}_0 \quad \text{as} \quad \alpha \to 0^+ \quad \text{in} \quad C^1_{bdd}(B, C^-_{\bar{\mu},\varrho,a}(X^\gamma)) \tag{10.2.23}$$

for any $B \subset X_{1,0}$ bounded and open.

Recall that

$$(\Phi_\alpha(x), \Psi_\alpha(x)) := (\bar{Y}_\alpha(x)(0), \tilde{H}_\alpha(\bar{Y}_\alpha(x))(0))$$

where $\tilde{H}_\alpha$ is now considered as a C^1 mapping from $C^-_{\bar{\mu},\varrho,a}(X^\gamma)$ into $C^-_\nu(Y^\beta)$ for some $\nu > \bar{\mu}$. In view of Lemma 10.2.5, statement (10.2.23) readily implies a C^1 - local uniform convergence of $(\Phi_\alpha, \Psi_\alpha)$ towards (Φ_0, Ψ_0) as stated in (10.2.1).

In accordance to Lemma 10.1.3, we remind that the assumption $\theta < 1$ (θ defined by (10.1.18)) is sufficient for the existence of a family of Lipschitz continuous invariant manifolds $\mathcal{M}_\alpha$ for semiflows $\mathcal{S}_\alpha, \alpha \in [0, \alpha_1]$. On the other hand, the assumption $\bar{\theta} = 2\theta < 1$ guarantees a "C^1- closeness" of $\mathcal{M}_\alpha$ and $\mathcal{M}_0$ which can be precisely expressed by (10.2.1). Clearly, one way how to ensure the condition $\bar{\theta} < 1$ is to require smallness of the constant $K > 0$.

Having developed the previous background we can state the main result of this part

THEOREM 10.2.7. *Assume that the hypothesis (H) holds. Then there are constants $\tau > 0$ and $\alpha_1 > 0$ such that, if $K(\lambda_-, \lambda_+, \mu, \gamma) < \tau$ then, for every $\alpha \in (0, \alpha_1]$,*

a) *there exists an invariant manifold $\mathcal{M}_\alpha$ ($\tilde{\mathcal{M}}_0$) for the semiflow $\mathcal{S}_\alpha$ ($\tilde{\mathcal{S}}_0$) generated by system (9.2.1)$_\alpha$. Moreover, $\dim \mathcal{M}_\alpha = \dim \tilde{\mathcal{M}}_0 < \infty$ and $\mathcal{M}_\alpha$ ($\tilde{\mathcal{M}}_0$) is the graph of a C^1 continuous mapping $X_{1,0} \ni x \mapsto (\Phi_\alpha(x), \Psi_\alpha(x)) \in X^\gamma \times Y^\beta$ ($X_{1,0} \ni x \mapsto \Phi_\alpha(x) \in X^\gamma$)*

b) *for any bounded and open subset $B \subset X_{1,0}$*

$$(\Phi_\alpha, \Psi_\alpha) \to (\Phi_0, \Psi_0) \text{ as } \alpha \to 0^+ \text{ in } C^1_{bdd}(B, X^\gamma \times Y^\beta)$$

REMARK 10.2.8. In addition to the hypothesis (H) we also assume that A_0 is a self-adjoint operator with eigenvalues

$$0 < \lambda_1 \le ... \le \lambda_n < \lambda_{n+1} \le ... \qquad \lambda_n \to \infty \quad \text{as} \quad n \to \infty$$

As it is usual in such a case, we will let $\lambda_- := \lambda_n,\ \lambda_+ := \lambda_{n+1}$ and $\mu := (\lambda_+ + \lambda_-)/2$. With this setting it should be obvious that the condition "K is small" reduces to the requirement

"$\lambda_n^\gamma(\lambda_{n+1}-\lambda_n)^{-1}$ is small enough". In case when $\lambda_m \approx m^2,\ m \in N$, the assumptions of Theorem 10.2.7 are satisfied, whenever $\gamma \in [0, 1/2)$ and n is large enough.

We end this section with a couple of remarks. First, we recall a notion of structural stability of a vector fields due to Andronov and Pontryagin. Let $\mathcal{M}$ be a C^1 compact manifold. Two vector fields G_1, G_2 on $\mathcal{M}$ are said to be topological equivalent if there exists a homeomorphism $\phi : \mathcal{M} \to \mathcal{M}$ which takes orbits of G_1 to orbits of G_2 preserving their orientation. A C^1 vector field G on $\mathcal{M}$ is said structurally stable if there is a neighborhood $\mathcal{O}(G)$ of G with respect to the C^1 topology such that each $\tilde{G} \in \mathcal{O}(G)$ is topological equivalent to G (see, [21], [32]).

Having recalled these broadly known definitions and using the form of induced ordinary differential equation $(10.1.26)_\alpha$ we can state the following consequence of Theorem 7.2.7 and Remark 10.1.4

Corollary 10.2.9. *Besides the hypotheses of Theorem 10.2.7 we also assume that systems* $(9.2.1)_\alpha, \alpha \in [0, \alpha_1]$, *admit a family of compact global attractors* $\mathcal{A}_\alpha \subset \mathcal{M}_\alpha,\ \alpha \in [0, \alpha_1]$ †, *and, moreover, the set*

$$\mathcal{B} := \{u \in X;\ (u, w) \in \mathcal{A}_\alpha,\ \text{for some}\ \ w \in Y,\ \alpha \in [0, \alpha_1]\}$$

is a bounded subset of X.

If system $(9.2.1)_0$ *is structurally stable then, for any* $\alpha > 0$ *small enough, the flow* $\mathcal{S}_\alpha$ *on* $\mathcal{A}_\alpha$ *is topological equivalent to that on* $\mathcal{A}_0$.

† We identify the attractor $\tilde{\mathcal{A}}_0 \subset X^\gamma$ with its natural extension

$$\mathcal{A}_0 := \{(u, B_0^{-1} f(u)),\ \ u \in \tilde{\mathcal{A}}_0\} \subset X^\gamma \times Y^\beta.$$

Section 11.

Applications

The purpose of this section is twofold. First, we will study the singular limit dynamics of local invariant manifolds and compact attractors for the model of shearing motions of a non-Newtonian fluid. Section 11.2 is devoted to some second order abstract evolution equations arising in the mathematical theory of elastic systems.

11.1 The limiting behavior of invariant manifolds and global attractors for a model of shearing motions with ratio of Reynolds number to Deborah number very small

As our first application of results of Section 10, we will consider the non-Newtonian model of shearing motions including diffusion which was introduced in Section 2. Recall that the system of governing equations has the form

$$\begin{aligned} u_t &= \nu^2 u_{xx} - u + g(v_x) + fx \\ \alpha v_t &= v_{xx} + u_x \end{aligned} \tag{11.1.1}$$

with boundary conditions

$$v_x(t,0) = v(t,1) = 0; \quad u(t,0) = u_x(t,1) = 0 \quad for\ t \geq 0 \tag{11.1.2}$$

and initial data

$$v(0,x) = v_0(x); \quad u(0,x) = u_0(x) \quad for\ x \in [0,1]. \tag{11.1.3}$$

Throughout this section, we let $X = Y$ denote the real Hilbert space $L_2(0,1)$ with norm $\|.\|$ and its usual inner product $(.,.)$. In accordance to the notation of Sections

2 and 3 we will let A denote the self-adjoint positive operator in X with the domain $D(A) = \{u \in W^{2,2}(0,1);\ u(0) = u_x(1) = 0\}$ and $Au := -\nu^2 u_{xx}$ for any $u \in D(A)$ †. We also denote B the self-adjoint positive operator in Y its domain being $D(B) = \{w \in W^{2,2}(0,1);\ w_x(0) = w(1) = 0\}$ and $Bw := -w_{xx}$ for any $w \in D(B)$. One checks immediately that the linear operators A and B have the spectrum $\sigma(A) = \nu^2\sigma(B)$ and $\sigma(B) = \{\lambda_n,\ \lambda_n = (n-1/2)^2\pi^2,\ n \in N\}$. Eigenvectors of A and B are proportional to $\sin\sqrt{\lambda_n}x$ and $\cos\sqrt{\lambda_n}x$, respectively. Knowing these spectral properties and using Fourier series with respect to eigenvectors $\sin\sqrt{\lambda_n}x$ and $\cos\sqrt{\lambda_n}x,\ n \in N$ (cf. [23, Chapter 1]) one can prove that the linear operator

$$\mathfrak{f} : X^\gamma \to Y^{\gamma-1/2}; \quad \mathfrak{f}(u) := u_x$$

is well defined and bounded for a $\gamma > 1/4$. Moreover, $\|\mathfrak{f}(u)\|_{Y^{\gamma-1/2}} = \|u\|_{X^\gamma}$. Here, we have adopted the convention according to which we identify the space $Y^{-\kappa}$ with the dual space $(Y^\kappa)^*$. Let $g \in C^1_{bdd}(R)$. Then the mapping

$$\mathfrak{g} : X^\gamma \times Y^\beta \to X; \quad \mathfrak{g}(u,v)(x) := g(v_x(x)) + fx - u(x), \quad x \in [0,1]$$

is well defined and Lipschitz continuous for any $g \in C^1_{bdd}(R)$ and $\beta \geq 0$. In terms of $A, B, \mathfrak{f}, \mathfrak{g}$ system (11.1.1)-(11.1.3) can be rewritten abstractly as

$$\begin{aligned} u_t + Au &= \mathfrak{g}(u,v) \\ \alpha v_t + Bv &= \mathfrak{f}(u) \end{aligned} \qquad \alpha \geq 0 \tag{11.1.4$_\alpha$}$$

with initial data $u(0) = u_0,\ v(0) = v_0$. Henceforth, we will assume

$$g \in C^1_{bdd}(R), \qquad \gamma \in (\frac{1}{4}, \frac{1}{2}), \quad \beta \in (\frac{3}{4}, \frac{1}{2} + \gamma) \tag{11.1.5}$$

According to Section 9.2, system $(11.1.4)_\alpha,\ \alpha > 0$, generates a semiflow $\mathcal{S}_\alpha(t),\ t \geq 0$, on $X^\gamma \times Y^\beta$ and system $(11.1.4)_0$ generates a semiflow $\tilde{\mathcal{S}}_0(t),\ t \geq 0$ on X^γ. Taking into account the particular form of $\mathfrak{g}$, system $(11.1.4)_0$ becomes a scalar reaction-diffusion parabolic equation

$$u_t - \nu^2 u_{xx} + u + g(u) - fx = 0 \tag{11.1.6}$$

with boundary conditions $u(t,0) = u_x(t,1) = 0$ and initial data belonging to the phase space X^γ.

In what follows, we will prove that the semiflow $\mathcal{S}_\alpha,\ (\tilde{\mathcal{S}}_0)$ has a compact global attractor $\mathcal{A}_\alpha,\ (\tilde{\mathcal{A}}_0)$. In many situations, the bounded dissipativity and compactness of a semiflow ensure the existence of a compact global attractor. In order to establish boundedness and compactness of semiflows $\mathcal{S}_\alpha,\ \alpha > 0, (\tilde{\mathcal{S}}_0)$ we will make use of the standard method of *à priori* estimates.

† In contrast to Section 3 where the singular limit $\nu \to 0^+$ has been investigated we will henceforth assume the constant $\nu > 0$ to be fixed

Absorbing sets

For every function g satisfying (11.1.5) there is a $M > 0$ such that

$$\|g(v_x(x)) + fx\|_{L_2(0,1)} \le M \quad \text{for any} \quad v \in Y^\beta$$

Throughout this section, we will let $M > 0$ denote a constant always assumed to be independent on initial conditions of (11.1.4) and $\alpha \in [0, 1/2]$. Since $u' + u + Au = g(v_x(x)) + fx$ we have

$$u(t) = e^{-t}\exp\,(-At)u_0 + \int_0^t e^{-(t-s)}\exp\,(-A(t-s))[g(v_x(s,x)) + fx]\,ds$$

By Lemma 9.1.3, for each $\delta \in [0, 1-\gamma)$, we obtain

$$\|u(t)\|_{\gamma+\delta} \le Ct^{-\delta}e^{-\omega t}\|u_0\|_\gamma + C\,M\int_0^t (t-s)^{-(\gamma+\delta)}e^{-\omega(t-s)}\,ds \le$$

$$\le M(1 + t^{-\delta}e^{-\omega t}\|u_0\|_\gamma) \quad \text{for any} \quad t > 0 \tag{11.1.7}$$

where $0 < \omega < \lambda_1 = \pi^2/4$ is fixed. Notice that (11.1.7) also holds for the case $\alpha = 0$. Furthermore, for $\alpha > 0$, we have

$$v(t) = \exp\,(-Bt/\alpha)v_0 + \frac{1}{\alpha}\int_0^t \exp\,(-B(t-s)/\alpha)\;\mathfrak{f}(u(s))\,ds$$

Thus, for each $\delta \in [0, \gamma - \beta + 1/2)$ and $\alpha \in (0, 1/2]$, we obtain

$$\|v(t)\|_{\beta+\delta} \le$$

$$\le C\alpha^\delta t^{-\delta}e^{-\omega t/\alpha}\|v_0\|_\beta + \frac{1}{\alpha}\int_0^t \|B^{\beta+\delta-\gamma+1/2}\exp\,(-B(t-s)\alpha)B^{\gamma-1/2}\;\mathfrak{f}(u(s))\|\,ds \le$$

$$\le C\alpha^\delta t^{-\delta}e^{-\omega t/\alpha}\|v_0\|_\beta + \frac{C\,M}{\alpha}\int_0^t ((t-s)/\alpha)^{-(\beta+\delta-\gamma+1/2)}e^{-\omega(t-s)/\alpha}(1 + e^{-\omega s}\|u_0\|_\gamma)\,ds$$

for any $t > 0$. Here we have used inequality (11.1.7) with $\delta = 0$, i.e.

$$\|B^{\gamma-1/2}\;\mathfrak{f}\,(u(s))\| = \|u(s)\|_\gamma \le M(1 + e^{-\omega s}\|u_0\|_\gamma) \quad s \ge 0$$

Thus, for each $\delta \in [0, \gamma - \beta + 1/2)$, $\alpha \in (0, 1/2]$ and $t > 0$ we have

$$\|v(t)\|_{\beta+\delta} \le M(1 + e^{-\omega t}\|u_0\|_\gamma + t^{-\delta}e^{-\omega t/\alpha}\|v_0\|_\beta) \tag{11.1.8}$$

The estimates (11.1.7) and (11.1.8) with a fixed $0 \le \delta < \min\{1-\gamma, \gamma - \beta + 1/2\}$ enable us to conclude that there exists a bounded set

$$\mathcal{B} \subset X^\gamma \times Y^\beta \qquad (\tilde{\mathcal{B}} \subset X^\gamma) \tag{11.1.9}$$

which dissipates every bounded set $\mathcal{J} \subset X^\gamma \times Y^\beta$ $(\mathcal{J} \subset X^\gamma)$, i.e.

$$\mathcal{S}_\alpha(t)\mathcal{J} \subset \mathcal{B},\ \alpha \in (0, 1/2], \qquad (\tilde{\mathcal{S}}_0(t)\mathcal{J} \subset \tilde{\mathcal{B}}) \qquad \text{for any } \ t \geq T(\alpha, \mathcal{J}) > 0 \tag{11.1.10}$$

Since A^{-1} and B^{-1} are compact linear operators on $X = Y = L_2(0, 1)$ we know that the embeddings $X^{\gamma+\delta} \hookrightarrow X^\gamma$, $Y^{\beta+\delta} \hookrightarrow Y^\beta$ are compact for any $\delta > 0$ (cf. [23, Chapter 1]). Therefore $\mathcal{S}_\alpha(t_0)$ $(\tilde{\mathcal{S}}_0(t_0))$ is a compact mapping on $X^\gamma \times Y^\beta$ (X^γ) whenever $t_0 > 0$. Hence, by [3, Theorem 1.2], there exists a compact global attractor

$$\mathcal{A}_\alpha \subset X^\gamma \times Y^\beta,\ (\tilde{\mathcal{A}}_0 \subset X^\gamma) \tag{11.1.11}$$

for the semiflow $\mathcal{S}_\alpha,\ \alpha \in (0, 1/2]$, $(\tilde{\mathcal{S}}_0)$.

Local invariant manifolds

We now turn our attention to the problem of the existence of a family of invariant manifolds for the semiflows $\mathcal{S}_\alpha$ $(\tilde{\mathcal{S}}_0)$. First, we emphasize that the functions $\mathfrak{g}$ and $\mathfrak{f}$ are unbounded. Therefore we cannot apply Theorem 10.2.7 to system (11.1.4). As it is usual, the idea how to overcome this difficulty is to modify the equations of (11.1.4) far from the vicinity of the absorbing set $\mathcal{B}$ and then apply Theorem 10.2.7 to the modified system. The modification of (11.1.5) will enable us to deal with a global invariant manifold for the semiflow generated by a new system instead of a local one for the original semiflow. To do so, let θ denote a smooth cut-off function with the following properties

$$\theta \in C^\infty(R^+);\ \theta(\xi) = 1 \text{ for } \xi \in [0, 1];\ \theta(\xi) = 0 \text{ for } \xi \geq 2; \quad |\ \theta'\ | \leq 2$$

and define, for each $R > 0$, the modified functions

$$\mathfrak{g}_R(u, v) := \theta(\|u\|_\gamma^2/R^2)\mathfrak{g}\,(u, v);\quad \mathfrak{f}_R(u) := \theta(\|u\|_\gamma^2/R^2)\mathfrak{f}\,(u) \tag{11.1.12}$$

Since we have assumed $g \in C^1_{bdd}(R)$ and the function $u \mapsto \|u\|_\gamma^2$ is C^2 continuously differentiable the modified functions $\mathfrak{g}_R$ and $\mathfrak{f}_R$ satisfy the hypothesis (H3) from Section 9.2. Let $R > 0$ be fixed and such that

$$(u, v) \in \mathcal{B} \quad (u \in \tilde{\mathcal{B}}) \text{ implies } \ \|u\|_\gamma < R \tag{11.1.13}$$

In order to deal with local invariant manifolds for (11.1.4), we will consider a modified system instead of (11.1.4)

$$\begin{aligned} u_t + Au &= \mathfrak{g}_R(u, v) \\ \alpha v_t + Bv &= \mathfrak{f}_R(u) \end{aligned} \qquad \alpha \geq 0 \tag*{(11.1.14)$_\alpha$}$$

According to Theorem 10.2.7 and Remark 10.2.8 we have ensured the existence and convergence properties of a family of finite dimensional invariant manifolds $\mathcal{M}_\alpha^R$ $(\mathcal{M}_0^R)$ for semiflows generated by (11.1.14). Because the vector field of system (11.1.4) and that of (11.1.14) coincide inside the cylinder $\{(u, v) \in X^\gamma \times Y^\beta; \|u\|_\gamma < R\}$, the sets

$\mathcal{M}_\alpha := \mathcal{M}^R_\alpha \cap \mathcal{B},\ \alpha \in (0, 1/2]\ \ (\tilde{\mathcal{M}}_0 := \tilde{\mathcal{M}}^R_0 \cap \tilde{\mathcal{B}})$ are local invariant manifolds for the original semiflows generated by (11.1.4). Moreover, by Remark 10.1.4, $\mathcal{A}_\alpha \subset \mathcal{M}_\alpha$ and $\tilde{A}_0 \subset \tilde{\mathcal{M}}_0$.

The Morse-Smale property

The well-known result due to Palis and Smale [39] says that Morse-Smale vector fields on compact smooth manifolds are structurally stable. We recall that a C^1 vector field G (a semiflow $\mathcal{S}$ generated by G) is called Morse-Smale if

i) G has only finite numbers of steady states and periodic orbits all hyperbolic
ii) the set of non-wandering points coincides with the set of steady states and periodic orbits
iii) $W^u(p) \,\bar{\pitchfork}\, W^s(q)$ for any critical elements p, q (steady states or periodic orbits) †

(cf. [21]). In our case, the limiting equation (11.1.6) is a scalar reaction-diffusion parabolic equation in the one space dimension. For such parabolic equations it is known (cf. [1], [24]) that the stable and unstable manifolds of steady states intersect transversally.

Let us assume that each steady state $\bar{u}$ of (11.1.6) is hyperbolic. Since the steady state equation for (11.1.6) is a Sturm-Liouville problem the last assumption is equivalent to the claim that the spectrum of the linear operator $B_1[u] \equiv \nu^2 u_{xx} - u - g'(\bar{u}(x))u,\ u(0) = u_x(1) = 0$, does not contain zero as an eigenvalue for any steady state solution $\bar{u}$. As the set $\{u \in C^1_{bdd}(0, 1);\ u(0) = 0\}$ is continuously embedded into X^γ, for any $0 \le \gamma \le 1/2$, by Proposition 3.2.1, we know that the set $\mathcal{E}$ of steady states if bounded and hence finite. In such a case the asymptotic behavior of $\tilde{\mathcal{S}}_0$ is simple - each trajectory tends to some steady state. Indeed, one easily derives

$$\frac{1}{2}\frac{d}{dt}\Big\{\|u(t)\|^2 + \|u(t)\|^2_{1/2} + G(u(t))\Big\} + \|u_t(t)\|^2 = 0 \quad \text{for any} \quad t > 0$$

where $u(.)$ is a solution of (11.1.6) and $G(u(t, .)) := 2\int_0^1 \int_0^{u(t,x)} (g(\xi) - f\xi)\, d\xi\, dx$. Since g is assumed to be bounded the functional in brackets is bounded from below and nonincreasing along non-constant trajectories $\{u(t),\ t \ge 0,\ u(0) \in X^{1/2}\}$. On the other hand, for any solution $u(.)$ with $u(0) \in X^\gamma$ we have $u(t_0) \in X^1 \subset X^{1/2}$, for $t_0 > 0$. With this we can argue similarly as in (3.2.6) and Theorem 3.2.2. Hence any solution $u(.)$ of (11.1.6) converges in X^γ to some steady state. For any semiflow with such a gradient structure we have that the set of non-wandering points coincides with the set of steady states. Summarizing the above facts we obtain

Theorem 11.1.1. *For any $\alpha > 0$ ($\alpha = 0$) there exists a local invariant manifold $\mathcal{M}_\alpha$ ($\tilde{\mathcal{M}}_0$) and a compact global attractor $\mathcal{A}_\alpha \subset \mathcal{M}_\alpha$ ($\tilde{\mathcal{A}}_0 \subset \tilde{\mathcal{M}}_0$)) for the semiflow $\mathcal{S}_\alpha$ ($\tilde{\mathcal{S}}_0$) generated*

† The symbol $W^u(p)\bar{\pitchfork}W^s(q)$ denotes the transversal intersection of the stable manifold $W^s(q)$ of q and the unstable manifold $W^u(p)$ of p. We refer to [21] or [32] for definition of the set of non-wandering points

by system (11.1.1). If, in addition, each steady state of (11.1.6) is hyperbolic then the semiflow $\tilde{\mathcal{S}}_0$ on $\tilde{\mathcal{A}}_0$ is topological equivalent to that on $\mathcal{A}_\alpha$, for $\alpha > 0$ small.

Remark 11.1.2. Let us recall that the qualitative properties of the semiflow on the attractor of a scalar reaction diffusion equation are very well understood. For instance, in [9] Brunovský proved that the attractor of some scalar RDE is a smooth graph. Moreover, in [10] Brunovský and Fiedler completely characterized connections between any two steady states of some RDE. Theorem 11.1.1 tells us that topological properties of the attractor of (11.1.6) extend to the attractor of full system (11.1.1) whenever all steady states are hyperbolic and $\alpha > 0$ is sufficiently small. An information regarding topological equivalence of attractors $\mathcal{A}_\alpha$ and $\tilde{\mathcal{A}}_0$ enables us to investigate the asymptotic behavior of reduced problem (11.1.6) instead that of the full system of governing equations (11.1.1) for $\alpha > 0$ small enough. Notice that all numerical simulations of Section 6 were also performed for the parameter value $\alpha = 0$. The obtained results matched those for $\alpha \approx 10^{-9}$.

Remark 11.1.3. Besides functions of Van der Walls type satisfying (W) one can also treat more complicated functions $g \in C^2(R)$ having arbitrarily many loops. As an example of such a constitutive dependence between steady shear stress and steady strain rate, one can consider the Spriggs model of shearing motions with infinitely many constitutive equations with different relaxation times. The interesting reader is referred to the textbook by Chang Dae Han [12, p.41]. If the function g has more than one loop then following the approach of Section 3.3 one can establish the existence of another steady states of (3.3.1) having more then one abrupt transitions. In such a case, the attractor $\tilde{\mathcal{A}}_0$ will contain more than three steady states.

Finally, we will discuss generic hyperbolicity of steady states.

Remark 11.1.4. In the following we will denote $C^1(R)$ the linear topological space consisting of all continuously differentiable functions on R endowed either by strong norm or the weak C^1 topology (see [32]). We also denote $Y := \{g \in C^1(R),\ g(0) = 0\}$ the subspace of $C^1(R)$ equipped with the induced topology of $C^1(R)$. We will show that there is an open dense subset $\mathcal{G}$ of the set

$$\mathcal{M} := \{g \in Y; g'(0) > 0,\ ug(u) > 0 \text{ for } u \neq 0\}$$

such that all solutions of the problem (3.3.1) are hyperbolic provided that $g \in \mathcal{G}$. For the proof of the above statement we make use of an infinite dimensional version of the transversality theorem due to Quinn and Uhlenbeck (cf. [42, Theorème 1.1]).

Let $f > 0$ be fixed. Assume that $g : R \to R$ is a continuous function such that $ug(u) > 0$ for $u \neq 0$ and $g'(0) > 0$. Let $\bar{u}$ be an arbitrary solution of (3.3.1). By taking the inner product $(.,.)$ in $L_2(0,1)$ of (3.3.1) with $\bar{u}$ and using the sign property of g we obtain $-\nu^2 \int_0^1 (\bar{u}')^2 \geq -(fx, \bar{u})$. Hence $|\bar{u}(x)| \leq \nu^{-2} f$ for any $x \in [0,1]$. Since $\bar{u}$ is a solution of (3.3.1) satisfying the boundary conditions $\bar{u}(0) = \bar{u}_x(1) = 0$ and $g(u) < 0$ for $u < 0$ an obvious concavity argument (see Proposition 3.2.1) enables us to conclude that $0 < \bar{u}(x)$ for any $x \in (0,1]$.

Denote Z the Banach space $C^1_{bdd}(0, \nu^{-2} f)$. First, we will show that there is a dense subset $\mathcal{O}$ of the Banach space Z such that any problem (3.3.1) with $g(u) := u\exp\,(\varrho(u)), u \in [0, \nu^{-2} f]$, for some $\varrho \in \mathcal{O}$ has all solutions hyperbolic.

Indeed, for any $\varrho \in Z$ we define the C^1 mapping $F : X \times Z \to X$ by

$$F(u, \varrho) := u + A^{-1}[u(1 + \exp\,(\varrho(u))\,) - fx]$$

Since the operator $A^{-1} : X \to X$ is compact we obtain that F is a Fredholm mapping in u of index zero. Furthermore, using the *a priori* estimate of solutions of (3.3.1) one can easily verify that the set of $u \in X$ such that $F(u, \varrho) = 0$ with ϱ belonging to a compact subset of Z is relatively compact in X. Hence the mapping F is proper in the sense of [42, (1.3)]. Moreover, if 0 is a regular value of F (henceforth we will write $F \pitchfork \{0\}$) then, by [42, Theorème 1.1], the set $\mathcal{O} := \{\varrho \in Z; F(., \varrho) \pitchfork \{0\}\ \}$ is a dense open subset of Z. It means, however, that all solutions of (3.3.1) with $g(u) := u\exp\,(\varrho(u))$ are hyperbolic whenever $\varrho \in \mathcal{O}$.

It remains to prove that $F \pitchfork \{0\}$ which means that the total differential

$$X \times Z \ni (u, \varrho) \mapsto DF(\bar{u}, \bar{\varrho})(u, \varrho) := D_u F(\bar{u}, \bar{\varrho})u + D_\varrho F(\bar{u}, \bar{\varrho})\varrho \in X \tag{11.1.15}$$

is onto at every point $(\bar{u}, \bar{\varrho}) \in F^{-1}(0)$. Let $(\bar{u}, \bar{\varrho}) \in F^{-1}(0)$. Then $\bar{u}$ is a solution of (3.3.1) with $\bar{g}(u) := u\exp\,(\bar{\varrho}(u))$. Denote B the linearization of (3.3.1) at $\bar{u}$, i.e. $Bu \equiv \nu^2 u_{xx} - u - \bar{g}'(\bar{u}(.))u,\ \ u(0) = u_x(1) = 0$ where $D(B) = D(A) = \{u \in W^{2,2}(0, 1);\ u(0) = u_x(1) = 0\} \subset X$. The surjectivity of the linear operator defined by (11.1.15) is equivalent to that of the linear operator

$$D(B) \times Z \ni (u, \varrho) \mapsto Bu - \bar{u}(.)\exp\,(\bar{\varrho}(\bar{u}(.)))\varrho(\bar{u}(.)) \in X \tag{11.1.16}$$

In case $0 \notin \sigma(B)$ the equation

$$Bu - \bar{u}(.)\exp\,(\bar{\varrho}(\bar{u}(.)))\varrho(\bar{u}(.)) = w \tag{11.1.17}$$

has a solution $(u, \varrho) \equiv (B^{-1}w, 0)$ for any $w \in X$. On the other hand, if $0 \in \sigma(B)$ then equation (11.1.17) has a solution iff the element $w+\bar{u}(.)\exp\,(\bar{\varrho}(\bar{u}(.)))\varrho(\bar{u}(.))$ is orthogonal to $\mathrm{Ker}(B)$ for some $\varrho \in Z$. Since B is a Sturm-Liouville operator we have $\mathrm{Ker}(B) = \mathrm{span}\{u_0\}$ for some $u_0 \not\equiv 0$. Now suppose to the contrary that

$$(w + \bar{u}(.)\exp\,(\bar{\varrho}(\bar{u}(.)))\varrho(\bar{u}(.)),\ u_0) \neq 0 \text{ for any } \ \varrho \in Z$$

Then $\int_0^1 \varrho(\bar{u}(x))\tilde{u}_0(x)\,dx = 0$ for any $\varrho \in Z$ where $\tilde{u}_0(x) := \bar{u}(x)\exp\,(\bar{\varrho}(\bar{u}(x)))u_0(x)$. With regard to the assumption $f > 0$ and $g'(0) > 0$ we obtain $\bar{u}'(0) > 0$. Hence there are $a, b > 0$ such that $\bar{u}^{-1}([0, b]) = [0, a]$ and, moreover, $\bar{u}$ is one-to-one on $[0, a]$. Since the set $C^1_{bdd}(0, \nu^{-2} f)$ is dense in $L_2(0, \nu^{-2} f)$ we obtain $\int_0^a \varrho(\bar{u}(x))\tilde{u}_0(x)\,dx = 0$ for any $\varrho \in Z$. According to the Stone-Weierstrass theorem the set $\{\phi \in L_2(0, a),\ \phi(.) = \varrho(\bar{u}(.)),\ \ \varrho \in Z\}$ is dense in $L_2(0, a)$. Therefore $\tilde{u}_0 \equiv 0$ on $[0, a]$. But this yields $u_0 \equiv 0$ on $[0, a]$. Hence $u_0 \equiv 0$ on the whole interval $[0, 1]$, a contradiction. This way we have shown that the linear operator defined in (11.1.16) is onto. Hence $F \pitchfork \{0\}$.

The above result provides a "density argument" in the proof. Indeed, let us denote

$$\mathcal{G} := \{g \in \mathcal{M} \text{ such that all solutions of (3.3.1) with } g \text{ are hyperbolic } \}$$

Take an arbitrary $\bar{g} \in \mathcal{M}$. Obviously, in any neighborhood of $\bar{g}$ we can find a $\tilde{g} \in \mathcal{M}$ such that there exists $\tilde{g}''(0) \neq \pm\infty$. Moreover, we have $0 < \bar{u}(x) \leq \nu^{-2} f,\ x \in (0,1]$, for any solution $\bar{u}$ of (3.3.1) with $g \in \mathcal{M}$. Hence all consideration concerning either density or openness of $\mathcal{G}$ in $\mathcal{M}$ do not depend on whether we operate with strong norm or the weak C^1 topology on R. Furthermore, it should be obvious that in the proof of density of $\mathcal{G}$ in $\mathcal{M}$ it suffices to find a $g \in C^1_{bdd}(0, \nu^{-2} f)$, arbitrarily close to $\tilde{g}$ in $C^1_{bdd}(0, \nu^{-2} f)$ and such that all solutions of the problem (3.3.1) with g are hyperbolic. To do so, let us define $\tilde{\varrho} := \log(\tilde{g}(u)/u) \in C^1_{bdd}(0, \nu^{-2} f)$. Then there is a $\varrho \in C^1_{bdd}(0, \nu^{-2} f)$ sufficiently close to $\tilde{\varrho}$ with the property that all solutions of (3.3.1) with $g(u) := u\exp\,(\varrho(u))$ are hyperbolic. It completes the proof of density of $\mathcal{G}$ in $\mathcal{M}$.

As it usual in similar circumstances, the proof of openess of $\mathcal{G}$ in $\mathcal{M}$ is easier than that of density and, for instance, one can argue in the same way as in [40, Section 4]. We omit this detail of the proof.

11.2 Second order evolution equations arising in some elastic systems with structural damping

Finally, we will study second order abstract evolution equations of the form

$$\alpha u'' + A^\kappa u' + Au = f(u)$$

$$u(0) = u_0,\ u'(0) = v_0 \tag{11.2.1$_\alpha$}$$

where A (the elastic operator) is a self-adjoint positive operator in a real Hilbert space $\mathcal{X},\ \kappa \in [1/2, 1),\ \alpha \geq 0$ and $f : \mathcal{X}^\varrho \to \mathcal{X}$ is a nonlinear function for some $\varrho \in [\kappa, 1)$. The operator A^κ may represent dissipation in elastic systems (cf. [13]).

In recent years, many authors have studied problems having the general form (11.2.1) (see, e.g. Chen, Triggiani [13], [14] and other references therein). As a motivation for studying systems like (11.2.1) one can consider some specific beam equations with damping, e.g.

$$u_{tt} - \beta \Delta u_t + \Delta^2 u = m(\int_\Omega |\nabla u|^2) \Delta u$$

$$u = \Delta u = 0 \text{ on } \partial\Omega,\ u(0,x) = u_0(x),\ \beta u_t(0,x) = v_0(x),\ x \in \Omega \tag{11.2.2}$$

where $\Omega \subset R^N$ is a smoothly bounded domain, $\beta > 0$ is a damping coefficient, $m : R^+ \to R$ is a nondecreasing differentiable function measuring nonlocal character of structural damping of a beam or string (see, for instance, Biler [7], [8], Ševčovič [43]). If we let

$\mathcal{X} := L_2(\Omega),\ Au := \Delta^2 u,\ D(A) = \{W^{4,2}(\Omega),\ u = \Delta u = 0 \text{ on } \partial\Omega\}$ then problem (11.2.2) can be rewritten abstractly as problem (11.2.1) with $\varrho = \kappa = 1/2$. After a suitable rescaling time ($\tau := t/\beta$) one obtains $\alpha = 1/\beta^2$ and the singular limit $\alpha \to 0^+$ corresponds to the situation when β tends to infinity (see [44]).

Another class of beam equations with damping has been extensively investigated, for instance, by Ball [4], Čuešov [16], Feireisl [19], Fitzgibbon [20], Ševčovič [44]. Notice that some wave equations with damping can be also rewritten as (11.2.1) (Webb [49]). However, in these problems we have either $\kappa = 1$ ([4], [20], [44], [49]) or $\kappa = 0$ ([34], [16], [19]). Let us emphasize that the method explained below covers neither the case $\kappa = 1$ nor $\kappa \in [0, 1/2)$.

Throughout this section we will assume the following hypothesis

$$(E)\begin{cases} A : D_{\mathcal{X}}(A) \subseteq \mathcal{X} \to \mathcal{X} \text{ is a self-adjoint positive unbounded operator in} \\ \text{a real Hilbert space } \mathcal{X}. \text{ The resolvent } A^{-1} \text{ is a compact operator on } \mathcal{X} \\ \\ \kappa \in [1/2, 1),\ \alpha \geq 0 \\ \\ f \in C^{1+\eta}_{bdd}(\mathcal{X}^\varrho, \mathcal{X}) \text{ for some } \varrho \in [\kappa, 1) \text{ and } \eta \in (0, 1]. \end{cases}$$

We recall that an operator A satisfying (E) has the spectrum consisting of eigenvalues

$$\sigma(A) = \{\lambda_n;\ n \in N\} \qquad 0 < \lambda_1 \leq \lambda_2 \leq \dots;\quad \lambda_n \to \infty \text{ as } n \to \infty$$

We denote ϕ_n the eigenvector of A corresponding to $\lambda_n,\ n \in N$. According to [23, Chapter 1] A is a sectorial operator in $\mathcal{X}$ and the fractional powers of A and $\mathcal{X}$ can be characterized as

$$\mathcal{X}^\xi = D_{\mathcal{X}}(A^\xi) = \{u \in \mathcal{X};\ \sum_{n=1}^{\infty} \lambda_n^{2\xi}(u, \phi_n)^2 < \infty\};\quad \|u\|_\xi^2 = \|A^\xi u\|^2 = \sum_{n=1}^{\infty} \lambda_n^{2\xi}(u, \phi_n)^2 \tag{11.2.3}$$

Knowing the above spectral decompositions, one can readily show that, for any $r, s \geq 0$, the operator A^r is a self-adjoint positive operator in the Hilbert space $X := \mathcal{X}^s$, its domain being $D_X(A^r) := D_{\mathcal{X}}(A^{r+s})$. The fractional power space $X^\gamma,\ \gamma \in [0, 1]$, subject to the sectorial operator A^r consists of the domain $D_{\mathcal{X}}(A^{s+\gamma r})$ and norm on X^γ is given by $\|u\|_{X^\gamma} = \|A^{s+\gamma r} u\|_{\mathcal{X}}$ for any $u \in X^\gamma$. Moreover, $\sigma(A^r) = \{\lambda_n^r; n \in N\}$.

(11.2.4)

Now we return to system (11.2.1). We will make use a change of variables in such a way that the resulting system fits into the abstract setting investigated in Section 10. To do so, we let X, Y denote real Hilbert spaces

$$X := [D_{\mathcal{X}}(A^{(1-\omega)\kappa})]_{\mathcal{X}} = \mathcal{X}^{(1-\omega)\kappa}, \quad Y := \mathcal{X} \tag{11.2.5}$$

where

$$\omega \in (0, (1-\varrho)/\kappa) \tag{11.2.6}$$

is fixed. Let linear operators $A_\alpha, B_\alpha,\ \alpha \in [0,\alpha_0]$, in X and Y, respectively, be defined as follows

$$A_\alpha := \frac{A^\kappa}{2\alpha}\left(1-\left(1-4\alpha A^{1-2\kappa}\right)^{1/2}\right), \quad B_\alpha := \frac{1}{2}\left(1+\left(1-4\alpha A^{1-2\kappa}\right)^{1/2}\right)A^\kappa$$

for $\alpha \in (0,\alpha_0]$, $\alpha_0 > 0$ small, and

$$A_0 := A^{1-\kappa}, \quad B_0 := A^\kappa \tag{11.2.7}$$

their domains being

$$D_X(A_\alpha) := D_X(A^{1-\kappa}), \quad D_Y(B_\alpha) := D_Y(A^\kappa) \quad \text{for any} \quad \alpha \in [0,\alpha_0] \tag{11.2.8}$$

Since A_0 and B_0 are self-adjoint positive operators in X and Y, respectively, with regard to [23, Chapter 1], we have that they are sectorial ones. Notice that $A_\alpha,\ \alpha \in (0,\alpha_0]$, is well defined. Indeed, using (11.2.3) we obtain

$$\frac{A^{2\kappa-1}}{2\alpha}\left(1-\left(1-4\alpha A^{1-2\kappa}\right)^{1/2}\right) \in L(X,X),$$

for $\alpha \in (0,\alpha_0]$, and hence

$$A_\alpha = \frac{A^{2\kappa-1}}{2\alpha}\left(1-\left(1-4\alpha A^{1-2\kappa}\right)^{1/2}\right)A^{1-\kappa}$$

Furthermore,

$$A_\alpha^{-1} = \frac{A^{\kappa-1}}{2}\left(1+\left(1-4\alpha A^{1-2\kappa}\right)^{1/2}\right)$$

Therefore, $A_0 A_\alpha^{-1} \to I$ in $L(X,X)$. Similarly, $B_\alpha B_0^{-1} \to I$ in $L(Y,Y)$. Hence the families of operators $\{A_\alpha,\ \alpha \in [0,\alpha_0]\}$ and $\{B_\alpha,\ \alpha \in [0,\alpha_0]\}$ fulfill the hypotheses (H1)-(H2) and (H1) on Hilbert spaces X and Y, respectively.

In terms of A_α and B_α, system (11.2.1) can be rewritten as a system of two abstract equations

$$\begin{aligned} u' + A_\alpha u &= w \\ \alpha w' + B_\alpha w &= f(u) \qquad \alpha \in [0,\alpha_1] \\ u(0) = u_0; \quad w(0) &= w_0 \end{aligned} \tag{11.2.9$_\alpha$}$$

in the space $X \times Y$. Let us take

$$\gamma := \frac{\varrho - (1-\omega)\kappa}{1-\kappa} \quad \text{and} \quad \beta := 1-\omega$$

Then $\gamma, \beta \in (0,1)$ and the functions

$$\begin{aligned} &g : X^\gamma \times Y^\beta \to X; \quad g(u,w) := w \\ &f : X^\gamma \to Y \end{aligned} \tag{11.2.10}$$

satisfy the hypothesis (H3) from Section 9 (here X^γ, Y^β denote the fractional power spaces with respect to sectorial operators $A_0 = A^{1-\kappa}$, $B_0 = A^\kappa$, respectively). Indeed, taking into account (11.2.4), we obtain

$$X^\gamma = D_{\mathcal{X}}(A^{(1-\omega)\kappa+\gamma(1-\kappa)}) = D_{\mathcal{X}}(A^\varrho) = \mathcal{X}^\varrho$$
$$Y^\beta = D_{\mathcal{X}}(A^{\beta\kappa}) = D_{\mathcal{X}}(A^{(1-\omega)\kappa}) = X.$$

Hence, $g \in L(X^\gamma \times Y^\beta, X)$ and $f \in C^{1+\eta}_{bdd}(X^\gamma, Y)$

Having developed this background we can apply Theorem 10.2.7 to semiflows generated by systems $(11.2.9)_\alpha, \alpha \in [0,\alpha_0]$, in the phase space $X^\gamma \times Y^\beta$. With regard to Remark 10.2.8 and (11.2.4), the assumptions of Theorem 10.2.7 are fulfilled whenever $\inf_{n\in N} \lambda_n^{(1-\kappa)\gamma}/(\lambda_{n+1}^{1-\kappa}-\lambda_n^{1-\kappa}) = 0$. Because $(1-\kappa)\gamma = \varrho-\kappa+\omega\kappa =: \delta$ and $\omega \in (0,(1-\varrho)/\kappa)$, the last condition becomes

$$\inf_{n\in N} \frac{\lambda_n^\delta}{\lambda_{n+1}^{1-\kappa} - \lambda_n^{1-\kappa}} = 0 \quad \text{for some} \quad \delta \in (\varrho-\kappa, 1-\kappa) \qquad (11.2.11)$$

Theorem 11.2.1. *Assume that the hypothesis (E) and (11.2.11) are satisfied. Let* $\gamma := \delta/(1-\kappa)$ *and* $\beta := (\delta-\varrho)/\kappa$. *Then the conclusions of Theorem 10.2.7 hold for semiflows generated by systems* $(11.2.9)_\alpha, \alpha \in [0,\alpha_0]$, $\alpha_0 > 0$ *small enough, in the phase space* $X^\gamma \times Y^\beta$.

Remark 11.2.2. Let us consider system (11.2.2). It is known ([43]) that there exists a compact global attractor for a semiflow generated by (11.2.2). Then, analogously as in Section 11.1, one can modify the function $f(u) := m(\|\nabla u\|^2)\Delta u$ far from a neighborhood of an attractor. Hence the assumption $f \in C^{1+\eta}_{bdd}(X^\varrho, Y)$ is not restrictive when we deal with local invariant manifolds instead of global ones. By classical spectral results (see, e.g. Courant, Hilbert [17]), it follows that $\lambda_n \approx n^{4/N^2}$, where λ_n, $n \in N$ are eigenvalues of the self-adjoint operator $A := \Delta^2$ subject to "hinged ends" boundary conditions $u = \Delta u = 0$ on $\partial\Omega$. In system (11.2.2) we have $\kappa = \varrho = 1/2$. Hence the condition (11.2.11) is satisfied whenever $N = 1$ and $\delta \in (0, 1/4)$.

Remark 11.2.3. Theorem 11.2.1 remains true for the case when the fractional power operator A^κ is replaced by a general self-adjoint linear operator B which commutes with A and is comparable with A^κ (cf. [14]), i.e. there are constants $a, b > 0$ such that

$$a(A^\kappa u, u) \le (Bu, u) \le b(A^\kappa u, u) \quad \text{for any} \quad u \in D(A^\kappa) = D(B)$$

Section 11.

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