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A NOTE ON EXISTENCE OF SOLUTIONS OF QUASILINEAR PERIODIC BOUNDARY VALUE PROBLEMS IN BANACH SPACES

BY M. KUBICOVÁ AND D. ŠEVČOVIČ

1. Introduction.

In this note we study the problem of the existence of a solution of the following abstract quasilinear periodic boundary value problem

$$\text{(QPP)} \qquad x' = A(t,x)x + b(t,x); \quad x(0) = x(\omega) \quad t \in [0,\omega]$$

where $A(t,x)$ is a bounded linear operator and b is a function with values in a reflexive separable Banach space X. The aim is to extend the proof of the existence theorem for (QPP) of Lasota and Opial, who in their paper [7] have considered an analogous problem in the Euclidean space R^n.

The problem of the existence of periodic solutions in Banach spaces has been investigated either in the case when the linear operator $A(t,x)$ is densely defined and generates a compact semigroup (see, for example, Becker [1], Vrabie [10]) or in the assumption that $A(t,x)$ satisfies conditions of a dissipative type (Deimling [4], Lightbourne [8]). We treat here the situation when $A(t,x)$ is a dissipative operator and A, b satisfy certain continuity assumptions on the coefficients. Lasota and Opial have proved the existence of a solution of a quasilinear periodic boundary value problem in R^n using the Schauder fixed point theorem. The proof of [7, Theorème 2] relies on the Arzelà-Ascoli theorem which, however, in the case of an infinite dimensional Banach space requires an additional assumption (cf. [6, Th. 1.6.9]). In this paper, the method of the proof, following the approach used in [7], is based on the freezing of the coefficients of the problem (QPP), i.e. for a given function x we solve the linear periodic boundary value problem $z' = A(t, x(t))z + b(t, x(t))$; $t \in [0, \omega]$. The mapping $x \mapsto z$ need not be compact in the strong norm topology of the Banach space of continuous functions and therefore we will have to work with a locally convex topological space of weakly continuous functions. The existence of a solution of the problem (QPP) is then assured by the Tichonoff Fixed Point Theorem.

2. Main results.

Let X be a Banach space with norm $\|.\|$ and $B \subseteq X$. Denote by $LB(X)$ the Banach space of bounded linear operators on X, $L\big([0,\omega], X\big)$ the space of Bochner integrable functions from $[0,\omega]$ into X, $C([0,\omega], X)$ the Banach space of all continuous functions from $[0,\omega]$ into X with the sup norm, $C([0,\omega], B)$ the subset of $C([0,\omega], X)$ consisting of all functions with values in B, $C_w([0,\omega], X)$ the locally convex linear topological space of all continuous functions from $[0,\omega]$ into $(X, \sigma(X, X^*))$. The topology τ on $C_w([0,\omega], X)$ is determined by the system of seminorms $q_f(z) = \sup_{t\in[0,\omega]} |f(z(t))|$, $f \in X^*$. Finally, let $C_w([0,\omega], B)$ denote the subset of $C_w([0,\omega], X)$ consisting of all functions with values in B.

Let X, Y be Banach spaces. A function $f : X \to Y$ is called *w-s* continuous iff for any weakly convergent sequence $x_n \rightharpoonup x$ we have $f(x_n) \to f(x)$ strongly in Y when $n \to \infty$. We will say that a function $f : [0,\omega] \times X \to Y$ satisfies *the generalized Caratheodory conditions* iff a): the mapping $t \mapsto f(t, x)$ is strongly measurable for any $x \in X$ and b): the mapping $x \mapsto f(t, x)$ is *w-s* continuous for almost every $t \in [0,\omega]$.

Observe that for any function $x \in C_w([0,\omega], X)$ there exists a sequence $\{x_n\}$ of step functions pointwise weakly converging to x, i.e. $x_n(t) \rightharpoonup x(t)$ as $n \to \infty$ for any $t \in [0,\omega]$. Hence we can conclude that the composite function $t \mapsto f(t, x(t))$ is strongly measurable, whenever f satisfies the generalized Caratheodory conditions and $x \in C_w([0,\omega], X)$.

(1)

First we state results concerning the following linear periodic boundary value problem.

(LPP) $$x' = A(t)x + b(t); \quad x(0) = x(\omega), t \in [0,\omega]$$

in a reflexive Banach space X. As usual, by a solution of the equation $x' = f(t,x), t \in [0,T]$ in a reflexive Banach space X we understand an absolutely continuous function $x(.)$ satisfying this equation almost everywhere on $[0,T]$. Assume that the following hypotheses hold:

(lpp_1) $A \in L([0,\omega], LB(X))$,
(lpp_2) $b \in L([0,\omega], X)$,
(lpp_3) there exists $l \in L([0,\omega], R)$ such that $\int_0^\omega l(s)\,ds < 0$,
$(A(t)x,x)_- \le l(t)\|x\|^2$ for all $x \in X$ and almost all $t \in [0,\omega]$.

Here $(.,.)_-$ denotes the semi-inner product in the Banach space X, i.e.

$$(x,y)_- = \inf\{f(x);\ f \in X^*, \|f\| = \|y\|, f(y) = \|y\|^2\}$$

In the estimates below we will use the following basic properties of the semi-inner product $(.,.)_-$ in the Banach space X:

$$(x+y,z)_- \le (x,z)_- + (y,z)_- \text{ for all } x,y,z \in X$$

$$|(x,y)_-| \le \|x\|\|y\| \text{ for all } x,y \in X$$

if $x:(a,b) \to X$ is Frechét differentiable at t then

$$\|x(t)\|D_t^-\|x(t)\| \le (x'(t),x(t))_- \tag{2}$$

where $D_t^-\|x(t)\|$ denotes the upper left Dini number (see, [2, p.35]).

For later purposes we extend functions A, b and l periodically onto the interval $[0,\infty)$ and identify A, b and l with these extensions. Let $\tilde{x} \in X$ be fixed. Supposing (lpp_1) and (lpp_2) hold, there is the unique solution $x(t)$ of the linear initial value problem $x' = A(t)x + b(t);\quad t \ge 0,\ x(0) = \tilde{x}$. By taking the semi-inner product of $x'(t) = A(t)x(t) + b(t)$ with $x(t)$, integrating with respect to t, using the Gronwall lemma and (2) we obtain

$$\|x(t)\| \le \|x(0)\|e^{\int_0^t l(s)\,ds} + \int_0^t e^{\int_r^t l(s)\,ds}\|b(r)\|\,dr \text{ for any } t \ge 0. \tag{3}$$

Since the functions l and b are extended $\omega-$ periodically on $[0,\infty)$ we conclude that

$$\begin{aligned} \|x(n\omega)\| &\le \|x(0)\|e^{n\alpha} + \sum_{i=1}^{n}\int_{(i-1)\omega}^{i\omega} e^{\int_r^{n\omega} l(s)\,ds}\|b(r)\|\,dr \\ &\le \|x(0)\|e^{n\alpha} + c\int_0^\omega \|b(\xi)\|\,d\xi \text{ for any } n \in N. \end{aligned} \tag{4}$$

where $\alpha := \int_0^\omega l(t)\,dt < 0$ and the constant $c := e^{\int_0^\omega |l|} \sum_{j=1}^\infty e^{j\alpha} > 0$ depends only on the function l.

Let us define an operator $T_\omega : X \to X$; $T_\omega(\tilde{x}_0) := x(\omega)$ where x is a solution of the linear initial value problem $x' = A(t)x + b(t)$; $t \geq 0$, $x(0) = \tilde{x}_0$. By the hypotheses (lpp_1) and (lpp_2), the operator T_ω is well-defined and contractive with a contraction constant being $e^\alpha < 1$. By the Banach fixed point theorem there is the unique fixed point $\tilde{x}$ of the operator T_ω. Hence the problem (LPP) has the unique solution.

Let $x(.)$ be the solution of the (LPP). From (4) we obtain

$$\|x(0)\| \leq \frac{c}{(1-e^\alpha)} \int_0^\omega \|b(r)\|\,dr.$$

By (3) we have

$$\sup_{t\in[0,\omega]} \|x(t)\| \leq c_1 \int_0^\omega \|b(r)\|\,dr \tag{5}$$

where $c_1 := (1 + c/(1-e^\alpha))e^{\int_0^\omega |l|} > 0$ is a constant.

Now we are in a position to examine the quasilinear periodic boundary value problem (QPP) in a reflexive separable Banach space X. Assume that the following hypotheses hold:

(qpp_1) $A : [0,\omega] \times X \to LB(X)$ satisfies the generalized Caratheodory conditions

(qpp_2) $b : [0,\omega] \times X \to X$ satisfies the generalized Caratheodory conditions

(qpp_3) there exists $l \in L([0,\omega], R)$ such that $\int_0^\omega l(s)\,ds < 0$,
$(A(t,x)y,y)_- \leq l(t)\|y\|^2$ for all $x, y \in X$ and almost all $t \in [0,\omega]$,

(qpp_4) there exists $p \in L([0,\omega], R)$ such that
$\|A(t,x)\| \leq p(t)$ for all $x \in X$ and almost all $t \in [0,\omega]$ and

(qpp_5) $\lim_{n\to\infty} \frac{1}{n} \int_0^\omega \sup_{\|y\|\leq n} \|b(t,y)\|\,dt = 0.$

The main result of this paper reads as follows

THEOREM. *Let X be a separable reflexive Banach space. Suppose that (qpp_1) - (qpp_5) hold. Then the quasilinear periodic boundary value problem (QPP) has at least one solution in X.*

Proof: The main idea of the proof is similar, in spirit, to that of Lasota & Opial [7, Th.2]. Let us denote $\mathcal{X} = C_\omega([0,\omega], X)$. Let us for now choose an arbitrary $n \in N$ and consider a set $B_n = \{x \in X; \|x\| \leq n\}$. In the case X being reflexive separable we have that B_n is weakly compact and the weak topology is metrizable on B_n by a metric d such that $d(x, y)$ depends only on the difference $x - y$ and $d(x, y) \leq \|x - y\|$ (see, [5,Lemma 7.2.1]). Let us also denote $\mathcal{D}_n = C_w([0, \omega], B_n)$. Then the topology τ restricted to $\mathcal{D}_n$ is metrizable by the metric $\tilde{d}(x, y) := \sup_{t \in [0,\omega]} d(x(t), y(t))$.

Let us define an operator $S : \mathcal{D}_n \to \mathcal{X}$ $S(x) := z$ where z is a solution of the linear periodic boundary value problem (LPP) with $A(t) = A(t, x(t))$ and $b(t) = b(t, x(t))$. The number $n \in N$ appearing in the definition of the set $\mathcal{D}_n$ will be determined later. Due to the hypotheses (qpp_1) - (qpp_5) it follows that the mappings

$$t \mapsto A(t) \quad \text{and} \quad t \mapsto b(t) \tag{6}$$

satisfy the hypotheses (lpp_1) - (lpp_3). Indeed, the hypotheses (lpp_1) and (lpp_2) follow from $(qpp_1) - (qpp_2)$ and (1) while (lpp_3) is obvious by the assumption (qpp_3). Hence the operator S is well-defined.

We will prove that there exists such $n \in N$ that operator S fulfills on $\mathcal{D}_n$ all assumptions of the Tichonoff Fixed Point Theorem. The set $\mathcal{D}_n$ is a convex, bounded and closed subset of $\mathcal{X}$. Now we will show that the operator S maps $\mathcal{D}_n$ into itself for a certain $n \in N$. In fact, we prove that there is an $n \in N$ such that $S\mathcal{D}_n \subset C([0, \omega], B_n)$. The method used in this step is analogous to the one used by Lasota and Opial in their paper. We will proceed by contradiction. Suppose that there is a sequence $\{x_n\}_{n=1}^{\infty} \subset C_w([0, \omega], B_n)$ such that $z_n = Sx_n \notin C([0, \omega], B_n)$ for an arbitrary $n \in N$. However, as $z_n \in C([0, \omega], X)$ we inevitably get for all $n \in N$ that $\|z_n\| = \sup_{t \in [0,\omega]} \|z_n(t)\| > n$, i.e. $\|z_n\|/n > 1$, so from the assumption (qpp_5) and (5) we obtain

$$\begin{aligned} 1 \leq \lim_{n \to \infty} \frac{1}{n} \|z_n\| &\leq c_1 \lim_{n \to \infty} \frac{1}{n} \int_0^{\omega} \|b(t, x_n(t))\| \, dt \\ &\leq c_1 \lim_{n \to \infty} \frac{1}{n} \int_0^{\omega} \sup_{\|y\| \leq n} \|b(t, y)\| \, dt = 0, \end{aligned}$$

a contradiction. Therefore there exists $n \in N$ such that $S\mathcal{D}_n \subset \mathcal{D}_n$. From now on we will consider this n as fixed in all further considerations and use it while examining the set $\mathcal{D}_n$.

Furthermore, we will show that the operator $S : \mathcal{D}_n \to \mathcal{X}$ is continuous. Indeed, let us consider an arbitrary $x \in \mathcal{D}_n$ and an arbitrary sequence $\{x_m\}_{m=1}^{\infty} \subset \mathcal{D}_n$ such that $x_m \to x$ in the topology τ. Denote $u_m := z_m - z$ where $z_m := Sx_m$ for all $m \in N$ and $z := Sx$.

Obviously, for all $m \in N$, u_m is a solution of the linear periodic boundary value problem

$$u'_m = \tilde{A}_m(t)u_m + \tilde{b}_m(t), \quad u_m(0) = u_m(\omega)$$

in X where

$$\tilde{A}_m(t) := A(t, x_m(t))$$
$$\tilde{b}_m(t) := \big(A(t, x_m(t)) - A(t, x(t))\big)\, z(t) + b(t, x_m(t)) - b(t, x(t))$$

By (6), both $\tilde{A}_m$ and $\tilde{b}_m$ fulfill the hypotheses (lpp_1) - (lpp_3) with a function $l \in L([0,\omega], R)$ independent of $m \in N$.

From (qpp_1) and (qpp_2) we have $\|\tilde{b}_m(t)\| \to 0$ for almost all $t \in [0,\omega]$. Recall that $\sup_{t\in[0,\omega]} \|u_m(t)\| \le c_1 \int_0^\omega \|\tilde{b}_m(t)\|\, dt$ where $c_1 \in R$ is a non-negative constant independent of $m \in N$. Thus $\sup_{t\in[0,\omega]} \|u_m(t)\| \to 0$. It means that $Sx_m \to Sx$ in the space $C([0,\omega], X)$. Since the norm topology of $C([0,\omega], X)$ is stronger than the topology τ restricted to $C([0,\omega], X)$ we have $Sx_m \to Sx$ in $\mathcal{X}$. Hence $S : \mathcal{D}_n \to \mathcal{D}_n$ is continuous.

Finally, we will show that the set $\mathcal{M} := S(\mathcal{D}_n) \subset \mathcal{D}_n$ is equicontinuous, i.e. for all $\varepsilon > 0$ there is $\delta > 0$ such that $|t - s| < \delta$, $t, s \in [0,\omega]$, implies $d(z(t), z(s)) < \varepsilon$ for all $z \in \mathcal{M}$. Indeed, let $x \in \mathcal{D}_n$ and $z = Sx$. The function z as a solution of a differential equation is absolutely continuous and therefore for all $s, t \in [0,\omega]$ such that $s < t$ we get

$$\|z(t) - z(s)\| = \|\int_s^t z'(\xi)\, d\xi\| \le \int_s^t \|A(\xi, x(\xi))\|\|z(\xi)\|\, d\xi + \int_s^t \|b(\xi, x(\xi))\|\, d\xi.$$

Therefore,

$$d(z(t), z(s)) \le \|z(t) - z(s)\| \le n \int_s^t p(\xi)\, d\xi + \int_s^t \sup_{\|y\|\le n} \|b(\xi, y)\|\, d\xi.$$

Both integrated functions are Lebesgue integrable and hence the set $\mathcal{M}$ is equicontinuous. As (B_n, d) is a compact metric space, by the Arzelà-Ascoli theorem (cf. [6, Th. 1.6.9]) we have that the set $\mathcal{M} \subset C_w([0,\omega], B_n)$ is relatively τ-compact. Hence the operator $S : \mathcal{D}_n \to \mathcal{X}$ is compact. Having assured that all the assumptions of the Tichonoff Fixed Point Theorem are fulfilled we can deduce that the operator S has a fixed point in $\mathcal{D}_n$. It means, however, that the quasilinear periodic BVP (QPP) has at least one solution in X.

$\diamond$

3. An example.

Let us recall that Deimling has in his paper [4] considered a similar periodic boundary value problem. In a Banach space X he has examined the existence of a solution of the following abstract problem:

$$(\mathrm{P}) \qquad u' = f(t,u); \quad u(0) = u(\omega) \quad t \in [0,\omega]$$

Deimling's result reads as follows

[4, Theorem 1]. . *Let X be a Banach space; $D \subset X$ compact and convex; $f : [0,\infty) \times D \to X$ continuous and ω-periodic. Suppose also that f satisfies the boundary condition of a dissipative type*

$$(7) \qquad \liminf_{\lambda\to 0+} \frac{1}{\lambda} dist\big(x + \lambda f(t,x),\ D\big) = 0 \quad \text{for } t \geq 0,\ x \in \partial D$$

Then the abstract differential equation $u' = f(t,u)$ has in D an ω-periodic solution.

By the following example we will demonstrate a case of an abstract quasilinear periodic boundary value problem that satisfies all of the hypotheses (qpp_1) - (qpp_5) of our Theorem. Nevertheless, for any $D \subset X$ compact and convex it does not satisfy the dissipative condition (7) from the above stated Theorem due to Deimling.

EXAMPLE. Let us consider the Hilbert space l^2. We will consider the problem (P^*) of the form

$$(P^*) \qquad x' = A(t,x)x + b(t); \quad x(0) = x(2\pi) \quad t \in [0,2\pi]$$

where for all $t \in [0,2\pi]$ and all $x \in l^2$ the linear operator $A(t,x) : l^2 \to l^2$ is defined by $A(t,x) = \psi(t)\varphi\left(\sum_{n=1}^{\infty} \frac{x_n^2}{n^2}\right) Id$. Here Id is the identity operator on l^2, and the functions φ and ψ are defined by

$$\varphi(r) = \pi + \arctan(r)$$

$$\psi(t) = \begin{cases} \sin t; & t \in [0,\pi] \\ 4\sin t; & t \in [\pi, 2\pi]. \end{cases}$$

The function b is defined as follows

$$b(t) = (1,0,0,\dots,0,\dots)\sin^2(t).$$

Now it is a routine to verify that the assumptions (qpp_1) - (qpp_5) are satisfied. Hence, by Theorem, the problem (P^*) has at least one solution in l^2.

One the other hand, we will show that our example does not fulfill the dissipativity assumption (7) for any non-empty compact and convex subset $D \subset l^2$. Indeed, let us consider a non-empty compact and convex set $D \subset l^2$. In the trivial case $D = \theta$ the condition (7) is not satisfied. Let us now consider the case $D \neq \{\theta\}$. The compactness of the set D implies the existence of an element $\bar{x} \in \partial D$ such that $\|\bar{x}\| = \max_{x \in D} \|x\|$. Let us now take an arbitrary $t \in [0, \pi]$. We denote $a(t, x) := \psi(t)\varphi\left(\sum_{n=1}^{\infty} \frac{x_n^2}{n^2}\right)$. For all $\lambda > 0$ and all $y \in D$ we have

$$(8) \qquad \| \left(\bar{x} + \lambda\big(A(t, \bar{x})\bar{x} + b(t)\big) - y\right) \| \geq \|\bar{x} + \lambda a(t, \bar{x})\bar{x} - y\| - \lambda\|b(t)\|.$$

Due to (8) as well as the fact $1 + \lambda a(t, \bar{x}) > 0$ for $t \in (0, \pi)$ we obtain

$$\big(1 + \lambda a(t, \bar{x})\big)\|\bar{x}\| = \|\bar{x} + \lambda a(t, \bar{x})\bar{x}\| \leq$$

$$\leq \|\bar{x} + \lambda a(t, \bar{x})\bar{x} - y\| + \|y\| \leq \|\bar{x} + \lambda a(t, \bar{x})\bar{x} - y\| + \|\bar{x}\|$$

and consequently $\lambda a(t, \bar{x})\|\bar{x}\| \leq \|\bar{x} + \lambda a(t, \bar{x})\bar{x} - y\|$ for all $y \in D$. Coming back to (8) we see now that $\|\bar{x} + \lambda\big(A(t, \bar{x})\bar{x} + b(t)\big) - y\| \geq \lambda a(t, \bar{x})\|\bar{x}\| - \lambda\|b(t)\|$ for all $y \in D$ and $\lambda > 0$. In the case $\liminf_{\lambda \to 0+} \frac{1}{\lambda} \text{dist}\big(\bar{x} + \lambda\big(A(t, \bar{x})\bar{x} + b(t)\big),\ D\big) = 0$ for all $t \in [0, \pi]$ we obtain from the above inequality $0 \geq a(t, \bar{x})\|\bar{x}\| - \|b(t)\|$. Hence we would have $\|\bar{x}\|\frac{\pi}{2}\sin t \leq a(t, \bar{x})\|\bar{x}\| \leq \sin^2 t$ for all $t \in [0, \pi]$. By letting $t \to 0+$ we obtain $\|\bar{x}\| = 0$, a contradiction. Hence we have shown that for any $D \subset X$ compact and convex our example does not fulfill the dissipativity condition (7).

Summarizing, we have that the problem (P^*) has according to our Theorem 1 at least one solution in l^2 although the assumptions of Deimling's theorem are not satisfied.

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Department of Mathematical Analysis
Faculty of Mathematics and Physics
Comenius University
Mlynská dolina, 842 15 Bratislava
Slovak Republic